PROPORTIONALLY FAIR RESOURCE ALLOCATION FOR WIRELESS NETWORKS

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ABSTRACT

Wireless communication links are often coupled by interference (e.g. smart antenna beamforming system with overlapping beams or non-orthogonal CDMA). It is thus desirable to include physical layer aspects and power control in the quality-of-service model. But this complicates the task of resource allocation. In this paper, we investigate the problem of weighted proportional fairness for log-convex interference functions. By introducing the concept of a dependency matrix, we characterize the coupling between interference functions. This facilitates conditions for the existence of a proportionally fair operating point. We also show under which condition an optimizer exists and we provide a sufficient condition for uniqueness of this optimizer.

1. INTRODUCTION

The motivation for *proportional fair* resource allocation [1] is the observed utility-inefficiency of min-max fairness. A min-max fair scheduler allocates the resources in such a way that all K communication links achieve the same quality-of-service (QoS). This strategy does not perform well in the presence of bottleneck links. If one link is very weak, then it may require all of the available resource in order to achieve an acceptable QoS. Min-max fairness is therefore known for its bad overall system efficiency.

Proportional fairness avoids this effect by putting more emphasis on the links with good channel conditions. This was first studied in the context of wireline networks [1, 2]. Later, it was also successfully applied in the wireless context (see e.g. [3]). Proportional fairness is especially useful for elastic traffic, since it exploits good channel states, and avoids the bad ones, while still preserving a certain degree of fairness.

It was shown in [2, 4] that proportional fairness corresponds to the optimum of a specific utility or cost function. Sometimes, this problem is generalized by introducing individual weighting factors $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_K]^T$, normalized such that $\|\boldsymbol{\alpha}\|_1 = \sum_k \alpha_k = 1$. The weights $\boldsymbol{\alpha}$ can model individual user requirements and possibly depend on system parameters like priorities, queue lengths, etc. By appropriately choosing $\boldsymbol{\alpha}$ it is possible to trade off overall efficiency against fairness. We can assume $\boldsymbol{\alpha} > 0$. The choice $\alpha_k = 0$ simply means that this link is excluded from the optimization.

This problem of weighted proportional fairness can be written as

$$\inf_{[\mathrm{QoS}_1,\dots,\mathrm{QoS}_K]\in\mathcal{Q}} \left(\sum_{k=1}^{K} \alpha_k \cdot \log \mathrm{QoS}_k \right) , \tag{1}$$

where $\mathcal{Q}$ is the quality-of-service (QoS) feasible region (see the illustration in Fig. 1).

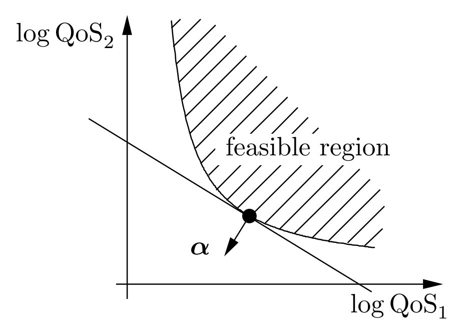

Fig. 1. Illustration of weighted proportional fairness (1). A suitable trade-off point is found by adjusting the weights $\boldsymbol{\alpha}$.

Sometimes, a maximization can be required instead of a minimization, e.g. when QoS stands for throughput or another performance measure that we wish to maximize. In this case, we can still use the form (1) with QoS_k^{-1} instead of QoS_k.

In this paper we will characterize the infimum (1), whose existence is often ensured by the properties of the region $\mathcal{Q}$. However, the region $\mathcal{Q}$ can be complicated, especially for

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wireless systems with mutual interference, which possibly depends on signal processing techniques and other strategies for interference reduction. For example, smart antenna beamforming systems are often non-orthogonal, i.e. users are coupled by overlapping beams. So before we can start characterizing (1), we need to specify the underlying QoS model.

1.1. Quality-of-Service Model

Some notational conventions are: Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component. Likewise, $A_{mn} = [\boldsymbol{A}]_{mn}$ is a component of the matrix $\boldsymbol{A}$. The notation $\boldsymbol{y} \geq 0$ means that $y_l \geq 0$ for all components l. $\boldsymbol{y} \geq \boldsymbol{x}$ means component-wise inequality. The set of non-negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$.

The system consists of propagation channels and transmit/receive strategies, which possibly include signal processing and coding techniques for interference reduction and robustness. Since all these functionalities are interdependent, a joint (cross-layer) optimization would be preferable. However, the analytical treatment of such a complicated model is difficult, if not impossible. It is therefore important to formulate an abstract model, which is simple enough to allow for efficient analysis and optimization, but which still incorporates the most important effects that dominate the system performance.

In this paper, we assume that QoS stands for the signal-to-interference ratio (SIR). The signal-to-interference ratios $\mathrm{SIR}_1, \ldots, \mathrm{SIR}_K$ depend on the transmission powers

$$\boldsymbol{p} = [p_1, \ldots, p_K]^T .$$

Generally, we can write

$$\mathrm{SIR}_k(\boldsymbol{p}) = \frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} \tag{2}$$

for all communication links $k = 1, 2, \ldots, K$, where $\mathcal{I}_k(\boldsymbol{p})$ is the interference (and possibly noise) power experienced by the kth communication link.

In this context, proportional fairness means that we seek the maximum of the utility function $\sum_k \alpha_k \log \mathrm{SIR}_k(\boldsymbol{p})$. Equivalently, we can ask for the minimum of the inverse function

$$-\sum_k \alpha_k \log \mathrm{SIR}_k(\boldsymbol{p}) = \sum_k \alpha_k \log\Big(\frac{1}{\mathrm{SIR}_k(\boldsymbol{p})}\Big)$$

The 'cost' $(1/\mathrm{SIR}_k)$ can be regarded as a high-SIR approximation of the normalized minimum mean square error, i.e., $\mathrm{MMSE}_k = 1/(1+\mathrm{SIR}_k) \approx 1/\mathrm{SIR}_k$. It also models the slope of the BER curve for a system with diversity order one. Another interpretation is the delay (average customer time) $D(\mathrm{SIR})$ for an $M/M/1$ queuing system in the low-SNR regime [5]. Given an arrival rate λ and a service rate ν, the delay is $D(\mathrm{SIR}) = 1/(\nu - \lambda)$. For low SIR, we can approximate $\nu = \log(1 + \mathrm{SIR}_k) \approx \mathrm{SIR}_k$, so $D(\mathrm{SIR}) \approx 1/\mathrm{SIR}$ for small λ.

1.2. Log-Convex Interference Functions

We consider the following interference model. The interference experienced by the kth user is modeled by a function $\mathcal{I}_k : \mathbb{R}_+^K \mapsto \mathbb{R}_+$, which is defined by the following framework of axioms.

A1. $\mathcal{I}(\boldsymbol{p}) \geq 0$ and there exists a $\boldsymbol{p}' > 0$ such that $\mathcal{I}(\boldsymbol{p}') > 0$
A2. $\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p})$ for $\alpha > 0$
A3. $\mathcal{I}(\boldsymbol{p}') \geq \mathcal{I}(\boldsymbol{p})$ if $\boldsymbol{p}' \geq \boldsymbol{p}$
A4. $\mathcal{I}(\mathrm{e}^{\boldsymbol{s}})$ is log-convex on $\mathbb{R}^K$ (substituting $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$)

A function $f(x)$ is said to be *log-convex* if $\log f(x)$ is convex. If $\mathcal{I}(\boldsymbol{p})$ is convex, then $\mathcal{I}(\mathrm{e}^{\boldsymbol{s}})$ is log-convex.

It was shown in [6] that A1 implies that $\mathcal{I}(\boldsymbol{p}) > 0$ for all $\boldsymbol{p} > 0$. So property A1 ensures that interference always depends on at least one power, which is not much of a restriction.

It should be noted that Yates' framework of *standard interference functions* [7] is an important special case of the generic model A1–A4. The function $\mathcal{I}(\boldsymbol{p})$ is standard if one component of $\boldsymbol{p}$ (e.g. noise) is constant and if $\mathcal{I}(\boldsymbol{p})$ is strictly increasing in this component [6].

Example 1 *(Linear Interference Model).* A common approach to interference modeling is the usage of linear interference functions

$$\mathcal{I}_k(\boldsymbol{p}) = [\boldsymbol{V}\boldsymbol{p}]_k, \quad k = 1, 2, \ldots, K , \tag{3}$$

where $\boldsymbol{V} \geq 0$ is a fixed *link gain matrix* containing interference coupling coefficients. This linear function is both convex and concave. It also fulfills the axioms A1–A3 so it is a special case of the framework under investigation.

Example 2 *(Spectral radius).* Consider the linear model (3). The resulting SIR region is defined as

$$\mathcal{S} = \{\boldsymbol{\gamma} : \lambda_p(\boldsymbol{\gamma}) \leq 1\} , \tag{4}$$

where $\lambda_p(\boldsymbol{\gamma}) = \rho\big(\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}\big)$ is the spectral radius (here: the maximum eigenvalue).

The SIR region $\mathcal{S}$ is a sub-level set of the function $\lambda_p(\boldsymbol{\gamma})$. This was exploited in [8–11], where the power vector was substituted by $\exp\{\boldsymbol{s}\}$. The interference functions $\mathcal{I}_k(\exp\{\boldsymbol{s}\})$ are log-convex. Also the spectral radius $\lambda_p(\exp\{\boldsymbol{x}\})$ (choosing the substitution $\boldsymbol{\gamma} = \exp\{\boldsymbol{x}\}$) is log-convex on $\mathbb{R}^K$, thus the log-SIR region $\log(\mathcal{S})$ is a convex set. This is a useful property which can be exploited for resource allocation techniques operating on the boundary of the region (see e.g. [5]).

The spectral radius $\lambda_p(\exp\{\boldsymbol{x}\})$ itself is a log-convex interference function. This shows that the proposed framework is not limited to interference in a physical sense, but it can be applied to other types of coupled systems as well.

Moreover, properties of the interference function $\lambda_p(\boldsymbol{\gamma})$ are closely connected with properties of the level set (4). So the analysis of interference functions can also help to better understand QoS feasible regions.

Example 3 *(Robustness).* Another example is the worst-case model

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c\in\mathcal{C}} [\boldsymbol{V}(c)\boldsymbol{p}]_k, \quad \forall k \ , \tag{5}$$

where the parameter c, chosen from a closed bounded set $\mathcal{C}$, can stand for the impact of error effects. Performing power allocation with respect to the worst-case interference, such as (5), guarantees a certain degree of robustness (see e.g. [12,13] and the references therein).

Example 4 *(Elementary log-convex interference function).* It was shown in [14] that *every* log-convex interference function has a product representation with fundamental log-convex building blocks

$$\mathcal{I}(\boldsymbol{p}) = C \cdot \prod_l (p_l)^{w_l} \ , \quad \boldsymbol{p} > 0, \ \boldsymbol{w} \in \mathcal{W} \ , \tag{6}$$

where $\mathcal{W} = \{\boldsymbol{w} \geq 0 : \|\boldsymbol{w}\|_1 = 1\}$. This will be used later in Section 4.

1.3. Problem Formulation and Contributions

With the above interference model, the problem of weighted proportional fairness (1) can be rewritten as

$$F(\boldsymbol{\alpha}, \mathcal{I}) = \inf_{\boldsymbol{p}>0} \sum_{k=1}^{K} \alpha_k \cdot \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} \ . \tag{7}$$

Note that the optimization is not over $\mathrm{QoS}_1, \ldots, \mathrm{QoS}_K$ directly, as in (1), but over the transmission powers $\boldsymbol{p}$, which are linked to the QoS via the relation $\mathrm{QoS}_k(\boldsymbol{p}) = \mathcal{I}_k/p_k$. This approach is motivated by the close interaction between physical layer and upper layers for wireless systems. For instance, signal processing strategies for robustness and interference rejection (e.g. beamforming [15]) can be included in the definition of $\mathcal{I}_k$.

It can be observed from (7) that if one or more interference values $\mathcal{I}_k(\boldsymbol{p})$ tend to zero, then the infimum $F(\boldsymbol{\alpha}, \mathcal{I})$ tends to minus infinity, which means that there is no proportionally fair operating point. This leads to the following questions:

P1. Under which condition is $F(\boldsymbol{\alpha}, \mathcal{I}) > -\infty$?

P2. If $F(\boldsymbol{\alpha}, \mathcal{I}) > -\infty$ holds, then under which condition does a $\hat{\boldsymbol{p}} > 0$ exist such that

$$F(\boldsymbol{\alpha}, \mathcal{I}) = \sum_{k=1}^{K} \alpha_k \cdot \log \frac{\mathcal{I}_k(\hat{\boldsymbol{p}})}{\hat{p}_k} \quad ? \tag{8}$$

P3. Under which condition does exactly one $\hat{\boldsymbol{p}} > 0$ exist such that (8) holds?

It will be shown in the following that problems P1–P3 are closely connected with the following three resource allocation problems N1–N3, where the optimization is performed with respect to

$$q_k(\boldsymbol{p}) = \log\Big(\frac{1}{\gamma_k(\boldsymbol{p})}\Big), \quad k = 1, 2, \ldots, K \ , \tag{9}$$

where $\gamma_k(\boldsymbol{p})$ is the SIR as a function of $\boldsymbol{p}$. We can write $\boldsymbol{q} = -\log \boldsymbol{\gamma}$, where the logarithm is taken component-wise. Then, $\boldsymbol{\gamma}(\boldsymbol{q}) = \exp(-\boldsymbol{q})$ is the SIR vector required to achieve certain QoS values $\boldsymbol{q}$. Using this definition, we can define the feasible region

$$\mathcal{F}_q = \{\boldsymbol{q} : C\big(\boldsymbol{\gamma}(\boldsymbol{q})\big) \leq 1\} \ , \tag{10}$$

where $C(\boldsymbol{\gamma}) := C\big(\boldsymbol{\gamma}(\boldsymbol{q})\big)$ is the min-max optimum

$$C(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>0} \Big(\max_k \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p})}{p_k} \Big) \ . \tag{11}$$

N1. Under which condition is

$$\inf_{\boldsymbol{q}\in\mathcal{F}_q} \sum_{k=1}^{K} \alpha_k q_k > -\infty \quad ? \tag{12}$$

N2. Under which condition does a $\hat{\boldsymbol{q}} \in \mathcal{F}_q$ exist such that

$$\sum_{k=1}^{K} \alpha_k \cdot \hat{q}_k = \inf_{\boldsymbol{q}\in\mathcal{F}_q} \sum_{k=1}^{K} \alpha_k \cdot q_k > -\infty \quad ? \tag{13}$$

N3. Under which condition does exactly one $\hat{\boldsymbol{q}} \in \mathcal{F}_q$ exist such that (13) holds?

In the remainder of this paper we will show in which way P1–P3 and N1–N3 are connected. We start by analyzing problems P2 and N2 in the following section.

2. EXISTENCE OF AN OPTIMIZER

The behavior of the given problems is determined by the multiuser interference. In order to find answers, we need to specify how users are coupled by interference.

2.1. Characterization of Interference Coupling

Consider the following asymptotic characterization for interference coupling:

Definition 1. The interference coupling is characterized by the *asymptotic matrix*

$$[\boldsymbol{A}_{\mathcal{I}}]_{kl} = \begin{cases} 1 & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that} \\ & \lim_{\delta\to\infty} \mathcal{I}_k(\boldsymbol{p} + \delta \boldsymbol{e}_l) = +\infty \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

where $\boldsymbol{e}_l$ is the all-zero vector with the lth component set to one.

The 1-entries in the kth row of $\boldsymbol{A}_{\mathcal{I}}$ mark the positions of the power components on which $\mathcal{I}_k$ depends. Notice that because of the special properties of the log-convexity of our interference functions, we have the following result:

Lemma 1. *For log-convex interference functions (properties A1–A4) we have $\boldsymbol{A}_{\mathcal{I}} = \boldsymbol{D}_{\mathcal{I}}$, where*

$$[\boldsymbol{D}_{\mathcal{I}}]_{kl} = \begin{cases} 1 & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that} \\ & \mathcal{I}_k(\boldsymbol{p} + \delta \boldsymbol{e}_l) \text{ is not constant for some} \\ & \text{values } \delta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The *dependency matrix* $\boldsymbol{D}_{\mathcal{I}}$ can be used to characterize the achievability of SIR targets $\boldsymbol{\gamma}$. We say that $\boldsymbol{\gamma} > 0$ is achievable if there exists a power allocation vector $\boldsymbol{p} > 0$ such that $\mathrm{SIR}_k(\boldsymbol{p}) = q_k, \forall k$.

For log-convex interference functions, achievability is ensured by an irreducible dependency matrix $\boldsymbol{D}_{\mathcal{I}}$.

Lemma 2. *Let $\boldsymbol{D}_{\mathcal{I}}$ be irreducible, then for all $\gamma > 0$ there exists a $\boldsymbol{p} > 0$ such that*

$$C(\boldsymbol{\gamma}) p_k = \gamma_k \mathcal{I}_k(\boldsymbol{p}), \quad \forall k \ . \tag{15}$$

Note that the fixed-point characterization (15) can be rewritten as $\mathrm{SIR}_k(\boldsymbol{p}) = \gamma_k / C(\boldsymbol{\gamma})$ for all $k = 1, 2, \ldots, K$. That is, all boundary points $\{\boldsymbol{\gamma} : C(\boldsymbol{\gamma}) = 1\}$ can be achieved by appropriately chosen power vectors. This result will play an important role for the following analysis.

2.2. Proportional Fairness and Geometry of the $\mathcal{F}_q$ Region

In general, the problems P1–P3 and N1–N3 are not equivalent, even not for the relatively simple linear case (see Example 5). But our first theorem shows that for given interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$, the set of weighting vectors $\boldsymbol{\alpha}$ for which P1 holds, coincides with the set of weighting vectors for N1. That is, both problems have the same optimum (if existent).

Theorem 1. *For an arbitrary weighting vector $\boldsymbol{\alpha} > 0$,*

$$F(\boldsymbol{\alpha}, \mathcal{I}) = \inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_{k=1}^{K} \alpha_k q_k \ . \tag{16}$$

The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *For an arbitrary weighting vector $\boldsymbol{\alpha} > 0$,*

$$\inf_{\boldsymbol{p} > 0} \sum_{k=1}^{K} \alpha_k \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} > -\infty \ \Leftrightarrow \ \inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_{k=1}^{K} \alpha_k q_k > -\infty \tag{17}$$

2.3. Existence of an Optimizer

If there exists a $\hat{\boldsymbol{p}} > 0$ that solves P2, then $\hat{\boldsymbol{q}} = -\log \boldsymbol{\gamma}(\hat{\boldsymbol{p}})$ solves problem N2. But the converse is not true, as will be demonstrated by the following example.

Example 5. Consider the linear model (3) with the coupling matrix

$$\boldsymbol{V} = \left[\begin{array}{c|cc} \boldsymbol{V}^{(1)} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \boldsymbol{V}^{(1,2)} & \boldsymbol{V}^{(2)} \end{array} \right] .$$

The blocks on the main diagonal have the form $\boldsymbol{V}^{(1)} = \begin{bmatrix} 0 & \rho_1 \\ \rho_1 & 0 \end{bmatrix}$ and $\boldsymbol{V}^{(2)} = \begin{bmatrix} 0 & \rho_2 \\ \rho_2 & 0 \end{bmatrix}$, where $\rho_1, \rho_2 > 0$. The off-diagonal matrix $\boldsymbol{V}^{(1,2)}$ is strictly positive. For both diagonal blocks, we can define diagonal SIR target matrices $\boldsymbol{\Gamma}^{(1)} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$ and $\boldsymbol{\Gamma}^{(1)} = \begin{bmatrix} \gamma_3 & 0 \\ 0 & \gamma_4 \end{bmatrix}$. Then we have a spectral radius

$$\rho(\boldsymbol{\Gamma} \boldsymbol{V}) = \max\big\{\rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)}), \rho(\boldsymbol{\Gamma}^{(2)} \boldsymbol{V}^{(2)})\big\} \ . \tag{18}$$

Consider the first spectral radius $\rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)})$, which is a root of the characteristic polynomial $\lambda^2 - \gamma_1 \gamma_2 \rho_1^2 = 0$. With $\gamma_k = \exp(-q_k)$ we have

$$\rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)}) = \exp\Big(-\frac{q_1 + q_2}{2}\Big) \cdot \rho_1 \ .$$

That is,

$$\rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)}) \leq 1 \quad \Leftrightarrow \quad -q_1 + 2 \log \rho_1 \leq q_2 \ ,$$

and

$$\rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)}) = 1 \quad \Leftrightarrow \quad -q_1 + 2 \log \rho_1 = q_2 \ .$$

In analogy,

$$\rho(\boldsymbol{\Gamma}^{(2)} \boldsymbol{V}^{(2)}) = 1 \quad \Leftrightarrow \quad -q_3 + 2 \log \rho_2 = q_4 \ .$$

We now show that instead of minimizing over the set $\mathcal{F}_q$, we can equivalently minimize over

$$\mathcal{F}_q^E = \{\boldsymbol{q} : \rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)}) = \rho(\boldsymbol{\Gamma}^{(2)} \boldsymbol{V}^{(2)}) = 1\} \ .$$

Generally, $\mathcal{F}_q^E \subseteq \mathcal{F}_q$, which follows from (18). But for arbitrary $\boldsymbol{\alpha} > 0$, we have

$$\inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_{k=1}^{K} \alpha_k q_k = \inf_{\boldsymbol{q} \in \mathcal{F}_q^E} \sum_{k=1}^{K} \alpha_k q_k \ . \tag{19}$$

This can be shown by writing

$$\inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_{k=1}^{K} \alpha_k q_k = \inf_{\boldsymbol{q} \in \mathbb{R}^K : \rho(\boldsymbol{\Gamma}(\boldsymbol{q}) \boldsymbol{V}) = 1} \sum_{k} \alpha_k q_k \ .$$

Now, consider a $\hat{\boldsymbol{q}} > 0$ such that $\rho(\boldsymbol{\Gamma}(\hat{\boldsymbol{q}}) \boldsymbol{V}) = 1$, but $\hat{\boldsymbol{q}} \notin \mathcal{F}_q^E$, i.e., $\rho(\boldsymbol{\Gamma}^{(1)} \boldsymbol{V}^{(1)}) < 1$ can be assumed without loss of generality. Because of the continuity of the spectral radius, we can choose $\tilde{q}_k = \hat{q}_k - \lambda$, $k = 1, 2$, with some $\lambda > 0$. With $\tilde{q}_k = \hat{q}_k$, $k = 3, 4$, we have $\sum_k \alpha_k \tilde{q}_k < \sum_k \alpha_k \hat{q}_k$, i.e., if not $\hat{\boldsymbol{q}} \in \mathcal{F}_q^E$, then the utility function could be further minimized, which is a contradiction and shows (19).

Hence, we have

$$
\begin{aligned}
\inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_k \alpha_k q_k &= \alpha_1 q_1 + \alpha_2 (2 \log \rho_1 - q_1) + \\
&\quad + \alpha_3 q_3 + \alpha_4 (2 \log \rho_2 - q_3) \\
&= \alpha_2 2 \log \rho_1 + q_1 (\alpha_1 - \alpha_2) + \\
&\quad + 2\alpha_4 \log \rho_2 + q_3 (\alpha_3 - \alpha_4) \qquad (20)
\end{aligned}
$$

Thus,

$$
\inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_k \alpha_k q_k > -\infty \qquad (21)
$$

if and only if $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$.

This can be further developed. Using $\sum_k \alpha_k = 1$, we have $2\alpha_1 + 2\alpha_3 = 1$. That is, (21) holds if and only if $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$ with $2\alpha_1 + 2\alpha_3 = 1$. Hence, there exist infinitely many $\hat{\boldsymbol{q}}$ such that $\inf_{\boldsymbol{q} \in \mathcal{F}_q} \sum_k \alpha_k q_k = \sum_k \alpha_k \hat{q}_k$. They are all contained in the set $\mathcal{F}_q^E$. But there exists no $\hat{\boldsymbol{p}} > 0$ such that $\hat{q}_k = \log \frac{1}{\gamma_k(\hat{\boldsymbol{p}})}$. This is because $\hat{\boldsymbol{p}}$ would have to be the principal right eigenvector of $\boldsymbol{\Gamma}(\hat{\boldsymbol{q}})\boldsymbol{V}$. Then all maximal blocks would have to coincide with the isolated blocks, which does not hold for $\boldsymbol{\Gamma}^{(1)}\boldsymbol{V}^{(1)}$, since $\boldsymbol{V}^{(1,2)} > 0$.

Example 5 has shown that a solution of N2 need not provide a solution of P2. That is, the boundary points of $\mathcal{F}_q$ need not be achievable by power vectors. For general log-convex interference functions, it is not clear which properties the boundary points have. In this context, Lemma 2 is important since it shows that it is important to understand and exploit properties of the interference coupling.

This result can now be used in order to characterize the existence of an optimizer.

Theorem 2. *Let $\boldsymbol{D}_{\mathcal{I}}$ be irreducible, and $F(\boldsymbol{\alpha}, \mathcal{I})$ defined as by (7). There exists a $\hat{\boldsymbol{q}} \in \mathcal{F}_q$ such that*

$$
F(\boldsymbol{\alpha}, \boldsymbol{\mathcal{I}}) = \sum_k \alpha_k \hat{q}_k
$$

if and only if there exists a $\hat{\boldsymbol{p}}$ such that

$$
F(\boldsymbol{\alpha}, \boldsymbol{\mathcal{I}}) = \sum_k \alpha_k \log \frac{\mathcal{I}_k(\hat{\boldsymbol{p}})}{\hat{p}_k} \qquad (22)
$$

3. BOUNDEDNESS OF THE INFIMUM

Theorem 2 shows that if $\boldsymbol{D}_{\mathcal{I}}$ is irreducible, then both problems P2 and N2 are equivalent. Note, that this does not tell us when for a given $\boldsymbol{\alpha}$ there exists a $\hat{\boldsymbol{p}} > 0$ (and thus $\hat{\boldsymbol{q}} \in \mathcal{F}_q$) solving P2.

This question seems to be complicated in general. Previous investigations were confined to the case of equal weights $\boldsymbol{\alpha} = \frac{1}{K}\mathbf{1}$. If, in addition, certain monotonicity properties are fulfilled, then it can be shown that

$$
\inf_{\boldsymbol{p} > 0} \sum_k \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} > -\infty
$$

if and only if there exists a row permutation $\boldsymbol{\sigma} = [\sigma_1, \ldots, \sigma_K]$ such that $[\boldsymbol{D}_{\mathcal{I}}]_{\sigma_k, k} > 0$ for all $k = 1, 2, \ldots, K$. The same result holds when replacing the row permutation by a column permutation. In other words, the infimum is bounded if and only if there exists either a row or a column permutation such that the main diagonal of the permuted coupling matrix is strictly positive. In addition, it can be shown that for the existence of a proportionally fair power vector it is important that the coupling matrix is irreducible after simultaneous row and column permutations.

However, these results cannot be transferred to the problem at hand, where we assume an arbitrary weighting vector $\boldsymbol{\alpha} > 0$. This is illustrated by the following example.

Example 6. Consider linear interference functions $\mathcal{I}_k(\boldsymbol{p}) = [\boldsymbol{V}\boldsymbol{p}]_k$, $k = 1, 2, 3$, with an irreducible coupling matrix

$$
\boldsymbol{V} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad (23)
$$

For this matrix, we have

$$
PF(\boldsymbol{V}) = \inf_{\boldsymbol{p} > 0} \sum_{k=1}^{K} \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} > -\infty \ ,
$$

This follows from

$$
\begin{aligned}
\sum_{k=1}^{3} \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} &= \log \frac{p_2}{p_1} + \log \frac{p_1 + p_3}{p_2} + \log \frac{p_1 + p_2}{p_3} \\
&> \log \frac{p_2}{p_1} + \log \frac{p_3}{p_2} + \log \frac{p_1}{p_3} = 0 \ .
\end{aligned}
$$

But there is no optimizer $\hat{\boldsymbol{p}} > 0$ (and thus no $\hat{\boldsymbol{q}} \in \mathcal{F}_q$) such that $\sum_{k=1}^{K} \log \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_k}{\hat{p}_k} = PF(\boldsymbol{V})$.

In order to show this, consider

$$
\boldsymbol{V}\boldsymbol{V}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}
$$

The product $\boldsymbol{V}\boldsymbol{V}^T$ is irreducible and the function $\sum_{k=1}^{3} \log \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_k}{\hat{p}_k}$ is strictly convex if we substitute $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$ (see [16] and section 6).

But the infimum need not be achievable. This can be shown by contradiction (see also [17]). Suppose that there exists a $\hat{\boldsymbol{p}} > 0$ such that the function

$$
F(\boldsymbol{p}) = \sum_{k=1}^{3} \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k}, \quad \boldsymbol{p} > 0
$$

has a minimum at $\hat{\boldsymbol{p}}$. That is,

$$\frac{\partial F(\boldsymbol{p})}{\partial p_r}\bigg|_{\boldsymbol{p}=\hat{\boldsymbol{p}}} = 0, \quad r = 1, \dots, 3 \ . \tag{24}$$

Without loss of generality, we can assume $\sum_k \hat{p}_k = 1$. Since $V_{kk} = 0, \forall k$, we can rewrite (24) as

$$\frac{1}{\hat{p}_r} = \sum_{l \neq r} \frac{1}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_l} V_{lr}, \quad r = 1, 2, 3 \tag{25}$$

We can again exploit $V_{rr} = 0$ to write

$$\sum_{l=1}^{3} \frac{V_{lr}\hat{p}_r}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_l} = 1, \quad r = 1, 2, 3 \tag{26}$$

Taking the sum over the index r instead of l yields

$$\sum_{r=1}^{3} \frac{V_{lr}\hat{p}_r}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_l} = \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_l}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_l} = 1, \quad l = 1, 2, 3 \tag{27}$$

With (26) and (27) we have six equations characterizing the optimizer $\hat{\boldsymbol{p}}$. Now, we show they cannot be fulfilled for the special matrix (23). With (27) an $l = 1$ we have

$$\frac{V_{12}\hat{p}_2}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_1} = 1 \ . \tag{28}$$

With (26) and $r = 2$ we have

$$\frac{V_{12}\hat{p}_2}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_1} + \frac{V_{32}\hat{p}_2}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_3} = 1 \ . \tag{29}$$

Plugging (28) in (29) we obtain

$$1 = 1 + \frac{V_{32}\hat{p}_2}{[\boldsymbol{V}\hat{\boldsymbol{p}}]_3} \ .$$

Since $\hat{p}_2 > 0$ and $[\boldsymbol{V}\hat{\boldsymbol{p}}]_3 > 0$, we have $V_{32} = 0$ which is a contradiction, thus implying that $\hat{\boldsymbol{p}} > 0$ cannot exist for the given problem, with equal weights $\boldsymbol{\alpha} = \frac{1}{K}\mathbf{1}$.

Example 7. However, this does not imply that no optimizer exists for arbitrary $\boldsymbol{\alpha}$. For the same interference functions, with $\boldsymbol{V}$ defined as by (23), it can be shown that there exist weights $\hat{\boldsymbol{\alpha}} > 0$ such that

$$\inf_{\boldsymbol{p}>0} \sum_{k=1}^{3} \hat{\alpha}_k \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} > -\infty$$

and there exists a $\hat{\boldsymbol{p}} > 0$ such that

$$\inf_{\boldsymbol{p}>0} \sum_{k=1}^{3} \hat{\alpha}_k \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} = \sum_{k=1}^{3} \hat{\alpha}_k \log \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_k}{\hat{p}_k}$$

This is a direct consequence of [18]. The matrix $\boldsymbol{V}$ is irreducible, with right and left principal eigenvectors $\hat{\boldsymbol{p}} > 0$ and $\hat{\boldsymbol{z}} > 0$, respectively, i.e.,

$$\boldsymbol{V}\hat{\boldsymbol{p}} = \rho(\boldsymbol{V})\hat{\boldsymbol{p}} \tag{30}$$

$$\text{and} \quad \boldsymbol{V}^T\hat{\boldsymbol{z}} = \rho(\boldsymbol{V})\hat{\boldsymbol{z}} \ . \tag{31}$$

Assume that $\sum_k \hat{p}_k \hat{z}_k = 1$ and $\hat{\alpha}_k = \hat{p}_k \hat{z}_k$. Then for all $\boldsymbol{p} > 0$ holds

$$\sum_{k=1}^{K} \hat{\alpha}_k \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} \geq \log \rho(\boldsymbol{V}) = \sum_{k=1}^{K} \hat{\alpha}_k \log \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_k}{\hat{p}_k} \ ,$$

That is, $\hat{\boldsymbol{p}}$ is an optimizer.

This examples show that is generally difficult to characterize the existence of a proportionally fair operating point for arbitrary $\boldsymbol{\alpha} > 0$. Results for equal weights cannot be immediately generalized.

4. PROPORTIONAL FAIRNESS AND STRUCTURE OF LOG-CONVEX INTERFERENCE FUNCTIONS

We now study proportional fairness from a different perspective, by exploiting the structure of log-convex interference functions. Namely, every log-convex interference function has a decomposition based on the elementary interference functions, which were already introduced in Example 4.

Consider the set

$$\mathcal{L}(\mathcal{I}) = \left\{ \boldsymbol{w} \in \mathbb{R}_+^K : f_{\mathcal{I}}(\boldsymbol{w}) > 0 \right\} , \tag{32}$$

where

$$f_{\mathcal{I}}(\boldsymbol{w}) = \inf_{\boldsymbol{p}>0} \frac{\mathcal{I}(\boldsymbol{p})}{\prod_{l=1}^{K} (p_l)^{w_l}} \ , \quad \boldsymbol{w} \in \mathbb{R}_+^K \ . \tag{33}$$

Lemma 3. *Every log-convex interference function $\mathcal{I}(\boldsymbol{p})$, characterized by A1-A4, with $\boldsymbol{p} > 0$, can be represented as*

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{w} \in \mathcal{L}(\mathcal{I})} \Big(f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l=1}^{K} (p_l)^{w_l} \Big) \ . \tag{34}$$

Thus, we can use

$$\sum_{k=1}^{K} \alpha_k \cdot \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} = \log \left(\frac{\prod_l \big(\mathcal{I}_k(\boldsymbol{p})\big)^{w_l}}{\prod_l (p_l)^{w_l}} \right) \tag{35}$$

$$= \log \left(\frac{\mathcal{I}_w(\boldsymbol{p})}{\prod_l (p_l)^{w_l}} \right) \tag{36}$$

The function $\mathcal{I}_w(\boldsymbol{p})$ is characterized by the following lemma.

Lemma 4. *Let $\boldsymbol{w} > 0$ be an arbitrary weighting vector with $\sum_k \alpha_k = 1$. Then,*

$$\mathcal{I}_w(\boldsymbol{p}) = \prod_{l=1}^{K} \big(\mathcal{I}_l(\boldsymbol{p})\big)^{w_l} \tag{37}$$

is a log-convex interference function.

The result exploits the particular properties of log-convex interference functions. The elementary operation (35) combines log-convex interference functions and builds a new log-convex interference function. With Lemma 3, we can now use $f_{\mathcal{I}_w}$ in order to characterize when N1 and thus P1 are valid.

Lemma 4 is used to show the following result.

Theorem 3. *We have $F(\boldsymbol{\alpha},\boldsymbol{\mathcal{I}}) = \log\big(f_{\mathcal{I}_w}(\boldsymbol{w})\big)$. Thus,*

$$F(\boldsymbol{\alpha},\boldsymbol{\mathcal{I}}) > -\infty \quad \Leftrightarrow \quad f_{\mathcal{I}_w}(\boldsymbol{w}) > 0 \,. \tag{38}$$

and there exists a $\hat{\mathbf{p}}$ as a solution of P2 if and only if problem (33) has an optimizer.

It can be observed from Theorem 3 how the elementary component $f_{\mathcal{I}_w}$ plays an essential role in characterizing the existence of a supporting power allocation.

5. CONNECTION WITH GENERALIZED CROSSTALK MATRICES

In this section, we study a further way of characterizing the solvability of P2. Only for the sake of simplifying the discussion (see comment below), we assume that the log-convex interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are continuously differentiable. Then for every index k and for every $\boldsymbol{p} > 0$ there exists a vector $\boldsymbol{w}^{(k)} := \boldsymbol{w}^{(k)}(\boldsymbol{p})$, with $\boldsymbol{w}^{(k)} \geq 0$, $\|\boldsymbol{w}^{(k)}\|_1 = 1$, and $f_k(\boldsymbol{w}^{(k)}) > 0$ such that

$$\mathcal{I}_k(\boldsymbol{p}) = f_k(\boldsymbol{w}^{(k)}) \prod_l (p_l)^{w_l^{(k)}} \tag{39}$$

Defining the matrix

$$\boldsymbol{W}(\boldsymbol{p}) = [\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(K)}] \tag{40}$$

we have the following result.

Theorem 4. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be continuously differentiable, log-convex interference functions. For an arbitrary fixed $\boldsymbol{\alpha} > 0$, there exists a vector $\hat{\mathbf{p}} > 0$ solving P2 if and only if there exists a $\tilde{\mathbf{p}} > 0$ such that*

$$\boldsymbol{W}(\tilde{\mathbf{p}})^T \boldsymbol{\alpha} = \boldsymbol{\alpha} \tag{41}$$

where $\boldsymbol{W}(\tilde{\mathbf{p}})$ is defined by (40). Then the vector $\tilde{\mathbf{p}}$ is a solution of P2.

All matrices of the form (40) fulfill $\boldsymbol{W}$, i.e. they are row-stochastic with spectral radius $\rho(\boldsymbol{W}) = 1$. Relation (41) means that $\boldsymbol{\alpha}$ is the principal left-hand eigenvector of $\boldsymbol{W}$.

The theorem shows how the solvability of P2 is coupled with the structural properties of log-convex interference functions. In principal, we just need to check whether there is $\boldsymbol{W}(\boldsymbol{p})$ as defined by (40), which has a principal left eigenvector $\boldsymbol{\alpha}$.

Finally, it should be noted that Theorem 4 can be extended to arbitrary log-convex interference functions which are not necessarily continuously differentiable. But this requires techniques from semi-smooth analysis and a full discussion is beyond the scope of this paper.

6. UNIQUENESS

We now consider problems P3 and N3. That is, we ask under which condition an existing optimizer is unique. A sufficient condition for uniqueness is strict convexity of the cost function in (7). In this section we will show how strictness depends on the structure of the dependency matrix $\boldsymbol{D}_{\mathcal{I}}$.

It was already shown in [19] that the cost function

$$f(\boldsymbol{s}) = \sum_k \alpha_k \log \frac{\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})}{\mathrm{e}^{s_k}} \tag{42}$$

is convex for log-convex interference functions. In [16] strict convexity was characterized for the linear interference model, with a cost function $\sum_k \alpha_k \log([\boldsymbol{V}\boldsymbol{p}]_k/p_k)$. If $\boldsymbol{V}$ is irreducible, then $f(\boldsymbol{w})$ is strictly convex if and only if $\boldsymbol{V}\boldsymbol{V}^T$ is irreducible.

This result can be generalized to general log-convex interference functions defined by A1–A4. To this end, we need the following definition

Definition 2. Assume that for two arbitrary $\boldsymbol{p}^{(1)}$, $\boldsymbol{p}^{(2)}$, there exists at least one index l from the dependency set (indices for which $\boldsymbol{D}_{\mathcal{I}}$ is non-zero) such that $p_l^{(1)} \neq p_l^{(2)}$. An interference function $\mathcal{I}$ is called *separating* if

$$\mathcal{I}\big(\boldsymbol{p}(\lambda)\big) < \mathcal{I}\big(\boldsymbol{p}^{(1)}\big)^{1-\lambda} \cdot \mathcal{I}\big(\boldsymbol{p}^{(2)}\big)^{\lambda} \tag{43}$$

That is, $\mathcal{I}(\boldsymbol{p})$ is strictly convex on its domain.

Theorem 5. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be separating interference functions. The matrix $\boldsymbol{D}_{\mathcal{I}}$ is assumed to be irreducible. Then, the cost function (42) is strictly convex if and only if $\boldsymbol{D}_{\mathcal{I}}\boldsymbol{D}_{\mathcal{I}}^T$ is irreducible.*

If the assumptions in Theorem 5 are fulfilled, then strict convexity of the proportional fair cost function (42) only depends on the structure of $\boldsymbol{D}_{\mathcal{I}}$. Applying this result to the linear linear model with a coupling matrix $\boldsymbol{V}$, we see that only the positions of the non-zero entries matter. The actual values of the entries do not affect the strict convexity of function.

7. CONCLUSIONS

In this paper we have studied the problem of weighted proportional fairness for log-convex interference functions which are defined by an axiomatic framework. The advantage of the axiomatic approach is it generality and flexibility. By modeling the QoS as a function of the transmission powers, it is easy to include physical layer techniques for robustness and interference reduction.

But this chosen parameterization complicates the task of resource allocation. The examples in this paper illustrate cases for which the given optimization problem has no solution. Whether or not such effects occur depends on the interference coupling structure of the system. In this paper, we show that these effects largely depend on the structure of the dependency matrix $\boldsymbol{D}_{\mathcal{I}}$.

Some application examples (power control, robustness) have already been discussed in the paper. But the chosen axiomatic framework is very general, so the results are not restricted to wireless communications.

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