Characterizing Completions of Finite Frames

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***Abstract*—Finite frames are possibly-overcomplete generalizations of orthonormal bases. We consider the "frame completion" problem, that is, the problem of how to add vectors to an existing frame in order to make it better conditioned. In particular, we discuss a new, complete characterization of the spectra of the frame operators that arise from those completions whose newly-added vectors have given prescribed lengths. To do this, we build on recent work involving a frame's eigensteps, namely the interlacing sequence of spectra of its partial frame operators. We discuss how such eigensteps exist if and only if our prescribed lengths are majorized by another sequence which is obtained by comparing our completed frame's spectrum to our initial one.**

I. Introduction

Let M and N be positive integers, and let $\{\varphi_n\}_{n=1}^N$ be a finite sequence of vectors in $\mathbb{C}^M$. The corresponding *synthesis operator* is the $M \times N$ matrix $\Phi = [\,\varphi_1 \cdots \varphi_N\,]$ obtained by stacking these vectors as columns. Multiplying this matrix by its adjoint (conjugate-transpose) Φ^* yields the $N \times N$ *Gram matrix* $\Phi^*\Phi$ as well as the $M \times M$ *frame operator* $\Phi\Phi^*$:

$$\Phi\Phi^* x = \left(\sum_{n=1}^N \varphi_n\varphi_n^*\right)x = \sum_{n=1}^N \langle x, \varphi_n\rangle\varphi_n.$$

Note that when $\{\varphi_n\}_{n=1}^N$ is an orthonormal basis for $\mathbb{C}^M$, we have $M = N$ and $\Phi^*\Phi = \mathrm{I} = \Phi\Phi^*$. In this case, the above expression gives the traditional orthonormal expansion of x.

Frame theory generalizes the notion of an orthonormal basis in order to provide possibly-overcomplete (nonorthogonal) expansions of x. It does this by relaxing Parseval's identity. To be precise, $\{\varphi_n\}_{n=1}^N$ is a *frame* for $\mathbb{C}^M$ if there exist *lower* and *upper frame bounds* $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_{n=1}^N |\langle x, \varphi_n\rangle|^2 \le B\|x\|^2, \quad \forall x \in \mathbb{C}^M. \tag{1}$$

In this finite-dimensional setting, one can show that the optimal frame bounds A and B of any $\{\varphi_n\}_{n=1}^N$ are the least and greatest eigenvalues of $\Phi\Phi^*$, respectively. In particular, when $\{\varphi_n\}_{n=1}^N$ is a frame for $\mathbb{C}^M$, we have that $\Phi\Phi^*$ is invertible, having condition number at most B/A. This enables us to define the *canonical dual frame* $\{\tilde{\varphi}_n\}_{n=1}^N$, $\tilde{\varphi}_n := (\Phi\Phi^*)^{-1}\varphi_n$. Together, a frame and its dual provide the decompositions:

$$x = \sum_{n=1}^N \langle x, \tilde{\varphi}_n\rangle\varphi_n = \sum_{n=1}^N \langle x, \varphi_n\rangle\tilde{\varphi}_n, \quad \forall x \in \mathbb{C}^M.$$

In recent years, these "painless nonorthogonal expansions" have been exploited in a variety of finite-dimensional signal processing applications in which redundancy is useful [5].

Much of the recent research on finite frames has focused on constructing frames that satisfy a given list of desired, application-motivated constraints. Sometimes these constraints are nonlinear. For example, we often want our frames to be *tight*, namely have $A = B$ in (1), which happens precisely when $\Phi\Phi^* = A\mathrm{I}$. Tightness ensures that Φ is as well-conditioned as possible, and makes it easy to compute the canonical dual: $\tilde{\varphi}_n = \frac{1}{A}\varphi_n$. Moreover, finite tight frames are easy to construct: we simply need the rows of Φ to be orthogonal and have constant norm. In short, tight frames behave much more like orthonormal bases than frames do in general, while still permitting overcompleteness.

In order to find overcomplete frames which are even more faithful to the concept of an orthonormal basis, we can further restrict ourselves to *unit norm tight frames* (UNTFs), that is, tight frames $\{\varphi_n\}_{n=1}^N$ for $\mathbb{C}^M$ that have the additional property that $\|\varphi_n\| = 1$ for all n. Whereas the synthesis operator Φ of an orthonormal basis satisfies $\Phi\Phi^* = \mathrm{I} = \Phi^*\Phi$, a UNTF instead has that $\Phi\Phi^* = A\mathrm{I}$ and that the diagonal entries of $\Phi^*\Phi$ are 1; the fact that these two matrices have the same trace implies A is necessarily $\frac{N}{M}$.

UNTFs are known to exist for every $N \ge M$. For example, one may form Φ by extracting M rows from an $N \times N$ discrete Fourier transform matrix. However, the problem of constructing *every* UNTF was open for many years, due to the fact that the entries of Φ must satisfy a large system of intertwined quadratic equations. This problem was recently solved in [1] and [3]. In fact, as detailed in the next section, [1] and [3] give an explicit, closed-form algorithm for constructing every sequence of vectors $\{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ whose frame operator $\Phi\Phi^*$ has a given spectrum $\{\lambda_m\}_{m=1}^M$ and whose Gram matrix $\Phi^*\Phi$ has diagonal entries $\{\mu_n\}_{n=1}^N$.

In this paper, we outline recent results from [4] and [7] that generalize the techniques of [1] and [3] to address the problem of *frame completions*. To be precise, given some initial sequence of vectors $\{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and some desired lengths $\{\mu_{N+p}\}_{p=1}^P$, we consider the problem of *completing* $\{\varphi_n\}_{n=1}^N$ by adding P new vectors $\{\varphi_{N+p}\}_{p=1}^P$ to this collection with the property that $\|\varphi_{N+p}\|^2 = \mu_{N+p}$ for all p.

We, like several other teams of researchers, are interested in the best (tightest) possible completions. Several cases of the optimal frame completion problem have already been solved, such as the case where the lengths permit a tight completion [2], and the case where all the added vectors have equal length [6]. To our knowledge, the general case of this problem (arbitrary lengths, tightness unobtainable) remains open. Our work here serves to characterize *every* possible completion that can be formed using a given sequence of lengths. Our longer-term goal is to use this characterization in order to find the optimal completion in the general case.

II. EIGENSTEPS

Let M and N be any positive integers and let $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ be any nonnegative nonincreasing sequences. Recent work given in [1] and [3] provides a method for explicitly constructing every finite sequence of vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{C}^M$ whose frame operator $\Phi\Phi^*$ has spectrum $\{\lambda_m\}_{m=1}^M$ and whose vectors have lengths $\|\varphi_n\|^2 = \mu_n$ for all n. This method is based on the concept of *eigensteps*. To be precise, given any such frame and any $k = 0, \dots, N$, let $\{\lambda_{k;m}\}_{m=1}^M$ denote the spectrum of its kth *partial frame operator*

$$\Phi_k\Phi_k^* = \sum_{n=1}^{k} \varphi_n\varphi_n^*. \tag{2}$$

In practice, we arrange these values in an $M \times (N+1)$ table:

$$\begin{bmatrix} \lambda_{0;M} & \lambda_{1;1} & \cdots & \lambda_{N;M} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{0;1} & \lambda_{1;M} & \cdots & \lambda_{N;1} \end{bmatrix}. \tag{3}$$

In order to arise from a sequence $\{\varphi_n\}_{n=1}^N$ whose frame operator has spectrum $\{\lambda_m\}_{m=1}^M$ and whose elements have lengths $\|\varphi_n\|^2 = \mu_n$, the values in this table necessarily satisfy four rules. First, for $k = N$, we have $\Phi_N\Phi_N^* = \Phi\Phi^*$ and so $\lambda_{N;m} = \lambda_m$ for all m. This means the last column of (3) corresponds to our final desired spectrum. Our second rule comes from the fact that when $k = 0$, we regard the empty sum defining $\Phi_0\Phi_0^*$ to be a matrix of zeros, and so $\lambda_{0;m} = 0$ for all m. This means the first column of (3) is all zeros.

The third rule is that for any k, the sum of the entries in the kth column of (3) is necessarily the sum of the first k of our μ_n's; this follows from the fact that

$$\sum_{m=1}^{M} \lambda_{k;m} = \mathrm{Tr}(\Phi_k\Phi_k^*) = \mathrm{Tr}(\Phi_k^*\Phi_k) = \sum_{n=1}^{k} \mu_n.$$

The fourth rule is the least obvious. For any $k = 1, \dots, N$, note that the kth partial frame operator is the sum of the previous one with an outer product: $\Phi_k\Phi_k^* = \Phi_{k-1}\Phi_{k-1}^* + \varphi_k\varphi_k^*$. As such, a classical result from matrix analysis tells us that the spectrum of $\Phi_k\Phi_k^*$ necessarily interlaces on that of $\Phi_{k-1}\Phi_{k-1}^*$. To be precise, we say that a finite sequence of real numbers $\{\gamma_m\}_{m=1}^M$ *interlaces* on another such sequence $\{\beta_m\}_{m=1}^M$, denoted $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$, provided

$$\beta_M \le \gamma_M \le \beta_{M-1} \le \gamma_{M-1} \le \cdots \le \beta_2 \le \gamma_2 \le \beta_1 \le \gamma_1.$$

That is, $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$ when $\gamma_{m+1} \le \beta_m \le \gamma_m$ for all $m = 1, \dots, M$, provided we adopt the convention that $\gamma_{M+1} := 0$. As mentioned above, a classical result from matrix analysis gives that the spectra of the partial frame operators necessarily satisfy $\{\lambda_{k-1;m}\}_{m=1}^M \sqsubseteq \{\lambda_{k,m}\}_{m=1}^M$ for all $k = 1, \dots, N$. This means that each pair of neighboring columns in (3) necessarily satisfy a zigzag of inequalities, each entry being no more than its neighbor to its right, which in turn is no more than its neighbor to its lower left. Gathering these four rules together, we arrive at the definition of a sequence of eigensteps:

Definition 1: A sequence $\{\lambda_{k;m}\}_{k=1,\ m=1}^{N\ \ \ M}$ is a sequence of *eigensteps* for given nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ if:

(i) $\lambda_{0;m} = 0$ for all $m = 1, \dots, M$,
(ii) $\lambda_{N;m} = \lambda_m$ for all $m = 1, \dots, M$,
(iii) $\sum_{m=1}^M \lambda_{k;m} = \sum_{n=1}^k \mu_n$ for all $k = 1, \dots, N$
(iv) $\{\lambda_{k-1;m}\}_{m=1}^M \sqsubseteq \{\lambda_{k;m}\}_{m=1}^M$ for all $k = 1, \dots, N$.

To summarize, if $\{\varphi_n\}_{n=1}^N$ is any sequence of vectors in $\mathbb{C}^M$ whose frame operator has spectrum $\{\lambda_m\}_{m=1}^M$ and for which $\|\varphi_n\| = \mu_n$ for all n, then the spectra of its partial frame operators (2) necessarily form a corresponding sequence of eigensteps.

Remarkably, these relatively-simple necessary conditions on the existence of such frames are also sufficient. Indeed, as shown in [1], given a valid sequence of eigensteps, one can explicitly construct a sequence of vectors $\{\varphi_n\}_{n=1}^N$ with the desired spectrum and lengths. The approach is iterative: given $\{\varphi_n\}_{n=1}^k$ such that $\Phi_k\Phi_k^*$ has the desired spectrum $\{\lambda_{k;m}\}_{m=1}^M$, such that $\|\varphi_n\| = \mu_n$ for all $n = 1, \dots, k$, and such that the eigenvectors of $\Phi_k\Phi_k^*$ are explicitly known, the algorithm shows how to choose φ_{k+1} as a linear combination of these eigenvectors so that $\Phi_{k+1}\Phi_{k+1}^*$ has spectrum $\{\lambda_{k+1;m}\}_{m=1}^M$ and such that $\|\varphi_{k+1}\|^2 = \mu_{k+1}$; the algorithm then goes on to explicitly update the eigenvectors of $\Phi_k\Phi_k^*$ into those of $\Phi_{k+1}\Phi_{k+1}^*$, as needed for the next iteration. Apart from possible rotations and reflections during each step of the process, the vectors constructed by the algorithm are unique. As such, eigensteps corresponding to a given $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ can be viewed as the truly meaningful "parameters" of all vector sequences whose frame operator has that spectrum and whose elements have those lengths.

For example, in order to use these ideas to construct a UNTF of $N = 5$ elements for $M = 3$-dimensional space, we want a 3×6 table of eigensteps whose last column has the desired spectrum $\lambda_1 = \lambda_2 = \lambda_3 = \frac{N}{M} = \frac{5}{3}$ and whose zeroth column is zero; we also want the entries in the kth column to sum to $k = \sum_{n=1}^k \mu_n$, and for the values in any column to interlace on those in the preceding one. An example of such a table is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & \frac{1}{3} & \frac{4}{3} & \frac{5}{3} & \frac{5}{3} \\ 0 & 1 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} \end{bmatrix}. \tag{4}$$

We emphasize that this table does not contain the frame vectors themselves, but rather the spectra of the partial frame operators. The process of transforming this table into the actual frame elements is nontrivial [1]. For example, to define φ_2, we need to find a vector that makes the correct angle with φ_1 in order for $\varphi_1\varphi_1^* + \varphi_2\varphi_2^*$ to have spectrum $\{\frac{5}{3}, \frac{1}{3}, 0\}$.

Moreover, we also note that the above table corresponds to just one way of constructing a 3×5 UNTF. There are infinitely many others, meaning there are infinitely many UNTFs of five elements in three-dimensional space, even modulo rotations. In fact, one can show in this case that every sequence of eigensteps is of the form

$$\begin{bmatrix} 0 & 0 & 0 & x & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & y & \frac{4}{3}-x & \frac{5}{3} & \frac{5}{3} \\ 0 & 1 & 2-y & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} \end{bmatrix}, \qquad (5)$$

where, in order to satisfy the interlacing requirements, we need to take our parameters (x, y) from the convex set

$$0 \le x \le \tfrac{2}{3}, \quad \max\{x, \tfrac{1}{3}\} \le y \le \min\{\tfrac{4}{3} - x, \tfrac{2}{3} + x\}.$$

This problem of constructing *every* sequence of eigensteps for a given $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$—thereby in effect constructing every sequence of vectors with this spectrum and set of lengths—is addressed in [3]. Here, it is important to note that there does not exist a set of eigensteps for every possible choice of $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$. Indeed, at a bare minimum, the second and third conditions of eigensteps require the λ_m's and the μ_n's to have the same sum; this corresponds to our final frame operator and Gram matrix having the same trace. Moreover, as evidenced in (4), the first and fourth conditions of eigensteps require us to have a "triangle of zeros" at the beginning of our table (3). That is, we necessarily have that $\lambda_{k;m} = 0$ for all $m > k$. When combined with our third and fourth conditions of eigensteps, this implies that the partial sums of our μ_n's are less than those of our λ_m's; for any $k = 1, \dots, \min\{M, N\}$, we necessarily have

$$\sum_{n=1}^{k} \mu_n = \sum_{m=1}^{M} \lambda_{k;m} = \sum_{m=1}^{k} \lambda_{k;m} \le \sum_{m=1}^{k} \lambda_m.$$

Together, these facts state that in order for eigensteps to exist, our desired spectrum $\{\lambda_m\}_{m=1}^M$ must necessarily majorize our desired lengths $\{\mu_n\}_{n=1}^N$.

To be precise, we say that a nonnegative nonincreasing sequence $\{\lambda_m\}_{m=1}^M$ *majorizes* another such sequence $\{\mu_n\}_{n=1}^N$, denoted $\{\mu_n\}_{n=1}^N \preceq \{\lambda_m\}_{m=1}^M$, if

$$\sum_{n=1}^{N} \mu_n = \sum_{m=1}^{M} \lambda_m,$$
$$\sum_{n=1}^{k} \mu_n \le \sum_{m=1}^{k} \lambda_m, \ \forall k = 1, \dots, \min\{M, N\}.$$

As we have just discussed, in order for a table of eigensteps to exist for a given $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$—that is, in order for there to exist a sequence of vectors whose frame operator has spectrum $\{\lambda_m\}_{m=1}^M$ and whose elements have lengths $\{\mu_n\}_{n=1}^N$—we necessarily have $\{\mu_n\}_{n=1}^N \preceq \{\lambda_m\}_{m=1}^M$. Remarkably, the converse of this statement is also true; this fact has been known for a long time, being a straightforward application of the classical Schur-Horn Theorem to the Gram matrix $\Phi^*\Phi$. However, the traditional proof of the converse is nonconstructive. The main contribution of [3] is to give a constructive proof of this converse and moreover, generalize the idea behind that construction so as to explicitly parameterize the convex polytope of *every* possible sequence of eigensteps.

To elaborate, the main idea of [3] is a new algorithm, dubbed *Top Kill*, for producing a valid sequence of eigensteps from a given $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$. The algorithm is iterative, starting with the final desired spectrum $\{\lambda_{N;m}\}_{m=1}^M = \{\lambda_m\}_{m=1}^M$ and working backwards from it to produce $\{\lambda_{N-1;m}\}_{m=1}^M$, then $\{\lambda_{N-2;m}\}_{m=1}^M$, etc., until finally arriving at $\{\lambda_{1;m}\}_{m=1}^M$. This algorithm has an intuitive, geometric motivation behind it (see [3]) that we do not have the space to discuss here.

In brief, however, note that looking at eigenstep tables such as (4), one quickly realizes that it is harder to get positive numbers in higher rows than it is in lower ones. This is because interlacing requires us to first build a suitable "foundation." That is, a number in a given column can only be a big as the one to its lower left, which, in turn, can only be as big as the one to its lower left, and so on. This can make it difficult to build the upper levels of our spectrum, especially if we do not plan ahead.

The Top Kill algorithm handles this issue by (i) working backwards from right to left, so that you are always explicitly using the final spectrum $\{\lambda_m\}_{m=1}^M$ you are trying to build and (ii) recognizing the higher rows are the most difficult to fill, and as such, making them the first thing we want to "kill" off. Indeed, the example table given in (4) is the result of applying Top Kill for $\lambda_m = \frac{5}{3}$ for all m and $\mu_n = 1$ for all n: to build each column from its neighbor to the right, we remove $\mu_n = 1$ units of "area," removing as much as possible from the highest row before removing from the second highest, and so on. Put another way, Top Kill's goal is to take x and y in (5) to be as small as possible, namely $(x, y) = (0, \frac{1}{3})$.

As detailed in [3], it turns out that the Top Kill algorithm will produce a valid sequence of eigensteps if and only our λ_m's majorize our μ_n's. In particular, if, for a given $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$, there is any way to produce a sequence of eigensteps, then Top Kill will produce such a sequence. That is, if anything works, then Top Kill does. This surprising fact led us to look for ways of generalizing Top Kill so that it could be applied to the frame completion problem.

III. Frame Completions

Recall the frame completion problem: given $\{\varphi_n\}_{n=1}^N$ in $\mathbb{C}^M$ and a set of desired lengths $\{\mu_{N+p}\}_{p=1}^P$, we want to add P new measurement vectors to $\{\varphi_n\}_{n=1}^N$ so that the frame operator of $\{\varphi_n\}_{n=1}^{N+P}$ has spectrum $\{\lambda_m\}_{m=1}^M$ and such that $\|\varphi_{N+p}\|^2 = \mu_{N+p}$ for all $p = 1, \dots, P$. Letting $\mu_n := \|\varphi_n\|^2$ for all n, note that in accordance with the theory of eigensteps in [1], any such frame completion necessarily corresponds to an $M \times (N + P + 1)$ table of eigensteps. Moreover, letting $\{\alpha_m\}_{m=1}^M$ denote the spectrum of the frame operator of the "initial frame" $\{\varphi_n\}_{n=1}^N$, we necessarily have that the values $\{\alpha_m\}_{m=1}^M$ lie in the $k = N$ column of this table. We thus see that each of our desired frame completions corresponds to a way of extending an existing $M \times (N+1)$ table of eigensteps by adding P new interlacing columns with the appropriate column sums.

As detailed in [4] and [7] it turns out that the existence of such "continued eigensteps" can be characterized in terms of majorization. However, it is not as simple as requiring that the final λ_m's majorize the μ_n's. Indeed, any such characterization must take into account the initial spectrum $\{\alpha_m\}_{m=1}^M$. At this point, we recall the motivation behind the Top Kill algorithm: when building eigensteps from a spectrum of zeros, we are faced with an "upper triangle" of zeros that makes it difficult to get large numbers in the higher rows of our table. However, this may not be the case when building eigensteps on top of an initial spectrum $\{\alpha_m\}_{m=1}^M$. Rather, it turns out that in this setting, what truly matters is how high the desired spectrum is *relative to the initial spectrum.*

A nonobvious concept such as this is best explained in pictures. In Figure 1(a), we see a given initial spectrum $\{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$. In (b), this spectrum is overlaid with a desired completion $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$. Suppose we want to know whether or not our initial frame can be completed to one with spectrum $\{\lambda_m\}_{m=1}^3$ by adding four new frame vectors having lengths $\{\mu_{N+1}, \mu_{N+2}, \mu_{N+3}, \mu_{N+4}\} = \{2, 1, \frac{1}{4}, \frac{1}{4}\}$. To answer this question, we "chop" up the λ_m's according to m and the α_m's; see (c). In (d), we then label the area in each chopped region according to its height above the initial spectrum. The total "amount" of $\{\lambda_m\}_{m=1}^3$ that lies one unit above the existing spectrum is our first "diagonal sum":

$$\mathrm{DS}_1 := (\tfrac{13}{4} - \tfrac{7}{4}) + (\tfrac{7}{4} - \tfrac{3}{4}) + (\tfrac{3}{4} - \tfrac{1}{2}) = \tfrac{11}{4}.$$

Meanwhile, our second diagonal sum represents the total amount that lies two units above the existing spectrum:

$$\mathrm{DS}_2 := (\tfrac{9}{4} - \tfrac{7}{4}) + (\tfrac{5}{4} - 1) = \tfrac{3}{4}.$$

Finally, as there is no component of the λ_m's that lies three units above the existing spectrum, we take $\mathrm{DS}_3 = 0$.

The main result of our forthcoming paper [4] states that a given sequence $\{\lambda_m\}_{m=1}^M$ is realizable as the spectrum of a completion of a frame with initial spectrum $\{\alpha_m\}_{m=1}^M$ via the addition of P new measurements of lengths $\{\mu_{N+p}\}_{p=1}^P$ if and only if $\{\mathrm{DS}_m\}_{m=1}^M$ majorizes $\{\mu_{N+p}\}_{p=1}^P$. In particular, our example is constructible since $\mathrm{DS}_1 \geq \mu_{N+1}$, $\mathrm{DS}_1 + \mathrm{DS}_2 \geq \mu_{N+1} + \mu_{N+2}$, $\mathrm{DS}_1 + \mathrm{DS}_2 + \mathrm{DS}_3 \geq \mu_{N+1} + \mu_{N+2} + \mu_{N+3}$, and $\mathrm{DS}_1 + \mathrm{DS}_2 + \mathrm{DS}_3 = \frac{14}{4} = \mu_{N+1} + \mu_{N+2} + \mu_{N+3} + \mu_{N+4}$.

The necessity of this majorization follows from the fact that for any $k \leq \min\{M, P\}$, interlacing forces all of the "area" of $\{\lambda_{N+k;m}\}_{m=1}^M$ to lie at most k units above the initial spectrum $\{\alpha_m\}_{m=1}^M$. The part of the $\lambda_{N+k;m}$'s that lies outside of the α_m's envelope has a total area of $\sum_{n=N+1}^{N+k} \mu_n$. The amount of area in these diagonals will only grow as more frame vectors are added, meaning

$$\sum_{n=N+1}^{N+k} \mu_n = \sum_{m=1}^{k} \mathrm{DS}_{m;k} \leq \sum_{m=1}^{k} \mathrm{DS}_{m;P} = \sum_{m=1}^{k} \mathrm{DS}_m.$$

When all P vectors are added, the above inequalities become equalities.

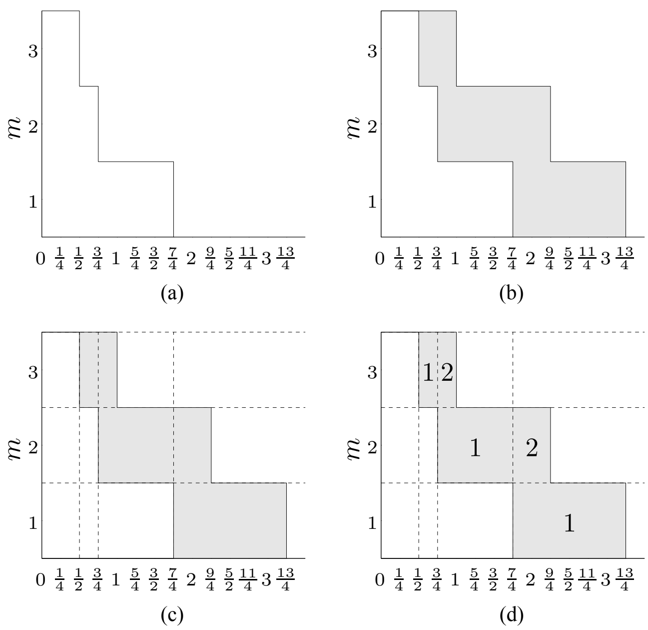

Fig. 1. Determining the relative height of a desired completion's spectrum above an existing one.

The sufficiency of this majorization condition follows from a variation of the Top Kill algorithm, called "Chop Kill." Here, we can build a valid sequence of eigensteps, provided we once again start with the desired spectrum and work backwards. However, rather than removing as much of the "top" of the spectrum as quickly as possible, we instead remove as much as possible from the outermost *diagonals*. This is consistent with the original motivation behind Top Kill: once we identify the hardest parts of our spectrum to construct, we work backwards, taking care of those parts as soon as possible.

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