A GENERAL FRAMEWORK FOR CONCAVE/CONVEX INTERFERENCE COORDINATION PROBLEMS AND NETWORK UTILITY OPTIMIZATION

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ABSTRACT

Many problems in wireless communications (e.g. multiuser beamforming or robust power allocation) can be traced back to convex or concave interference functions. While fixed-point iterations are available for general standard interference functions, a better, super-linear convergence behavior can be achieved by exploiting convexity. In this paper we show that that every convex (resp. concave) interference function can be expressed as an optimum over elementary linear functions weighted by coefficients. Based on this representation, both transmit powers and coefficients can be optimized iteratively. The proposed framework also shows a connection between interference functions and feasible regions. Furthermore, every convex or concave interference function can be interpreted as an optimum over a sub-level set.

1. AXIOMATIC INTERFERENCE MODEL

Axiomatic frameworks have a longstanding tradition in communication theory. A well-known example is the axiomatic framework for power control introduced by Yates [1], who proposed to characterize the multiuser interference coupling by means of *interference functions* $\mathcal{I}_1(\boldsymbol{p}), \dots \mathcal{I}_K(\boldsymbol{p})$, for K communication links in a multiuser system. The function $\mathcal{I}_k(\boldsymbol{p})$ yields the interference power of the kth user caused by the vector of transmission powers $\boldsymbol{p} = [p_1, \dots, p_K]^T$. The way how $\mathcal{I}_k(\boldsymbol{p})$ depends on $\boldsymbol{p}$ is only characterized by certain properties, like positivity, monotonicity, scalability [1].

This framework was recently generalized in [2]. In this work, a function $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is called an interference function if the following axioms are fulfilled.

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A1 $\mathcal{I}(\boldsymbol{p}) \geq 0$
A2 $\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p}) \quad \text{for } \alpha > 0$
A3 $\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\boldsymbol{p}') \text{ if } \boldsymbol{p} \geq \boldsymbol{p}'$.

Note, that an interference function characterized by A1–A3 can be 'standard' as defined in [1]. Assume that $\mathcal{I}(\boldsymbol{p})$ is strictly monotone with respect to one component of $\boldsymbol{p}$. This component could be a fixed noise power, for example. Then $\mathcal{I}(\boldsymbol{p})$ is standard with respect to the remaining components [2].

In this paper we do generally not require $\mathcal{I}(\boldsymbol{p})$ to be standard (except for some application examples). Instead, we focus on another interesting property: It is assumed that $\mathcal{I}(\boldsymbol{p})$ is convex or concave. This property appears in many practically relevant contexts, some of which will be discussed in the following section.

Notational conventions are: Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component. Likewise, $A_{mn} = [\boldsymbol{A}]_{mn}$ is a component of the matrix $\boldsymbol{A}$. The notation $\boldsymbol{y} \geq 0$ means that $y_l \geq 0$ for all components l. $\boldsymbol{y} \geq \boldsymbol{x}$ means component-wise inequality. $\boldsymbol{x} \circ \boldsymbol{y}$ denotes element-wise multiplication of vectors or matrices (Hadamard product). The set of non-negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$.

2. CONVEX/CONCAVE INTERFERENCE FUNCTIONS

We start with a few examples.

Example 1 *(Linear Interference Model).* A common approach to interference modeling is the usage of linear interference functions

$$\mathcal{I}_k(\boldsymbol{p}) = [\boldsymbol{V}\boldsymbol{p}]_k, \quad k = 1, 2, \dots, K \ , \tag{1}$$

where $\boldsymbol{V} \geq 0$ is a fixed *link gain matrix* containing interference coupling coefficients. This linear function is both convex and concave. It also fulfills the axioms A1–A3 so it is a special case of the framework under investigation.

The model (1) has been widely investigated and many interesting properties were derived, some of which can be directly connected to convexity. For example, $\mathcal{I}_k(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^K$ with the substitution $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$. This property was exploited in [3–6], where it was shown that the resulting log-SIR region, i.e., the set of jointly achievable signal-to-interference ratios $p_k/\mathcal{I}_k(\boldsymbol{p})$, is convex. This is a useful property which can be exploited for resource allocation techniques operating on the boundary of the region (see e.g. [7]).

Example 2 *(MMSE beamforming).* Model (1) can be extended by assuming that the interference coupling $\boldsymbol{V}$ depends on adaptive receive beamforming vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_K$, with $\|\boldsymbol{u}\| = 1$. In this case, the normalized coupling matrix $\boldsymbol{V}(\boldsymbol{u})$ is defined as

$$[\boldsymbol{V}(\boldsymbol{u})]_{kl} = \begin{cases} \frac{\boldsymbol{u}_k^H \boldsymbol{R}_l \boldsymbol{u}_k}{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k} & l \neq k \ , \\ 0 & k = l \ . \end{cases}$$

where $\boldsymbol{R}_l = \mathrm{E}[\boldsymbol{h}_l \boldsymbol{h}_l^H]$ is the spatial covariance matrix of the vector channel $\boldsymbol{h}_l$ containing the complex path attenuations between the lth transmitter and the receiving antenna array. If $\boldsymbol{h}_l$ is deterministic (perfect channel information), then $\boldsymbol{R}_l = \boldsymbol{h}_l \boldsymbol{h}_l^H$.

Under this model, we have an interference function

$$\mathcal{I}_k(\boldsymbol{p}, \sigma_n^2) = \min_{\|\boldsymbol{u}_k\|=1} \big[\boldsymbol{V}(\boldsymbol{u}) \cdot \boldsymbol{p}\big]_k + \frac{\sigma_n^2}{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k} \ . \quad (2)$$

It can be verified that (2) fulfills A1–A3 and is concave, since concavity is preserved under minimization. The SIR[1] has the well-known form

$$\begin{aligned} \mathrm{SIR}_k(\boldsymbol{p}, \sigma_n^2) &= \frac{p_k}{\mathcal{I}_k(\boldsymbol{p}, \sigma_n^2)} \\ &= \max_{\|\boldsymbol{u}_k\|=1} \frac{p_k \cdot \boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k}{\boldsymbol{u}_k^H \Big(\sum_{l \neq k} p_l \boldsymbol{R}_l + \sigma_n^2 \boldsymbol{I}\Big) \boldsymbol{u}_k} \ . \quad (3) \end{aligned}$$

Note, that the $\mathcal{I}_k$ only depends on the kth row of $\boldsymbol{V}(\boldsymbol{u})$. That is, for an arbitrary power vector $\boldsymbol{p} > 0$, optimum beamformers $\boldsymbol{u}_1, \dots, \boldsymbol{u}_K$ can be found independently, as the principal generalized eigenvectors of the respective matrix pairs $(\boldsymbol{R}_k, \sum_{l \neq k} p_l \boldsymbol{R}_l)$, for all $k = 1, \dots, K$. These beamformers maximize the individual SIR of all K users, so they are optimal linear MMSE equalizers (normalized to unit norm).

Example 3 *(Transmit beamforming).* It was shown in [8–10] that the same model (2) can be used in order to jointly optimize K transmit beamformers for a downlink channel. This approach is based on the duality between the uplink interference coupling characterized by $\boldsymbol{V}(\boldsymbol{u})$ and the downlink coupling characterized by the transpose $\boldsymbol{V}(\boldsymbol{u})^T$. This is an example where the framework of interference functions can also be successfully applied to the optimization of transmitters. This is not always possible, e.g. if neither rows nor columns of $\boldsymbol{V}(\boldsymbol{u})$ can be optimized independently.

Example 4 *(Zeroforcing beamforming).* Since the SIR (3) is invariant with respect to a scaling of $\boldsymbol{u}_k$, we can as well choose the normalization $\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k = 1$. Then, the interference function becomes

$$\mathcal{I}_k(\boldsymbol{p}, \sigma_n^2) = \min_{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k = 1} \big[\boldsymbol{V}(\boldsymbol{u}) \cdot \boldsymbol{p}\big]_k + \sigma_n^2 \cdot \|\boldsymbol{u}_k\|_2 \ . \quad (4)$$

Assuming that K is less or equal to the number of antennas, we can further constrain the set of beamformers by requiring $\boldsymbol{u}_k^H \boldsymbol{h}_l = 0$, $l \neq k$. This leads to a complete elimination of the interference, and only the effective noise term remains. We have a new interference function

$$\mathcal{I}_k(\boldsymbol{p}, \sigma_n^2) = \min_{\substack{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k = 1 \\ \boldsymbol{u}_k^H \boldsymbol{h}_l = 0, l \neq k}} \|\boldsymbol{u}_k\|_2 \cdot \sigma^2 \ . \quad (5)$$

The optimal zeroforcing beamformers solving (5) are obtained by a least squares approach: The beamformer $\boldsymbol{u}_k$ is given as the kth row of the pseudo-inverse $\boldsymbol{H}^\dagger$.

This function (5) fulfills A1–A3, so it is an interference function. But it does actually not deserve this name since it is simply a linear function of the noise power σ_n^2. But the example illustrates how additional constraints can be added without altering the fundamental properties A1–A3. Also concavity is preserved. Further examples for additional constraints can be found, e.g., in [11].

Example 5 *(Base station assignment).* The above models can be further extended by allowing the receiver to make a choice between different propagation paths of the channel. For example, consider the problem of combined beamforming and base station assignment [8,12–14]. The basic idea is to choose from a set of possible receivers (base stations) the one with the best link quality. Consider an uplink system with receiving base stations from a set $\mathcal{B}$. For the kth user, the system can choose an assignment $b_k \in \mathcal{B}_k$. The assignment b_k is the index of the base station which is to receive the signal. Then $\boldsymbol{R}_l^{(b_k)}$ is the covariance of the channel between the lth transmitter and the receiving antenna array which consists of all antennas belonging to base station b_k.

Since the choice of the receiving base station does not influence the interference of other users, we can optimize the assignments independently for all K communication links:

$$\mathcal{I}_k(\boldsymbol{p}, \sigma_n^2) = \min_{b_k \in \mathcal{B}_k} \left(\min_{\boldsymbol{u}_k : \|\boldsymbol{u}_k\|=1} \frac{\boldsymbol{u}_k^H \big(\sum_{l \neq k} p_l \boldsymbol{R}_l^{(b_k)} + \sigma_n^2 \boldsymbol{I}\big) \boldsymbol{u}_k}{\boldsymbol{u}_k^H \boldsymbol{R}_k^{(b_k)} \boldsymbol{u}_k} \right) \ . \quad (6)$$

[1] In case of the model (2), the ratio $p_k/\mathcal{I}_k(\boldsymbol{p}, \sigma_n^2)$ is often denoted as signal-to-interference-plus-noise ratio (SINR), but in order to keep the notation consistent, we prefer to use SIR instead. This notation is also commonly used, e.g. in the context of CDMA receiver design.

Since minimization preserves concavity, this is a concave interference function which fulfills A1–A3.

Note that a signal can be jointly received and coherently combined by several base stations. In this case, b_k is an assignment vector. Such a cooperation further increases the performance and helps mitigating inter-cell interference. The disadvantage is an increased signaling overhead and the need for synchronization.

Example 6 *(Generic Receiver Optimization).* The above examples all have a similar structure. They can be seen as special cases of the generic model

$$\mathcal{I}_k(\boldsymbol{p}) = \min_{z_k \in \mathcal{Z}_k} \big[\boldsymbol{V}(z) \cdot \boldsymbol{p}\big]_k, \quad k = 1, 2, \dots, K \;, \qquad (7)$$

where z_k is a receive strategy chosen from a closed bounded set $\mathcal{Z}_k$. The coupling matrix $\boldsymbol{V}(z)$ depends on $z = (z_1, \dots, z_K)$ in a row-wise fashion (or column-wise if duality is used). That is, the kth row of $\boldsymbol{V}(z)$ only depends on z_k. Also, $\mathcal{Z}_k$ must be such that the minimum exists (e.g. continuous).

This notion of a receive strategy is quite abstract, and it includes all the previous examples as special cases. This generalization was proposed in [2,15,16]. It can be verified that (7) is a concave interference function which fulfills A1–A3. If, in addition, one component of $\boldsymbol{p}$ stands for constant noise, and if $\mathcal{I}_k(\boldsymbol{p})$ is strictly monotone in this component (which is typically fulfilled in practical systems), then the problem of minimizing the total power subject to individual SIR constraints can be solved with super-linear convergence speed [17, 18]. Of course, the same holds for the previous examples of beamforming and base station assignment, which are special cases of the generic model (7).

One main contribution of this paper will be to show that these results can even be further generalized. In fact, *every* concave interference function has a matrix-based structure like (7). This reveals that concavity is one of the fundamental underlying properties that enables such an excellent convergence behavior, as observed in the beamforming context [10].

Example 7 *(Robustness).* Another example is the worst-case model

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c \in \mathcal{C}} [\boldsymbol{V}(c)\boldsymbol{p}]_k, \quad \forall k \;, \qquad (8)$$

where the parameter c, chosen from a closed bounded set $\mathcal{C}$, can stand for the impact of error effects. Performing power allocation with respect to the worst-case interference, such as (8), guarantees a certain degree of robustness (see e.g. [19,20] and the references therein).

The functions (8) are convex and fulfill A1–A3. If one component of $\boldsymbol{p}$ stands for constant noise power, and if $\mathcal{I}_k(\boldsymbol{p})$ is strictly monotone with respect to this component, then the problem of SIR-constrained power minimization can be solved with super-linear convergence [21]. Amongst other properties, this convergence behavior is enabled by convexity. One main contribution of this paper is to show that *every* convex interference function has a matrix-based structure (8).

These examples show that many problems in wireless communications are based on convex (resp. concave) interference functions. Efficient resource allocation algorithms are available for this type of interference model.

In the remainder of the paper it will be shown that every convex/concave interference function characterized by A1–A3 can be represented as an optimum over linear elementary functions. We start by analyzing the concave case in the next section.

3. STRUCTURE RESULT

In this section we show that every convex/concave interference functions can be decomposed into linear elementary functions.

3.1. Concave Functions

Consider a *concave* interference function $\mathcal{I}(\boldsymbol{p})$, characterized by A1–A3. A useful concept for analyzing concave/convex functions is the conjugate function [22]

$$\underline{\mathcal{I}}^*(\boldsymbol{w}) = \inf_{\boldsymbol{p}>0} \Big(\sum_{l=1}^{K} w_l p_l - \mathcal{I}(\boldsymbol{p}) \Big) \;. \qquad (9)$$

The special properties A1–A3 lead to the following result:

Lemma 1. *The conjugate function (9) is either minus infinity or zero, i.e.,*

$$\underline{\mathcal{I}}^*(\boldsymbol{w}) > -\infty \quad \Leftrightarrow \quad \underline{\mathcal{I}}^*(\boldsymbol{w}) = 0 \qquad (10)$$

In the following section we will use the conjugate function and Lemma 1 to show that every concave interference function can be represented in matrix-form.

We know from Lemma 1 that the set of vectors $\boldsymbol{w}$ which lead to a finite conjugate $\underline{\mathcal{I}}^*(\boldsymbol{w}) > -\infty$ is

$$\mathcal{N}_0(\mathcal{I}) = \{\boldsymbol{w} \in \mathbb{R}^K_+ : \underline{\mathcal{I}}^*(\boldsymbol{w}) = 0\} \;. \qquad (11)$$

The following result shows that every $\boldsymbol{w} \in \mathcal{N}_0(\mathcal{I})$ is associated with a hyperplane which upper bounds the interference function.

Lemma 2. *For all $\boldsymbol{w} \in \mathcal{N}_0(\mathcal{I})$, we have*

$$\mathcal{I}(\boldsymbol{p}) \le \sum_l w_l p_l \;, \quad \forall \boldsymbol{p} > 0 \;. \qquad (12)$$

This leads to the first main result, which shows that concave interference functions can always be characterized as the minimum sum of weighted powers.

Theorem 1. *Let $\mathcal{I}$ be an arbitrary concave interference function, then*

$$\mathcal{I}(\boldsymbol{p}) = \min_{\boldsymbol{w} \in \mathcal{N}_0(\mathcal{I})} \sum_{k=1}^{K} w_k p_k \;, \quad \textit{for all } \boldsymbol{p} > 0. \qquad (13)$$

Theorem 1 shows that an arbitrary concave interference function $\mathcal{I}$ can be characterized as the minimum of a weighted sum of powers, optimized over the set $\mathcal{N}_0(\mathcal{I})$. We will now further analyze the relationship between $\mathcal{I}$ and $\mathcal{N}_0(\mathcal{I})$.

Lemma 3. *Let $\mathcal{I}$ be a concave interference function, then the set $\mathcal{N}_0(\mathcal{I}) \subset \mathbb{R}_+^K$ is closed convex.*

The set $\mathcal{N}_0(\mathcal{I})$ can be further characterized:

Lemma 4. *Assume there exists a $\hat{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$, then every $\boldsymbol{w} \geq \hat{\boldsymbol{w}}$ is contained in $\mathcal{N}_0(\mathcal{I})$, as illustrated in Fig 1.*

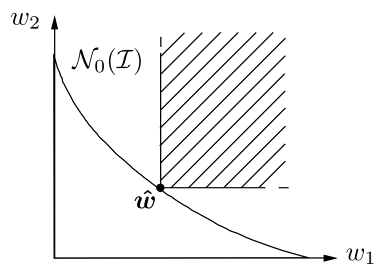

Fig. 1. Illustration of the monotonicity property stated by Lemma 4

Thus far, we have focused our attention on the interference function $\mathcal{I}$. From Lemmas 3 and 4 we know that every concave interference function is associated with a set with the described properties.

Now, we show the converse, namely that every such set is uniquely associated with a concave interference function. To this end, consider a closed convex set $\mathcal{V}$ with the scaling property stated in Lemma 4. For this set we define the interference function

$$\mathcal{I}_{\mathcal{V}} = \min_{\boldsymbol{w} \in \mathcal{V}} \sum_l w_l p_l \tag{14}$$

It can be verified that the function $\mathcal{I}_{\mathcal{V}}$ is concave and fulfills the properties A1–A3. The following result shows that there is a one-to-one relationship between every concave interference function $\mathcal{I}$ and the convex set $\mathcal{N}_0(\mathcal{I})$.

Lemma 5. *Let $\mathcal{V}$ be a closed convex set, with the monotonicity property stated in Lemma 4, then*

$$\mathcal{V} = \mathcal{N}_0(\mathcal{I}_{\mathcal{V}}) \tag{15}$$

This can be further extended:

Corollary 1. *Let $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$, be two arbitrary closed bounded convex sets, with the monotonicity property stated in Lemma 4. If $\mathcal{I}_{\mathcal{N}^{(1)}}(\boldsymbol{p}) = \mathcal{I}_{\mathcal{N}^{(2)}}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$, then $\mathcal{N}^{(1)} = \mathcal{N}^{(2)}$.*

These results allow for an interesting interpretation. Each concave interference function $\mathcal{I}$ is uniquely associated with a convex set $\mathcal{N}_0(\mathcal{I})$, over which we minimize a weighted "cost function" $\sum_k w_k p_k$. This is a familiar problem, which occurs for example in the context of network resource allocation. Suppose that w_k stands for some QoS measure, like bit error rate, or delay. For certain choices of system parameters the QoS region is convex (see e.g. [23]). The weights p_l can be chosen so as to reflect certain priorities among the users. Then, $\mathcal{I}(\boldsymbol{p})$ is the minimum network cost obtained by optimizing over the boundary of the QoS region $\mathcal{N}_0(\mathcal{I})$, as illustrated in Fig. 2. This shows the close connection between the axiomatic interference theory and resource allocation problems.

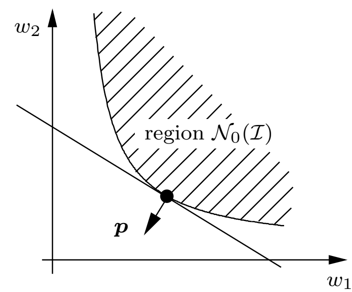

Fig. 2. The concave interference $\mathcal{I}(\boldsymbol{p})$ is the minimum of a weighted cost function optimized over the convex set $\mathcal{N}_0$. For each $\boldsymbol{p}$ a different optimum is obtained.

3.2. Convex Functions

Now, similar properties will be shown for a convex interference function $\mathcal{I}(\boldsymbol{p})$. Most results are in analogy to the concave function. However, there are slight differences, which will be pointed out in the following.

The conjugate function for the convex case is [24]

$$\bar{\mathcal{I}}^*(\boldsymbol{w}) = \sup_{\boldsymbol{p}>0} \Big(\sum_{l=1}^K w_l p_l - \mathcal{I}(\boldsymbol{p}) \Big) . \tag{16}$$

As for the concave function, we have the following result:

Lemma 6. *The conjugate function (16) is either infinity or zero, i.e.,*

$$\bar{\mathcal{I}}^*(\boldsymbol{w}) < +\infty \quad \Leftrightarrow \quad \bar{\mathcal{I}}^*(\boldsymbol{w}) = 0 . \tag{17}$$

The following set contains all $\boldsymbol{w}$ for which the conjugate function (16) is finite.

$$\mathcal{W}_0(\mathcal{I}) = \{ \boldsymbol{w} \in \mathbb{R}_+^K : \bar{\mathcal{I}}^*(\boldsymbol{w}) = 0 \} \tag{18}$$

Each $\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})$ is associated with a hyperplane which lower bounds the interference function.

Lemma 7. *For each $\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})$,*

$$\sum_{l=1}^K w_l p_l \leq \mathcal{I}(\boldsymbol{p}) , \quad \forall \boldsymbol{p} > 0 . \tag{19}$$

Using this result, it can be shown that a convex interference function can always be characterized as the maximum sum of weighted powers.

Theorem 2. *Let $\mathcal{I}$ be an arbitrary convex interference function, then*

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})} \sum_{k=1}^{K} w_k \cdot p_k \ , \quad \textit{for all } \boldsymbol{p} > 0. \tag{20}$$

We now study the properties of $\mathcal{W}_0(\mathcal{I})$.

Lemma 8. *Let $\mathcal{I}$ be a convex interference function, then the set $\mathcal{W}_0(\mathcal{I}) \subset \mathbb{R}_+^K$ is closed bounded convex.*

Lemma 9. *Assume there exists a $\hat{\boldsymbol{w}} \in \mathcal{W}_0(\mathcal{I})$, then every $\boldsymbol{w} \leq \hat{\boldsymbol{w}}$ is contained in $\mathcal{W}_0(\mathcal{I})$, as illustrated in Fig 3.*

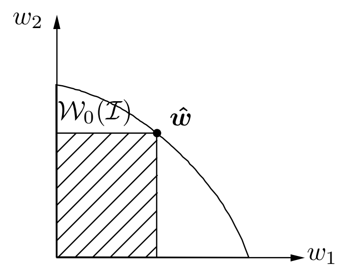

Fig. 3. Illustration of the monotonicity property stated by Lemma 9

The results show that every convex interference function $\mathcal{I}$ can be interpreted as the maximum of the linear function $\sum_l p_l w_l$ over a closed bounded convex set $\mathcal{W}_0(\mathcal{I})$. Conversely, if $\mathcal{V}$ is a closed bounded convex set, with the monotonicity property stated by Lemma 9, then the function

$$\mathcal{I}_\mathcal{V}(\boldsymbol{p}) = \max_{\boldsymbol{w} \in \mathcal{V}} \sum_l p_l w_l \tag{21}$$

is a convex interference function which fulfills A1–A3. Thus, there is a one-to-one relationship between convex interference functions and closed bounded convex sets characterized by Lemma 9. This is specified by the following theorem.

Lemma 10. *Let $\mathcal{V}$ be a closed bounded convex set, with the monotonicity property stated in Lemma 9, then*

$$\mathcal{V} = \mathcal{W}_0(\mathcal{I}_\mathcal{V}) \tag{22}$$

In analogy to the concave case, the convex interference function can be interpreted as the maximum of a weighted utility function optimized over the convex set $\mathcal{W}_0$, as illustrated in Figure 4.

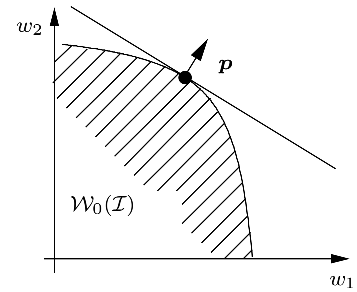

Fig. 4. The convex interference $\mathcal{I}(\boldsymbol{p})$ is the maximum of a weighted utility function optimized over the convex set $\mathcal{W}_0$. For each $\boldsymbol{p}$ a different optimum is obtained.

4. CONNECTION BETWEEN INTERFERENCE FUNCTIONS AND LEVEL SETS

In the previous section it was shown that every convex interference function can be characterized as a maximum of a linear function. The maximization is over a convex set.

Conversely, a convex interference function is obtained from an arbitrary closed bounded convex set $\mathcal{W}$ with the described monotonicity property. It was already mentioned that the set $\mathcal{W}$ can be interpreted as a quality-of-service (QoS) region, over which we wish to maximize a utility function

$$f(\boldsymbol{p}) = \max_{\boldsymbol{q} \in \mathcal{W}} \sum_{k=1}^{K} p_k q_k \ . \tag{23}$$

Here, the weights $\boldsymbol{p} = [p_1, \ldots, p_K]$ stand for individual user priorities, which possibly depend on queue lengths, etc. By appropriately choosing $\boldsymbol{p}$ it is possible to trade off throughput against fairness.

It can be observed that the convex function $f(\boldsymbol{p})$ fulfills the axioms A1–A3, thus $f(\boldsymbol{p})$ itself is a convex "interference function". This is another example, which shows that the proposed axiomatic framework has an importance beyond power allocation.

Now, consider an arbitrary convex interference function $\mathcal{I}(\boldsymbol{p})$. We know from the previous results that $\mathcal{I}$ is associated with a convex set $\mathcal{V}$ such that

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{v} \in \mathcal{V}} \sum_{k=1}^{K} v_k p_k \ . \tag{24}$$

Now, an interesting question is whether the set $\mathcal{V}$ can also be interpreted as level set. As an example, consider the linear interference model introduced in Example 1. The SIR region associated with these K interference functions is the sub-level set

$$\mathcal{S} = \{\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K] : \rho(\boldsymbol{\gamma}) \leq 1\} \ ,$$

where $\rho(\boldsymbol{\gamma})$ is the spectral radius (maximal eigenvalue) of the matrix $\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}$. The indicator function $\rho(\boldsymbol{\gamma})$ is an interference function itself. We have

$$\rho(\boldsymbol{\gamma}) = \min_{\hat{\boldsymbol{\gamma}} \in \mathcal{S}} \max_{1 \le k \le K} \frac{\gamma_k}{\hat{\gamma}_k} \;. \tag{25}$$

Consider again the convex function $\mathcal{I}(\boldsymbol{p})$, which is associated with the convex set $\mathcal{V}$. Also $\mathcal{I}(\boldsymbol{p})$ has a min-max representation of the form (25), with a sub-level set defined by an indicator function $\mathcal{I}(\boldsymbol{p})$. But $\mathcal{I}(\boldsymbol{p})$ is also convex, so it can be expressed as (24). However, this representation is based on the convex set $\mathcal{V}$, with the monotonicity property: $\boldsymbol{v} \le \hat{\boldsymbol{v}}$, with $\hat{\boldsymbol{v}} \in \mathcal{V}$, implies $\boldsymbol{v} \in \mathcal{V}$.

In order to show the connection to sub-level sets, we introduce

$$\underline{L}(\mathcal{I}) = \{\hat{\boldsymbol{p}} > 0 : \mathcal{I}(\hat{\boldsymbol{p}}) \le 1\} \tag{26}$$

$$\overline{L}(\mathcal{I}) = \{\hat{\boldsymbol{p}} > 0 : \mathcal{I}(\hat{\boldsymbol{p}}) \ge 1\} \tag{27}$$

(28)

It can be observed that the set $\underline{L}(\mathcal{I})$ has monotonicity properties like $\mathcal{V}$. It is also closed convex. The function

$$\mathcal{I}_1(\boldsymbol{p}) = \max_{\boldsymbol{v} \in \underline{L}(\mathcal{I})} \sum_k v_k p_k \tag{29}$$

is also a convex interference function.

Now, an interesting question is how the set $\mathcal{V}$ is related to $\underline{L}(\mathcal{I})$. Generally, one might expect that both sets can be different. The answer is provided by the following theorem.

Theorem 3. *Let $\mathcal{I}$ be a convex interference function (24), synthesized from a closed convex set $\mathcal{V}$. Also, $\mathcal{V}$ has the following monotonicity property: if $\boldsymbol{v} \le \hat{\boldsymbol{v}}$, with $\hat{\boldsymbol{v}} \in \mathcal{V}$, then $\boldsymbol{v} \in \mathcal{V}$. Let $\mathcal{I}_1$ be defined as by (29), then*

$$\mathcal{V} = \underline{L}(\mathcal{I}_1). \tag{30}$$

Next, consider a concave interference function $\mathcal{I}$. There exists a closed convex set $\mathcal{V}$, with the monotonicity property $\boldsymbol{p} \ge \hat{\boldsymbol{p}}$, $\hat{\boldsymbol{p}} \in \mathcal{V}$ implies $\boldsymbol{p} \in \mathcal{V}$, such that

$$\mathcal{I}(\boldsymbol{p}) = \min_{\boldsymbol{v} \in \mathcal{V}} \sum_k v_k p_k \tag{31}$$

In analogy, the set $\overline{L}(\mathcal{I})$ has the same properties as the set $\mathcal{V}$. A concave interference function is defined by

$$\mathcal{I}_2(\boldsymbol{p}) = \min_{\boldsymbol{v} \in \overline{L}(\mathcal{I})} \sum_k v_k p_k \tag{32}$$

For the set $\mathcal{V}$, we have the following connection:

Theorem 4. *Let $\mathcal{I}$ be a concave interference function and $\mathcal{V}$ the set in the representation (31) with the aforementioned monotonicity property, then $\mathcal{V} = \overline{L}(\mathcal{I}_1)$.*

Theorems 3 and 4 allow for an interesting interpretation in terms of achievable regions. We have $\mathcal{V} = \underline{L}(\mathcal{I}_1)$, thus the optimization (24) can be regarded as the optimization of the network utility $\sum_{k=1}^{K} v_k \cdot p_k$ over the "feasible region" $\{\boldsymbol{p} : \mathcal{I}_1(\boldsymbol{p}) \le 1\}$. Here, $\mathcal{I}_1(\boldsymbol{p})$ can be seen as an indicator function depending on certain system parameters $\boldsymbol{p}$ values. A similar interpretation holds for concave interference functions. That is, concave/convex interference functions also have an interpretation in terms of an optimization over a level set. But in contrast to the general case, a linear elementary function can as well be used.

5. ALGORITHM

We can exploit that every concave (resp. convex) interference function can be expressed as (13) or (20), respectively. Assume that the first $K' = K - 1$ powers are caused by users with arbitrary concave interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_{K'}$. The last power component $p_K = \sigma_n^2$ is constant noise. All functions $\mathcal{I}_1, \ldots, \mathcal{I}_{K'}(\boldsymbol{p})$ are strictly monotonic with respect to p_K. We are interested in the global power minimum

$$\min_{\boldsymbol{p} > 0 : p_K = \sigma_n^2} \sum_{k=1}^{K'} p_k \quad \text{s.t. } p_k \ge \gamma_k \mathcal{I}_k(\boldsymbol{p}),\; k = 1, 2, \ldots, K' \;, \tag{33}$$

where γ_k is a target SIR. Collecting all targets in a matrix $\boldsymbol{\Gamma} = \mathrm{diag}\{\gamma_1, \ldots, \gamma_{K'}\}$, the global optimum of (33) is found by the following iteration:

$$\bar{\boldsymbol{p}}^{(n+1)} = \sigma_n^2 \big(\boldsymbol{\Gamma}^{-1} - \boldsymbol{A}^{(n)}\big)^{-1} \boldsymbol{b}^{(n)} \tag{34}$$

$$\boldsymbol{w}_k^{(n)} = \operatorname*{arg\,min}_{\boldsymbol{w}_k \in \mathcal{N}_0(\mathcal{I}_k)} \boldsymbol{w}_k^T \begin{bmatrix} \bar{\boldsymbol{p}}^{(n)} \\ \sigma_n^2 \end{bmatrix}, \quad k = 1, 2, \ldots K' \tag{35}$$

where $\boldsymbol{A}^{(n)}$ is the first $K' \times K'$ block of the $K' \times K$ matrix $[\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{K'}]^T$. The vector $\boldsymbol{b}^{(n)}$ is the last column of this matrix. The vector $\bar{\boldsymbol{p}}$ contains the K' powers of the users. The following result is an immediate consequence of the results in [17, 18].

Theorem 5. *The sequence $\boldsymbol{p}^{(n)}$ obtained by the iteration (34) has super-linear convergence.*

For convex functions, the algorithm is the similar, except that $\min$ is replaced by $\max$. In this case, $\mathcal{W}_0(\mathcal{I}_k)$ (defined in analogy) models possible interference uncertainties. Then minimizing the total power subject to $\boldsymbol{p}_k / \mathcal{I}_k(\boldsymbol{p}) \ge \gamma_k$ guarantees a certain degree of robustness.

6. CONCLUSIONS

Convex and concave interference functions have been shown to play a central role in the context of resource allocation for wireless communication systems. One well-known example is the problem of combined multiuser beamforming and power control, also known as space-division-multiple-access (SDMA).

In this paper we show two fundamental properties: Firstly, every convex interference function can be expressed as a maximum over linear elementary functions. This has an interesting interpretation in terms of a weighted network utility optimization, which is optimized over a convex set (the "feasible region"). Likewise, every concave interference function can be expressed as minimum over linear elementary functions. For this case, there is an analogy with the minimization of a weighted network cost function.

Secondly, it is shown that every convex (resp. concave) interference function can be interpreted as an optimum over a level set. This provides a link to results in resource allocation theory, where the QoS region is often expressed as a level set depending on an indicator function for feasibility.

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