Perfect Preconditioning of Frames by a Diagonal Operator

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***Abstract*—Frames which are tight might be considered optimally conditioned in the sense of their numerical stability. This leads to the question of perfect preconditioning of frames, i.e., modification of a given frame to generate a tight frame. In this paper, we analyze prefect preconditioning of frames by a diagonal operator. We derive various characterizations of functional analytic and geometric type of the class of frames which allow such a perfect preconditioning.**

I. Introduction

Frames are nowadays a common methodology in applied mathematics, computer science, and engineering, see [7], when non-unique, but stable decompositions and expansions are required. They are utilized in various applications which can roughly be subdivided into two categories. One type of applications utilize frames for decomposing data. In this case, typical goals are erasure-resilient transmission, data analysis or processing, and compression, with the advantage of frames being their robustness as well as their flexibility in design. The other type of applications requires frames for expanding data. This approach is extensively used in sparsity methodologies such as Compressed Sensing (see [9]), but also, for instance, as systems generating trial spaces for PDE solvers. Again, it relies on non-uniqueness of the expansion which promotes sparse expansions and on the flexibility in design.

A crucial requirement for all such applications is the numerical stability of the associated algorithms, which is optimally ensured by the subclass of tight frames. Thus, urgent questions are: When can a given frame be modified to become a tight frame? Obviously, the most careful modification – which also retains properties such as providing sparse representations for a class of data – is to rescale each frame vector. Thus, in this paper, we consider the question: When can the vectors of a given frame be rescaled to obtain a tight frame?

A. Tight Frames

Before continuing, let us first fix the notions we will use throughout. Letting $\mathcal{H}$ be a real or complex separable Hilbert space and letting J be a subset of $\mathbb{N}$, a set of vectors $\Phi = \{\varphi_j\}_{j\in J} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$, if there exist positive constants $A, B > 0$ (the *lower* and *upper frame bound*) such that

$$A\|x\|^2 \le \sum_{j\in J} |\langle x, \varphi_j\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \qquad (1)$$

A frame Φ is referred to as A-*tight* or just *tight*, if $A = B$ is possible in (1), and *Parseval*, if $A = B = 1$ is possible. Moreover, if $|J| < \infty$ (which implies that $\mathcal{H} = \mathbb{K}^N$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), the frame Φ is called *finite*.

Let $\Phi = \{\varphi_j\}_{j\in J} \subset \mathcal{H}$ now be a frame for $\mathcal{H}$. Signals are analyzed using a frame by application of the associated *analysis operator* $T_\Phi : \mathcal{H} \to \ell^2(J)$ defined by $T_\Phi x := \big(\langle x, \varphi_j\rangle\big)_{j\in J}$. Its adjoint T_Φ^*, the *synthesis operator* of Φ, maps then $\ell^2(J)$ surjectively onto $\mathcal{H}$. Concatenating both operators leads to the *frame operator* $S_\Phi := T_\Phi^* T_\Phi$ of Φ, given by

$$S_\Phi x = \sum_{j\in J} \langle x, \varphi_j\rangle \varphi_j, \quad x \in \mathcal{H},$$

which is a bounded and strictly positive selfadjoint operator in $\mathcal{H}$. These properties imply that Φ admits the reconstruction formula

$$x = \sum_{j\in J} \langle x, \varphi_j\rangle S_\Phi^{-1}\varphi_j \quad \text{for all } x \in \mathcal{H}.$$

To avoid numerical stability issues, it seems desirable to have $S_\Phi = const \cdot I_{\mathcal{H}}$ ($I_{\mathcal{H}}$ denoting the identity on $\mathcal{H}$, for $\mathcal{H} = \mathbb{K}^N$ we will use I_N). And in fact, this equation characterizes tight frames. Thus an A-tight frame admits the numerically optimally stable reconstruction given by

$$x = A^{-1} \cdot \sum_{j\in J} \langle x, \varphi_j\rangle \varphi_j \quad \text{for all } x \in \mathcal{H}.$$

B. Generating Parseval Frames

Since applications typically require specific frames, which might not automatically form a tight frame, an important problem is to introduce approaches for modifying a given frame in order to generate a tight frame. We might restrict our attention to generating Parseval frames, since this just requires a renormalization once we derived a tight frame. One key issue in this whole process is to modify the frame as careful as possible to not disturb its frame properties – which might be crucial for the application at hand – too much. As an example, think for instance of a frame which sparsifies a given test data set; a property one might want to keep.

A very common approach to generate a tight frame is to apply $S_\Phi^{-1/2}$ to each frame vector of a frame Φ, which in fact even yields a Parseval frame. However, this modification also changes the frame properties significantly, and to date it is still entirely unclear in which way. In particular, a sparse representation property would be completely destroyed.

The most careful modification of a frame is scaling its frame vectors. For instance, this procedure even preserves any sparse representation properties of the frame. In [12], a frame was coined *scalable*, if a scaling exists which leads to a Parseval frame. Notice that this notion is weakly related to the notion of signed frames, weighted frames as well as controlled frames (see, e.g., [13], [1], [14]).

Evidently, not every frame is scalable. For instance, a basis in $\mathbb{R}^2$ which is not an orthogonal basis is not scalable, since a frame with two elements in $\mathbb{R}^2$ is a Parseval frame if and only if it is an orthonormal basis. The relation to preconditioning is revealed by analyzing the finite-dimensional version of Proposition II.3 which shows that a frame Φ in $\mathbb{K}^N$ with analysis operator T_Φ is scalable if and only if there exists a diagonal matrix D such that DT_Φ is isometric. Since the condition number of such a matrix equals one, the scaling question is a particular instance of the problem of preconditioning of matrices.

The results in [12] were the leadoff results on this problem, which we present a survey about in this paper. The derived characterizations can be subdivided into the following two classes:

- Various characterizations of (strict) scalability of a frame for a general separable Hilbert space (see, e.g., Theorem II.5).
- Geometric characterization of scalability of finite frames (see Theorems III.1 and III.4).

We wish to note that recently, new results on this question from a slightly different angle have been derived in [6].

C. An Excursion to Numerical Linear Algebra

The problem of preconditioning is extensively studied in the numerical linear algebra community, see, e.g., [8], [10]. Preconditioners which are constructed by scaling appears in various forms in the numerical linear algebra literature. The most common approach is to minimize the condition number of the matrix multiplied by a preconditioning matrix – in our case of DT_Φ, where D runs through the set of diagonal matrices. It was for instance shown in [4], that this minimization problem can be reformulated as a convex problem. A major problem is however (see also [4]) that all algorithms solving this convex problem perform slowly, and, even worse, there exist situations in which the infimum is not attained. As additional references to this complex problem, we wish to mention [5], [2], [8], [11] and [15].

D. Outline

This paper is organized as follows. In Section II we focus on the situation of general separable Hilbert spaces and derive characterization of scalability and strict scalability. In Section III we then restrict to the situation of finite frames, and derive a yet different characterization of scalability as well as a geometric interpretation of scalable frames.

II. Strict Scalability of General Frames

This section is devoted to a very general characterization of (strictly) scalable frames.

A. Scalability and Frame Properties

The following definition makes the notion of scalability mathematically precise.

Definition II.1. A frame $\Phi = \{\varphi_j\}_{j\in J}$ for $\mathcal{H}$ is called *scalable* if there exist scalars $c_j \geq 0$, $j \in J$, such that $\{c_j\varphi_j\}_{j\in J}$ is a Parseval frame. If, in addition, $c_j > 0$ for all $j \in J$, then Φ is called *positively scalable*. If there exists $\delta > 0$, such that $c_j \geq \delta$ for all $j \in J$, then Φ is called *strictly scalable*.

For finite frames, it is immediate that positive and strict scalability coincide and that each scaling $\{c_j\varphi_j\}_{j\in J}$ of a finite frame $\{\varphi_j\}_{j\in J}$ with positive scalars c_j forms again a frame.

For infinite frames, the situation is significantly more involved. A partial answer was given in [1, Lemma 4.3], which proves that if there exist $K_1, K_2 > 0$ such that $K_1 \leq c_j \leq K_2$ holds for all $j \in J$, then also $\{c_j\varphi_j\}_{j\in J}$ is a frame. Our next result provides a complete characterization of when a scaling preserves the frame property. A crucial ingredient for this result is the *diagonal operator* D_c in $\ell^2(J)$ corresponding to a sequence $c = (c_j)_{j\in J} \subset \mathbb{K}$, which is defined by

$$D_c(v_j)_{j\in J} := \big(c_j v_j\big)_{j\in J}, \quad (v_j)_{j\in J} \in \operatorname{dom} D_c,$$

where

$$\operatorname{dom} D_c := \big\{(v_j)_{j\in J} \in \ell^2(J) : (c_j v_j)_{j\in J} \in \ell^2(J)\big\}.$$

It is a well-known fact that D_c is a (possibly unbounded) selfadjoint operator in $\ell^2(J)$ if and only if $c_j \in \mathbb{R}$ for all $j \in J$. If even $c_j \geq 0$ ($c_j > 0$, $c_j \geq \delta > 0$) for each $j \in J$, then the selfadjoint operator D_c is non-negative (positive, strictly positive, respectively).

The following result indeed provides a complete characterization of when a scaled frame constitutes again a frame. For stating this, as usual, we denote the domain, the kernel and the range of a linear operator T by $\operatorname{dom} T$, $\ker T$ and $\operatorname{ran} T$, respectively. Also, a closed linear operator T between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ will be called *ICR* (or an *ICR-operator*), if it is injective and has a closed range, i.e., if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in \operatorname{dom} T$.

Proposition II.2 ([12]). *Let $\Phi = \{\varphi_j\}_{j\in J}$ be a frame for $\mathcal{H}$ with analysis operator T_Φ and let $c = (c_j)_{j\in J}$ be a sequence of non-negative scalars. Then the following conditions are equivalent.*

(i) *The scaled sequence of vectors* $\Psi := \{c_j\varphi_j\}_{j\in J}$ *is a frame for* $\mathcal{H}$.
(ii) *We have* $\operatorname{ran} T_\Phi \subset \operatorname{dom} D_c$ *and* $D_c|\operatorname{ran} T_\Phi$ *is ICR.*

Moreover, in this case, the frame operator of the frame Ψ *is given by*

$$S_\Psi = (D_cT_\Phi)^*(D_cT_\Phi) = \overline{T_\Phi^* D_c} D_c T_\Phi,$$

where $\overline{T_\Phi^* D_c}$ *denotes the closure of the operator* $T_\Phi^* D_c$.

B. General Equivalent Condition

The following result seems to be quite obvious. However, in the general setting of an arbitrary separable Hilbert space, it is not straightforward at all.

Proposition II.3 ([12]). *Let* $\Phi = \{\varphi_j\}_{j\in J}$ *be a frame for* $\mathcal{H}$. *Then the following conditions are equivalent.*

(i) Φ *is (positively, strictly) scalable.*
(ii) *There exists a non-negative (positive, strictly positive, respectively) diagonal operator* D *in* $\ell^2(J)$ *such that*

$$\overline{T_\Phi^* D} D T_\Phi = I_{\mathcal{H}}. \tag{2}$$

We can now easily draw the conclusion that scalability is invariant under unitary transformations.

Corollary II.4. *Let* U *be a unitary operator in* $\mathcal{H}$. *Then a frame* $\Phi = \{\varphi_j\}_{j\in J}$ *for* $\mathcal{H}$ *is scalable if and only if the frame* $U\Phi = \{U\varphi_j\}_{j\in J}$ *is scalable.*

C. Main Result

Our main result provides several equivalent conditions for a frame Φ to be strictly scalable. For this, recall that a sequence $\{v_k\}_k$ of non-zero vectors in a Hilbert space $\mathcal{K}$ is called an *orthogonal basis* of $\mathcal{K}$, if $\inf_k \|v_k\| > 0$ and $(v_k/\|v_k\|)_k$ is an orthonormal basis of $\mathcal{K}$.

Theorem II.5 ([12]). *Let* $\Phi = \{\varphi_j\}_{j\in J}$ *be a frame for* $\mathcal{H}$ *such that* $\liminf_{j\in J} \|\varphi_j\| > 0$, *and let* $T = T_\Phi$ *denote its analysis operator. Then the following statements are equivalent.*

(i) *The frame* Φ *is strictly scalable.*
(ii) *There exists a strictly positive bounded diagonal operator* D *in* $\ell^2(J)$ *such that* DT *is isometric (that is,* $T^*D^2T = I_{\mathcal{H}}$*).*
(iii) *There exist a Hilbert space* $\mathcal{K}$ *and a bounded ICR operator* $L : \mathcal{K} \to \ell^2(J)$ *such that* $TT^* + LL^*$ *is a strictly positive bounded diagonal operator.*
(iv) *There exist a Hilbert space* $\mathcal{K}$ *and a frame* $\Psi = \{\psi_j\}_{j\in J}$ *for* $\mathcal{K}$ *such that the vectors*

$$\varphi_j \oplus \psi_j \in \mathcal{H} \oplus \mathcal{K}, \quad j \in J,$$

form an orthogonal basis of $\mathcal{H} \oplus \mathcal{K}$.

If one of the above conditions holds, then the frame Ψ *from* (iv) *is strictly scalable, its analysis operator is given by an operator* L *from* (iii)*, and with a diagonal operator* D *from* (ii) *we have*

$$L^*D^2L = I_{\mathcal{K}}, \quad \text{and} \quad L^*D^2T = 0. \tag{3}$$

We next analyze this result in the special case of finite frames. Although this restriction seems trivial, in fact restricting conditions (iii) and (iv) in Theorem II.5 to the situation of finite frames is not immediate.

Corollary II.6. *Let* $\Phi = \{\varphi_j\}_{j=1}^M$ *be a frame for* $\mathbb{K}^N$ *and let* $T = T_\Phi \in \mathbb{K}^{M\times N}$ *denote the matrix representation of its analysis operator. Then the following statements are equivalent.*

(i) *The frame* Φ *is strictly scalable.*
(ii) *There exists a positive definite diagonal matrix* $D \in \mathbb{K}^{M\times M}$ *such that* DT *is isometric.*
(iii) *There exists* $L \in \mathbb{K}^{M\times(M-N)}$ *such that* $TT^* + LL^*$ *is a positive definite diagonal matrix.*
(iv) *There exists a frame* $\Psi = \{\psi_j\}_{j=1}^M$ *for* $\mathbb{K}^{M-N}$ *such that* $\{\varphi_j \oplus \psi_j\}_{j=1}^M \in \mathbb{K}^M$ *forms an orthogonal basis of* $\mathbb{K}^M$.

III. Scalability of Real Finite Frames

Finally, we take a geometric viewpoint with respect to scalability. For this, we will focus on frames for $\mathbb{R}^N$ due to the fact that the proof of Theorem III.1 requires the utilization of Farkas' Lemma which only exists for real vector spaces.

A. Characterization Result

The following theorem provides a characterization of non-scalability of a finite frame specifically tailored to the finite-dimensional case. Condition (iii) of this result will be reinterpreted in Subsection III-B as a geometric condition for non-scalability.

Theorem III.1 ([12]). *Let* $\Phi = \{\varphi_j\}_{j=1}^M \subset \mathbb{R}^N \setminus \{0\}$ *be a frame for* $\mathbb{R}^N$. *Then the following statements are equivalent.*

(i) Φ *is not scalable.*
(ii) *There exists a symmetric matrix* $Y \in \mathbb{R}^{N\times N}$ *with* $\operatorname{tr}(Y) < 0$ *such that* $\varphi_j^T Y \varphi_j \geq 0$ *for all* $j = 1, \ldots, M$.
(iii) *There exists a symmetric matrix* $Y \in \mathbb{R}^{N\times N}$ *with* $\operatorname{tr}(Y) = 0$ *such that* $\varphi_j^T Y \varphi_j > 0$ *for all* $j = 1, \ldots, M$.

The following corollary, for whose proof we refer to [12], can be easily drawn from the previous result, showing that the set of non-scalable frames for $\mathbb{R}^N$ is open in the following sense.

Corollary III.2. *Let* $\Phi = \{\varphi_j\}_{j=1}^M \subset \mathbb{R}^N \setminus \{0\}$ *be a frame for* $\mathbb{R}^N$ *which is not scalable. Then there exists* $\varepsilon > 0$ *such that each set of vectors* $\{\psi_j\}_{j=1}^M \subset \mathbb{R}^N$ *with*

$$\|\varphi_j - \psi_j\| < \varepsilon \quad \text{for all } j = 1, \ldots, M \tag{4}$$

is a frame for $\mathbb{R}^N$ *which is not scalable.*

B. Geometric Interpretation

We now derive a geometric interpretation of the characterization result Theorem III.1, in particular of condition (iii). For this, first notice that each of the sets

$$C(Y) := \{x \in \mathbb{R}^N : x^TYx > 0\}, \quad Y \in \mathbb{R}^{N\times N} \text{ symmetric},$$

considered in Theorem III.1 (iii) forms an open cone with the additional property that $x \in C(Y)$ implies $-x \in C(Y)$.

Hence, from now on, we will analyze the impact of the condition $\mathrm{tr}(Y) = 0$ on the shape of these cones.

We will require the following particular class of conical surfaces. Their special relation to quadrics inspired us to coin those 'conical zero-trace quadrics'.

Definition III.3. Let the *class of conical zero-trace quadrics* $\mathcal{C}_N$ be defined as the family of sets

$$\left\{ x \in \mathbb{R}^N : \sum_{k=1}^{N-1} a_k \langle x, e_k \rangle^2 = \langle x, e_N \rangle^2 \right\}, \qquad (5)$$

where $\{e_k\}_{k=1}^N$ runs through all orthonormal bases of $\mathbb{R}^N$ and $(a_k)_{k=1}^{N-1}$ runs through all tuples of elements in $\mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^{N-1} a_k = 1$.

Utilizing this notion, we can state the following result on a geometric characterization of non-scalability.

Theorem III.4 ([12]). *Let $\Phi \subset \mathbb{R}^N \setminus \{0\}$ be a frame for $\mathbb{R}^N$. Then the following conditions are equivalent.*

(i) *Φ is not scalable.*
(ii) *All frame vectors of Φ are contained in the interior of a conical zero-trace quadric of $\mathcal{C}_N$.*
(iii) *All frame vectors of Φ are contained in the exterior of a conical zero-trace quadric of $\mathcal{C}_N$.*

By $\mathcal{C}_N^*$ we denote the subclass of $\mathcal{C}_N$ consisting of all conical zero-trace quadrics in which the orthonormal basis is the standard basis of $\mathbb{R}^N$. Thus, the elements of $\mathcal{C}_N^*$ are in fact quadrics of the form

$$\left\{ x \in \mathbb{R}^N : \sum_{k=1}^{N-1} a_k x_k^2 = x_N^2 \right\}.$$

with non-zero a_k's satisfying $\sum_{k=1}^{N-1} a_k = 1$.

This allows us to draw the following corollary from Theorem III.4 and Corollary II.4.

Corollary III.5. *Let $\Phi \subset \mathbb{R}^N \setminus \{0\}$ be a frame for $\mathbb{R}^N$. Then the following conditions are equivalent.*

(i) *Φ is not scalable.*
(ii) *There exists an orthogonal matrix $U \in \mathbb{R}^{N \times N}$ such that all vectors of $U\Phi$ are contained in the interior of a conical zero-trace quadric of $\mathcal{C}_N^*$.*
(iii) *There exists an orthogonal matrix $U \in \mathbb{R}^{N \times N}$ such that all vectors of $U\Phi$ are contained in the exterior of a conical zero-trace quadric of $\mathcal{C}_N^*$.*

Finally, in the 2- and 3-dimensional case Theorem III.4 reduces to the following results.

Corollary III.6. (i) *A frame $\Phi \subset \mathbb{R}^2 \setminus \{0\}$ for $\mathbb{R}^2$ is not scalable if and only if there exists an open quadrant cone which contains all frame vectors of Φ.*
(ii) *A frame $\Phi \subset \mathbb{R}^3 \setminus \{0\}$ for $\mathbb{R}^3$ is not scalable if and only if all frame vectors of Φ are contained in the interior of an elliptical conical surface with vertex 0 and intersecting the corners of a rotated unit cube.*

ACKNOWLEDGMENT

G. Kutyniok acknowledges support by the Einstein Foundation Berlin, by Deutsche Forschungsgemeinschaft (DFG) Grant SPP-1324 KU 1446/13 and DFG Grant KU 1446/14, by the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics", and by the DFG Research Center MATHEON "Mathematics for Key Technologies" in Berlin. F. Philipp is supported by the DFG Research Center MATHEON. K. A. Okoudjou was supported by ONR grants N000140910324 and N000140910144, by a RASA from the Graduate School of UMCP and by the Alexander von Humboldt foundation. He would also like to express his gratitude to the Institute for Mathematics at the University of Osnabrück for its hospitality while part of this work was completed.

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