STABLE AND CAUSAL LTI-PRECODERS AND EQUALIZERS FOR MIMO-ISI CHANNELS WITH OPTIMAL ROBUSTNESS PROPERTIES

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ABSTRACT

In this paper, we address the problems of precoding and equalization in frequency-selective MIMO channels by channel inversion. The channel is assumed to be stable and causal. As our main result, we give a closed form expression for a linear precoder which is stable and causal as well as time-invariant. In particular, our solution further is optimal in terms of robustness and not restricted to finite impulse response (FIR) systems.

1. INTRODUCTION

The appearance of multiple input multiple output (MIMO) systems in mobile communications introduced several new problems in channel precoding and equalization. The standard situation here is equalization of a MIMO-channel with inter symbol interference (ISI). There, due to multipath propagation, signal vectors transmitted successively arrive at the various receive antennas at different times over different paths and superpose. In order to cope with these highly complex effects, multicarrier techniques like orthogonal frequency division multiplexing (OFDM [1]) may be used. However, as it has been shown in [2], those techniques can be outperformed by single carrier approaches in certain settings. While there exist techniques to equalize a finite impulse response (FIR) transfer function with single carrier techniques, little is known on the optimal equalizer for general, non-FIR channels. At this point we want to contribute with this work.

In our paper we give a closed form solution for a precoder for a stable and causal MIMO-ISI channel, which itself is stable and causal. Furthermore, it is linear and time-invariant (LTI) and of minimum stability norm. A closed form solution of this precoder is especially interesting for two reasons.

Firstly, we encounter bad approximation properties for the space of all stable and causal transfer functions. It is known, that there exist stable and causal transfer functions, which cannot be approximated by arbitrary FIR functions [3]. Thus, it may occur that the precoder cannot be found by polynomial approximation. Also, since the space has no basis, approximation by filter banks (even of infinite length) is impossible in general [4].

The second reason is the minimum norm property. For MIMO-ISI channels a completely new phenomenon has been observed. It was shown in [5, 6] that if causality is required for precoders, channel conditions can emerge which force the stability norm of any precoder to grow at least exponentially with the number of transmit or receive antennas in the system. Since the stability norm is an inverse measure for robustness as well as for effective receive energy, a high stability norm results in bad robustness properties and low effective receive energy. Apparently, it is especially important in such scenarios to use a precoder that minimizes the impact of these unfavourable channel conditions, which is a minimum stability norm precoder. Note, that the minimum stability norm over all stable and causal LTI-precoders is the same as over all causal but not neccessarily stable, linear or time-invariant precoders [6]. Therefore, the requirement of being stable and LTI actually does not decrease the precoders performance. Since we can calculate an equalizer for a MIMO channel by calculating a precoder for its adjoint channel, we obtain a formula for equalizers, too. In that case, instead of being a measure for effective receive energy, the stability norm is a measure for noise enhancement.

There has been various previous work on single carrier equalization in MIMO-ISI channels. In the case of FIR channels, the problem has been analysed in [2]. Moreover, optimal precoding has been modeled as a convex optimization problem in [7], while it has been tackled by a semidefinite programming approach in [8]. In the general case of non-FIR channels, the focus has been set on the norm trade-off between causal and non-causal equalizers in [6]. Most notably, the minimum norm precoder was characterized geometrically as the precoder which maximizes the angle between two special spaces in [9]. A related problem (there, contrary to our approach, the equalizers don't have to invert the channel) has been solved in [10] for the case that there are as many transmit as receive antennas.

The structure of this paper is as follows. First, section

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2 comments on notations. Then, section 3 introduces our model of the MIMO-ISI channel in the $\mathscr{D}$-domain. Section 4 explains $\mathscr{D}$-domain precoding and defines the problem statement. In section 5 our formula for the stable and causal LTI-precoder having minimum norm is derived. Conclusions follow in section 6. Finally, the appendix in section 7 states some of the lesser known theorems used as well as proofs skipped in the main text for reading convenience.

1.1. General Remarks

The classic approach to linear MIMO-ISI equalization with single carrier techniques is the calculation of a FIR equalizer for a FIR channel. Here, the focus traditionally has been set on the algebraic properties of the equalizer (see e.g. [11, 12] and the references therein). However, also the analytic properties like stability, robustness or system perfomance are important and have to be investigated. As mentioned in the introduction, the stability norm is an important measure for precoder performance regarding those criteria. How difficult it is to investigate the analytical properties depends very much on the underlying MIMO-system [6].

1. In case of a square system the precoder is unique, given that the system is invertible. Then, of course, taking influence on the precoders stability norm is impossible. We see that in the square case, optimal precoding is trivial.
2. Assume, the system is precodeable and non-square, but precoders are not required to be causal. Here, the precoder having optimal stability norm is known to be the pseudoinverse of the transfer function, i.e. $\mathbf{H}^*(\mathbf{H}\mathbf{H}^*)^{-1}$. Again, for non-causal precoders, optimal precoding turns out to be simple.
3. The problem becomes intricate for precodable, non-square systems, where precoders are required to be causal. While there exists a solution with optimal stability norm, i.e. $T^*_{\mathbf{H}}(T_{\mathbf{H}}T^*_{\mathbf{H}})^{-1}$ ($T_{\mathbf{H}}$ is the Toeplitz operator associated to the transfer function), this solution incorporates some bad properties. As a most important one, it is not time-invariant. On time-invariant precoders virtually nothing is known. Now, optimal precoding becomes a difficult task.

It is interesting to see, that not only in equlization but also in other fields the causality constraint introduces severe problems. Recently, two interesting results have been obtained. In approximation theory, there exists no linear approximation method for certain stable and causal filters which itself is causal and stable. Only if the causality or the stability constraint is dropped, a linear approximation method may exist [3]. In the representation of linear systems, filter banks are of interest. With the help of the generalized Fourier series and a filter bank, a explicit representation of any filter can be gained. It turns out, that for an arbitrary filter bank for the disc algebra (which is a subset of the space of stable and causal filters) even the slightest non-causal distortion in the filter can render the representation by this filter bank impossible [4].

In both cases dropping of the causality constraint solves the issues. However, that is a bad idea in general, since causality is a crucial filter property. A filter has to be causal, in order to be realizable. Thus, causality constraints turn out to be a necessary evil, which have extensive theoretical impact in various areas.

2. NOTATIONS

In the following, the complex numbers will be denoted by $\mathbb{C}$ as well as complex matrices with n rows and m columns by $\mathbb{C}^{n\times m}$. For vectors we introduce the shorthand $\mathbb{C}^n := \mathbb{C}^{n\times 1}$. The complex unit disc will be referred to as $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a complex number z or matrix A, $\bar{z}$ and $\bar{A}$ are the (elementwise) complex conjugates, while for a set M, $\overline{M}$ refers to the closure of M.

Since matrix-valued functions are going to be used frequently, we denote them by bold, upper case letters like $\mathbf{A}$. An asterisk supscript A^* means taking adjoints. In the case of a matrix-valued function that is pointwise transpose and conjugation, $\mathbf{A}^*(z) = \mathbf{A}(z)^* = \overline{\mathbf{A}(z)^T}$. Vector-valued functions are given by bold, lower case letters like $\mathbf{a}$.

Hilbert spaces are always referred to by calligraphic capital letters like $\mathcal{H}$, the associated scalar product by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Note that in general, the adjoint A^* of a bounded linear operator A with domain dom$A$, which is acting between Hilbert spaces $\mathcal{F}$ and $\mathcal{G}$, is uniquely determined by the relation $\langle Af, g\rangle_{\mathcal{G}} = \langle f, A^*g\rangle_{\mathcal{F}}$ for any $f \in$ dom$A$.

3. SYSTEM MODEL

Let $\mathbf{H}$ be a causal MIMO system with N inputs and M outputs and a time domain input-output relation

$$y_n = \sum_{k=0}^{\infty} H_k x_{n-k} + \nu_n, \qquad n = 0, 1, 2, ... \tag{1}$$

Here the series of matrices $\{H_k\}_{k=0}^{\infty}$ identifies the channel impulse response, while the series of vectors $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ denote input and output signals, respectively. We set $x_k = 0$ for $k < 0$. Additional noise is provided by the series $\{\nu_k\}_{k=0}^{\infty}$.

The input and output signals are assumed to be of finite energy, i.e.

$$\sum_{k=0}^{\infty} \|x_k\|^2_{\mathbb{C}^N} < \infty \qquad \text{and} \qquad \sum_{k=0}^{\infty} \|y_k\|^2_{\mathbb{C}^M} < \infty.$$

While the impulse response is causal by definition ($H_k = 0$ for $k < 0$), we further assume its stability (finite energy input signals yield finite energy output signals).

In order to go on, the time domain relation of the MIMO system has to be transferred into the so-called $\mathscr{D}$-domain. To transfer an arbitrary sequence $\{f_k\}_{k=0}^{\infty}$, we consider the $\mathscr{D}$-transform

$$\{f_k\}_{k=0}^{\infty} \overset{\mathscr{D}}{\mapsto} f(z) := \sum_{k=0}^{\infty} f_k z^k, \qquad z \in \mathbb{D},$$

with inverse transform

$$f(z) \overset{-1}{\mapsto} \{f_k\}_{k=0}^{\infty}, \qquad f_k := \frac{1}{2\pi i} \int_{|z|=1} f(z) z^{-(k+1)} dz.$$

Note that choosing $z \in \partial\mathbb{D} = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ instead of $z \in \mathbb{D}$ would result in a Fourier transform. Similar to the Fourier transform, a convolution theorem applies. Thus, when applying the $\mathscr{D}$-transform to the time domain input-output relation (1), it is transferred into the $\mathscr{D}$-domain relation

$$\mathbf{y}(z) = \mathbf{H}(z)\mathbf{x}(z) + \nu(z), \qquad z \in \mathbb{D},$$

whereas $\mathbf{H}$ is the $\mathscr{D}$-transformed impulse response called transfer function and $\mathbf{x}$ and $\mathbf{y}$ are the $\mathscr{D}$-transformed input and output signals, respectively. The $\mathscr{D}$-transformed additional noise is given by the term ν.

In order to express to properties like causality, stability, finite energy or time-invariance in the $\mathscr{D}$-domain, the Hardy spaces have to be introduced.

Definition 1 *Let the vector valued Hardy space $H^2(\mathbb{C}^n)$ be the space of all functions*

$$f : \mathbb{D} \to \mathbb{C}^n, \qquad f(z) = \sum_{k=0}^{\infty} f_k z^k, \qquad f_k \in \mathbb{C}^n,$$

which have finite H^2-norm, i.e.

$$\|f\|^2_{H^2(\mathbb{C}^n)} = \sum_{k=0}^{\infty} \|f_k\|^2_{\mathbb{C}^n} < \infty,$$

and the matrix valued Hardy space $H^{\infty}(\mathbb{C}^{m\times n})$ be the space of matrix valued functions

$$F : \mathbb{D} \to \mathbb{C}^{m\times n}, \qquad F(z) = \sum_{k=0}^{\infty} F_k z^k, \qquad F_k \in \mathbb{C}^{m\times n},$$

with finite H^{∞}-norm, i.e.

$$\|F\|_{H^{\infty}(\mathbb{C}^{m\times n})} = \sup_{z\in\mathbb{D}} \sup_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|F(z)x\|_{\mathbb{C}^m}}{\|x\|_{\mathbb{C}^n}} < \infty.$$

The H^2- and H^{∞}-norms are also known as energy and stability norm, respectively.

It can be shown that a series is stable, causal and LTI in the time domain if and only if its $\mathscr{D}$-transform is H^{∞}. Similarly, a sequence is of finite energy, causal and LTI if and only if its $\mathscr{D}$-transform is H^2. Thus, we have $\mathbf{H} \in H^{\infty}(\mathbb{C}^{M\times N})$ as well as $\mathbf{x} \in H^2(\mathbb{C}^N)$ and $\mathbf{y} \in H^2(\mathbb{C}^M)$. For additional information on how time domain properties translate into spaces of functions in the $\mathscr{D}$-domain, we refer to [13].

4. PRECODING IN THE $\mathscr{D}$-DOMAIN

4.1. Definition and Properties

We start with a precise definition of a general linear precoder.

Definition 2 *We say that $\mathbf{G}$ is a precoder of $\mathbf{H}$, if and only if*

$$\mathbf{x}(z) = \mathbf{H}(z)\mathbf{G}(z)\mathbf{x}(z)$$

for any $\mathbf{x} \in H^2(\mathbb{C}^M)$ and $z \in \mathbb{D}$.

A precoder can be used to calculate the necessary transmit signal $\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x}$ such that any signal $\mathbf{x}$ is received at the output of the channel, i.e.

$$\mathbf{x}(z) \approx \mathbf{H}(z)\tilde{\mathbf{x}}(z) + \nu(z).$$

Precoders of interest will be those which are stable, causal and LTI. As mentioned in the previous section, the following definition is equivalent to the definition usually given in the time domain.

Definition 3 *A precoder $\mathbf{G}$ is called stable and causal, linear and time invariant (or simply a H^{∞}-precoder) if and only if $\mathbf{G} \in H^{\infty}(\mathbb{C}^{N\times M})$.*

For further investigations, the following two insights on H^{∞}-precoders are required.

4.1.1. Existence of H^{∞}-Precoders

Due to the matrix corona theorem, which can be found e.g. in [14], it is known that such a H^{∞}-precoder exists if and only if $\mathbf{H}$ fulfills the condition

$$\mathbf{H}(z)\mathbf{H}(z)^* \geq \delta^2 I \tag{2}$$

for a $\delta > 0$ and any $z \in \mathbb{D}$.

4.1.2. The Minimum Stability Norm

The minimum norm of all H^{∞}-precoders of $\mathbf{H}$ is given by the well-defined term

$$\delta_c^{-1} := \left(\inf_{\|y\|_{H^2(\mathbb{C}^M)}=1} \|P_+(\mathbf{H}^* y)\|_{H^2(\mathbb{C}^N)} \right)^{-1}.$$

Thereby P_+ denotes the Riesz projection into Hardy space, i.e.

$$P_+ f(z) = \sum_{k=0}^{\infty} f_k z^k \in H^2$$

for any given

$$f(z) = \sum_{k=-\infty}^{\infty} f_k z^k \text{ with } \sum_{k=-\infty}^{\infty} \|f_k\|^2 < \infty.$$

This can be seen with the help of Corollary 11 (see appendix). The corollary further shows that this bound is sharp, i.e. there exists a H^{∞}-precoder $\mathbf{G}$ of $\mathbf{H}$ with $\|\mathbf{G}\| = \delta_c^{-1}$.

4.2. Advantages of Minimum Norm Precoding

Choosing a precoder having minimum norm gives two major advantages in general.

4.2.1. *Optimal Robustness*

By the term robustness we understand sensitivity against errors in the channel measurement. Assume a perturbated transfer function

$$\hat{\mathbf{H}}(z) = \mathbf{H}(z) + \Delta(z),$$

where $\mathbf{H}(z)$ is the unperturbated channel and $\Delta(z)$ a small error. Because of

$$\|\mathbf{x} - \mathbf{H}\hat{\mathbf{G}}\mathbf{x}\|_2 = \|\Delta\hat{\mathbf{G}}\mathbf{x}\|_2 \leq \|\Delta\|_\infty \|\hat{\mathbf{G}}\|_\infty \|\mathbf{x}\|_2$$

the enhancement of the reconstruction error when using a precoder $\hat{\mathbf{G}}$ of $\hat{\mathbf{H}}$ is upper bounded by $\|\hat{\mathbf{G}}\|_\infty$. So choosing $\hat{\mathbf{G}}$ to have minimum norm gives the smallest bound on the resulting reconstruction errors.

4.2.2. *Low Loss in Effective Receive Power*

Since in practical applications the transmit power is upper bounded, $\|\tilde{\mathbf{x}}\|_2^2 \leq P_{max}$, and the power enhancement of the precoded signal is bounded by the precoders norm,

$$\|\tilde{\mathbf{x}}\|_2^2 = \|\mathbf{G}\mathbf{x}\|_2^2 \leq \|\mathbf{G}\|_\infty^2 \|\mathbf{x}\|_2^2,$$

we see that the precoder having minimum norm garantuees the highest amount of availible transmit power. If an equalizer instead of precoder is calculated, this will result in low noise enhancement instead.

4.3. Problem Statement

After having introduced precoding in the $\mathscr{D}$-domain and the advantages of minimum norm precoding, the problem statement can be given.

Let $\mathbf{H} \in H^\infty(\mathbb{C}^{M\times N})$ be a stable and causal MIMO-channel, which fulfills equation (2). We want to obtain a closed form expression for a stable and causal LTI-precoder $\mathbf{G}$ with optimal robustness properties. As seen in this section, that is a function $\mathbf{G} \in H^\infty(\mathbb{C}^{N\times M})$ which satisfies $\mathbf{HG} = I$ and further has minimum H^∞-norm.

5. THE MINIMUM NORM PRECODER

In this section a closed form expression for the minimum norm precoder will be derived. Since that is a problem which is hard to tackle, several non-standard tools from mathematics, mostly operator theory, are required. All non-standard operator theory theorems used can be found in the appendix. Nevertheless some understanding of linear operators and Hilbert spaces is necessary.

5.1. Derivation of the Formula

In section 4.1 the minimum norm for precoders has been given as δ_c^{-1}. We also showed that actually a precoder with norm δ_c^{-1} exists. From now on, consider a scaled transfer function $\tilde{\mathbf{H}} := \delta_c \mathbf{H}$ instead of $\mathbf{H}$. Then, a H^∞-precoder of $\tilde{\mathbf{H}}$ having norm at most one has in fact minimum norm. In this subsection we will show how to obtain such a precoder.

The following result is a corollary to a profound operator theory theorem, which can be found in the appendix. It states the existence of some Hilbert space $\mathcal{H}_0$ and a function W. Those are auxiliary means, which will be used for the construction of a minimum norm precoder. While their properties are not of interest in this part of the paper, Sections 5.2.2 and 5.2.3 explain in detail how to obtain $\mathcal{H}_0$ and W.

Corollary 4 *A H^∞-precoder of $\tilde{\mathbf{H}}$ having norm at most one exists if and only if there exists an auxiliary Hilbert space $\mathcal{H}_0$ and a holomorphic function $W : \mathbb{D} \to \mathcal{L}(\mathcal{H}_0, \mathbb{C}^M)$ such that*

$$\frac{\tilde{\mathbf{H}}(z)\tilde{\mathbf{H}}(w)^* - I}{1 - z\bar{w}} = W(z)W(w)^* \tag{3}$$

for any $z, w \in \mathbb{D}$, whereby $\mathcal{L}(\mathcal{H}_0, \mathbb{C}^M)$ denotes the space of bounded linear operators from $\mathcal{H}_0$ to $\mathbb{C}^M$.

Proof In Theorem 9 (see appendix) choose $d = 1$, $\mathcal{E}_1 = \mathcal{E}_3 = \mathbb{C}^M$, $\mathcal{E}_2 = \mathbb{C}^N$ and $B = I$. Since for $d = 1$ the generalized Schur class $\mathcal{S}_1$ is the class of H^∞-functions having norm at most one, the corollary follows. □

Now, assume equation (3) from corollary 4 holds for $\tilde{\mathbf{H}}$. Then we can define the following mapping.

Definition 5 *Set*

$$\begin{aligned}
D_0 &:= \overline{\mathit{span}} \left\{ \begin{bmatrix} \bar{w}W(w)^* \\ \tilde{\mathbf{H}}(w)^* \end{bmatrix} y : w \in \mathbb{D}, y \in \mathbb{C}^M \right\}, \\
R_0 &:= \overline{\mathit{span}} \left\{ \begin{bmatrix} W(w)^* \\ I \end{bmatrix} y : w \in \mathbb{D}, y \in \mathbb{C}^M \right\}, \\
V_0 &: \quad D_0 \to R_0, \sum_{k=0}^\infty c_k \begin{bmatrix} \bar{w}_k W(w_k)^* \\ \tilde{\mathbf{H}}(w_k)^* \end{bmatrix} y_k \\
&\quad \mapsto \sum_{k=0}^\infty c_k \begin{bmatrix} W(w_k)^* \\ I \end{bmatrix} y_k.
\end{aligned}$$

Note that $D_0 \subset \mathcal{H}_0 \oplus \mathbb{C}^N$ and $R_0 \subset \mathcal{H}_0 \oplus \mathbb{C}^M$. As not obvious, we proof the following proposition in the appendix.

Proposition 6 *V_0 is a well-defined isometry.*

The next theorem shows how to determine a precoder having norm at most one. The proof uses a method which appears in [15].

Theorem 7 *Let*

$$V_{00} = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] : \begin{array}{c} \mathcal{H}_0 \\ \oplus \\ \mathbb{C}^N \end{array} \rightarrow \begin{array}{c} \mathcal{H}_0 \\ \oplus \\ \mathbb{C}^M \end{array} .$$

be the continuation of V_0 with zero, i.e.

$$V_{00}h = \left\{ \begin{array}{ll} V_0 h & , h \in D_0 \\ 0 & , h \in D_0^{\perp} \end{array} \right. .$$

Then,

$$\tilde{\mathbf{G}}(z) := D^* + B^*(I - zA^*)^{-1} zC^*$$

is a H^∞-precoder of $\tilde{\mathbf{H}}$ having norm at most one.

Proof By definition of V_{00} it is

$$\left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \left[\begin{array}{c} \bar{w} W(w)^* \\ \tilde{\mathbf{H}}(w)^* \end{array} \right] y = \left[\begin{array}{c} W(w)^* \\ I \end{array} \right] y$$

for every $w \in \mathbb{D}$ and $y \in \mathbb{C}^M$. This yields two equations,

$$A\bar{w}W(w)^* + B\tilde{\mathbf{H}}(w)^* = W(w)^* \qquad (4)$$

as well as

$$C\bar{w}W(w)^* + D\tilde{\mathbf{H}}(w)^* = I. \qquad (5)$$

Equation (4) is equivalent to

$$B\tilde{\mathbf{H}}(w)^* = (I - A\bar{w})W(w)^*.$$

Since V_{00} is a partial isometry it satisfies $\|V_{00}\| \leq 1$ and in particular $\|A\| \leq 1$. Thus $(I - A\bar{w})$ is invertible for any $w \in \mathbb{D}$ by the Neumann series. Multiplying by $(I - A\bar{w})^{-1}$ now gives

$$W(w)^* = (I - A\bar{w})^{-1} B\tilde{\mathbf{H}}(w)^*.$$

Plugging $W(w)^*$ into (5) then results in

$$(C\bar{w}(I - A\bar{w})^{-1}B + D)\tilde{\mathbf{H}}(w)^* = I.$$

Taking adjoints and replacing w by z gives

$$\tilde{\mathbf{H}}(z)(D^* + B^*(I - zA^*)^{-1}zC^*) = I.$$

Thus,

$$\tilde{\mathbf{G}}(z) = D^* + B^*(I - zA^*)^{-1}zC^*, \qquad z \in \mathbb{D},$$

is a precoder of $\tilde{\mathbf{H}}$. Since that is a holomorphic function and by Lemma 10 (see appendix) it is upper bounded by one,

$$\|D^* + B^*(I - zA^*)^{-1}zC^*\| \leq 1,$$

$\tilde{\mathbf{G}}$ is a H^∞-precoder for $\tilde{\mathbf{H}}(z)$ having norm at most one. □

By observations previously made we know that $\tilde{\mathbf{G}}$ is a H^∞-precoder of $\tilde{\mathbf{H}}$ having minimum norm. Since that is the case if and only if $\delta_c\tilde{\mathbf{G}}$ is a H^∞-precoder of $\mathbf{H}$ having minimum norm, we get our main result.

Corollary 8 *The function $\mathbf{G}$ given by*

$$\mathbf{G}(z) := \delta_c \left(D^* + B^*(I - zA^*)^{-1}zC^* \right)$$

is a H^∞-precoder of $\mathbf{H}$ having minimum norm.

5.2. Structural Properties

The next question of interest is of course, what the operators A, B, C and D do look like. While we leave that topic open for future work, some starting points shall be given.

5.2.1. *The Operator D*

It is relatively simple to reveal some of the structure of D. Note that it is an operator from $\mathbb{C}^N$ to $\mathbb{C}^M$ and therefore independent of w. By setting $w = 0$, we obtain

$$\left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \left[\begin{array}{c} 0 \\ \tilde{\mathbf{H}}(0)^* \end{array} \right] y = \left[\begin{array}{c} H(0)^* \\ I \end{array} \right] y$$

and thus $D\tilde{\mathbf{H}}(0)^*y = y$ for any $y \in \mathbb{C}^M$. Therefore, it is

$$D\mid_{\mathrm{ran}\tilde{\mathbf{H}}(0)^*} = (\tilde{\mathbf{H}}(0)\tilde{\mathbf{H}}(0)^*)^{-1}\tilde{\mathbf{H}}(0).$$

5.2.2. *Reproducing Kernel Spaces*

It is also instructive to know how the function W and the Hilbert space $\mathcal{H}_0$, which were used in the construction of A, B, C and D in Corollary 4, can be chosen. Therefore, we have to introduce the concept of a Reproducing Kernel space. Some more details on Reproducing Kernel spaces and choosing $\mathcal{H}_0$ and W can be found in Section 3.3 of [16].

For our purposes we can use the following definition. Assume a Hilbert space $\mathcal{F}$ of $\mathbb{C}^n$-valued functions on $\mathbb{D}$. A Reproducing Kernel of $\mathcal{F}$ is a $\mathbb{C}^{n\times n}$-valued function

$$k(z, w) =: k_w(z), \qquad z, w \in \mathbb{D},$$

such that

1. $k_w y \in \mathcal{F}$
2. $\langle f, k_w y\rangle_{\mathcal{F}} = \langle f(w), y\rangle_{\mathbb{C}^n}$

for any $f \in \mathcal{F}$, $w \in \mathbb{D}$ and $y \in \mathbb{C}^n$. Every Reproducing Kernel is positive in the sense that

$$\sum_{i,j=1}^{d} \langle k_{z_j}(z_i)y_j, y_i\rangle \geq 0$$

for all $z_1, ..., z_d \in \mathbb{D}$, $y_1, ..., y_d \in \mathbb{C}^n$ and $d = 1, 2, ...$.

Now, for every positive kernel a fundamental result states that there exists a unique Hilbert space $\mathcal{H}(k)$ such that k is a Reproducing Kernel of $\mathcal{H}(k)$. It is possible to construct $\mathcal{H}(k)$ from k. Therefore, choose $\mathcal{H}(k)$ to be the function space which is the completition of

$$\mathrm{span}\left\{ K_w y : w \in \mathbb{D}, y \in \mathbb{C}^M \right\},$$

obtained after factoring out any elements of norm zero, equipped with a scalar product

$$\langle K_{w_1}y_1, K_{w_2}y_2\rangle_{\mathcal{H}(k)} := \langle K_{w_1}(w_2)y_1, y_2\rangle_{\mathbb{C}^M} .$$

5.2.3. $\mathcal{H}_0$ and $W(z)$

Now, $\mathcal{H}_0$ and $W(z)$ as used in Corollary 4 can be specified. It is always possible to choose $\mathcal{H}_0$ to be the Reproducing Kernel space created by the (positive) kernel

$$K_z(w) := \frac{\tilde{\mathbf{H}}(z)\tilde{\mathbf{H}}(w)^* - I}{1 - z\bar{w}},$$

i.e. $\mathcal{H}_0 = \mathcal{H}(K)$. Then, the point evaluation

$$W(z)f := f(z), \qquad z \in \mathbb{D},\ f \in \mathcal{H}_0,$$

is a valid choice for W. Due to the Reproducing Kernel properties it is

$$\langle W(w)f, y\rangle_{\mathbb{C}^M} = \langle f(w), y\rangle_{\mathbb{C}^M} = \langle f, K_w y\rangle_{\mathcal{H}_0}.$$

Thus, the adjoint is given by $W(w)^*y := K_w y$ and the desired decomposition

$$W(z)W(w)^* = W(z)K_w = K_w(z) = \frac{\tilde{\mathbf{H}}(z)\tilde{\mathbf{H}}(w)^* - I}{1 - z\bar{w}}$$

of $K_z(w)$ from Corollary 4 holds.

6. CONCLUSIONS

In our paper we derived an explicit formula of a stable and causal LTI-precoder with optimal robustness properties for precoding in stable and causal MIMO-ISI channels. The obtained solution is not limited to the FIR case, but applies to general transfer functions. By using the $\mathscr{D}$-transform, the problem has been modeled as the problem to find a minimum norm H^∞-right inverse of a H^∞-function. Future work should concentrate on further examination of the used operators structural properties.

7. APPENDIX

We now give the proof on V_0 being well-defined and isometric, which has been omitted in subsection 5.1.

Proof (of Proposition 6) We will first show that V_0 is actually a mapping in terms of every $d_0 \in D_0$ being mapped to exactly one $r_0 \in R_0$. Therefore, we have to show that from

$$\begin{bmatrix} \bar{w}_1 W(w_1)^* \\ \tilde{\mathbf{H}}(w_1)^* \end{bmatrix} y_1 = \begin{bmatrix} \bar{w}_2 W(w_2)^* \\ \tilde{\mathbf{H}}(w_2)^* \end{bmatrix} y_2$$

it follows

$$\begin{bmatrix} W(w_1)^* \\ I \end{bmatrix} y_1 = \begin{bmatrix} W(w_2)^* \\ I \end{bmatrix} y_2$$

for any $w_1, w_2 \in \mathbb{D}$ and $y_1, y_2 \in \mathbb{C}^M$. Because of equation (3) it is

$$W(z)W(w)^* + I = \tilde{\mathbf{H}}(z)\tilde{\mathbf{H}}(w)^* + z\bar{w}W(z)W(w)^*$$

for any $z, w \in \mathbb{D}$. Therewith, we get

$$\begin{aligned}
&\left\langle \begin{bmatrix} \bar{w}_1 W(w_1)^* y_1 \\ \tilde{\mathbf{H}}(w_1)^* y_1 \end{bmatrix}, \begin{bmatrix} \bar{w}_2 W(w_2)^* y_2 \\ \tilde{\mathbf{H}}(w_2)^* y_2 \end{bmatrix} \right\rangle_{\mathcal{H}_0 \oplus \mathbb{C}^N} \\
=\ & \langle \bar{w}_1 W(w_1)^* y_1, \bar{w}_2 W(w_2)^* y_2 \rangle_{\mathcal{H}_0} \\
& + \left\langle \tilde{\mathbf{H}}(w_1)^* y_1, \tilde{\mathbf{H}}(w_2)^* y_2 \right\rangle_{\mathbb{C}^N} \\
=\ & \left\langle (\tilde{\mathbf{H}}(w_2)\tilde{\mathbf{H}}(w_1)^* + w_2\bar{w}_1 W(w_2)W(w_1)^*) y_1, y_2 \right\rangle_{\mathbb{C}^M} \\
=\ & \langle (W(w_2)W(w_1)^* + I) y_1, y_2 \rangle_{\mathbb{C}^M} \\
=\ & \langle W(w_1)^* y_1, W(w_2)^* y_2 \rangle_{\mathcal{H}_0} + \langle y_1, y_2 \rangle_{\mathbb{C}^M} \\
=\ & \left\langle \begin{bmatrix} W(w_1)^* y_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} W(w_2)^* y_2 \\ y_2 \end{bmatrix} \right\rangle_{\mathcal{H}_0 \oplus \mathbb{C}^M}
\end{aligned} \tag{6}$$

for arbitrary $w_1, w_2 \in \mathbb{D}$ and $y_1, y_2 \in \mathbb{C}^M$. Since in Hilbert spaces one has $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$, now from

$$\left\| \begin{bmatrix} \bar{w}_1 W(w_1)^* y_1 \\ \tilde{\mathbf{H}}(w_1)^* y_1 \end{bmatrix} - \begin{bmatrix} \bar{w}_2 W(w_2)^* y_2 \\ \tilde{\mathbf{H}}(w_2)^* y_2 \end{bmatrix} \right\|^2_{\mathcal{H}_0 \oplus \mathbb{C}^N} = 0$$

it follows

$$\left\| \begin{bmatrix} W(w_1)^* y_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} W(w_2)^* y_2 \\ y_2 \end{bmatrix} \right\|^2_{\mathcal{H}_0 \oplus \mathbb{C}^M} = 0$$

and V_0 is a valid mapping. Next, V_0 is a linear operator by construction. Finally, V_0 is a isometry since for equation (6) we have

$$\langle V_0 h, V_0 h \rangle_{\mathcal{H}_0 \oplus \mathbb{C}^N} = \langle h, h \rangle_{\mathcal{H}_0 \oplus \mathbb{C}^N}$$

for any $h \in D_0$. $\square$

Next, we will state two results from operator theory which have been used in the proof of Theorem 7 in Subsection 5.1. The following result is due to Ball and Trent in [16] (Theorem 5.2).

Theorem 9 *Let $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ be three Hilbert spaces and suppose that A and B are given holomorphic functions on $\mathbb{D}^d$ with values in $\mathcal{L}(\mathcal{E}_2, \mathcal{E}_3)$ and $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_3)$ respectively. In order that there exist $F \in \mathcal{S}_d(\mathcal{E}_1, \mathcal{E}_2)$ with $A(z)F(z) = B(z)$ on $\mathbb{D}^d$, it is necessary and sufficiant that there exist auxiliary Hilbert spaces $\mathcal{C}_1, ..., \mathcal{C}_d$ and d holomorphic functions $H_1(z), ..., H_d(z)$ on $\mathbb{D}^d$, with $H_k(z)$ having value in $\mathcal{L}(\mathcal{C}_k, \mathcal{E}_3)$ for $k = 1, ..., d$, such that*

$$A(z)A(w)^* - B(z)B(w)^* = \sum_{k=1}^{d} (1 - z_k\bar{w}_k) H_k(z) H_k(w)^*$$

for all $z = (z_1, ..., z_d)$ and $w = (w_1, ..., w_d)$ in $\mathbb{D}^d$.

The next lemma can be found in [15].

Lemma 10 *Let*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{K} \end{matrix} \rightarrow \begin{matrix} \mathcal{H}_2 \\ \oplus \\ \mathcal{K} \end{matrix}$$

be a bounded linear operator, where $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{K}$ are Hilbert spaces and assume $\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq 1$. *Then for* $|z| < 1$,

$$\|A + zB(I - zD)^{-1}C\| \leq 1.$$

This last corollary follows directly from Theorem 9.2.1 in [17], which is also known as the Toeplitz Corona Theorem.

Corollary 11 *Assume $\delta > 0$. A H^∞-precoder $\mathbf{G}$ of $\mathbf{H}$ with $\|\mathbf{G}\|_\infty \leq \delta^{-1}$ exists if and only if it is*

$$\|P_+(\mathbf{H}^* y)\|_{H^2(\mathbb{C}^M)} \geq \delta \|y\|_{H^2(\mathbb{C}^N)}$$

for any $y \in H^2(\mathbb{C}^N)$.

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