Fundamental Limits of Phase Retrieval

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***Abstract*—Recent advances in convex optimization have led to new strides in the phase retrieval problem over finite-dimensional vector spaces. However, certain fundamental questions remain: What sorts of measurement vectors uniquely determine every signal up to a global phase factor, and how many are needed to do so? This paper presents several results that address these questions, specifically in the less-understood complex case. In particular, we characterize injectivity, we identify that the complement property is indeed necessary, we pose a conjecture that $4M - 4$ generic measurement vectors are necessary and sufficient for injectivity in M dimensions, and we describe how to prove this conjecture in the special cases where $M = 2, 3$. To prove the $M = 3$ case, we leverage a new test for injectivity, which can be used to determine whether any 3-dimensional measurement ensemble is injective.**

I. Introduction

Phase retrieval is the problem of recovering a signal from absolute values (squared) of linear measurements, called intensity measurements. However, non-injectivity is inherent to many measurement processes. For instance, intensity measurements with the identity basis effectively discard all phase information contained in the signal's entries. As a result, many researchers invoke a priori knowledge of the desired signal in order to restrict to a smaller signal class over which the intensity measurements might be injective. To avoid the various ad hoc methods that invariably follow, an alternative approach to phase retrieval, as introduced in 2006 by Balan, Casazza and Edidin [3], seeks injectivity by designing a larger ensemble of intensity measurements. Using this approach, Balan et al. [3] characterized injectivity in the real case and further leveraged algebraic geometry to show that $4M-2$ intensity measurements suffice for injectivity over M-dimensional complex signals. This has since sparked a search for practical phase retrieval guarantees. For example, viewing intensity measurements as Hilbert-Schmidt inner products between rank-1 operators, Candès, Strohmer and Voroninski [7] applied certain intuition from convex optimization to reconstruct the desired M-dimensional signal with semidefinite programming using only $\mathcal{O}(M \log M)$ random measurements. Another approach uses the polarization identity to discern relative phases between certain intensity measurements using $\mathcal{O}(M \log M)$ random measurements in concert with an expander graph [1].

Despite these recent advances in phase retrieval algorithms, there remains a lack of understanding about the fundamental requirements for intensity measurements to be injective. For example, it was widely believed that $3M - 2$ intensity measurements sufficed for injectivity, until recently disproved by Heinosaari, Mazzarella and Wolf [10] using embedding theorems from differential geometry. Heinosaari et al. were able to establish the necessity of $(4 + o(1))M$ measurements for injectivity, but the following problem still remains:

Problem 1. *What are the necessary and sufficient conditions for measurement vectors to lend injective intensity measurements?*

This paper addresses this problem by first providing the only known characterization of injectivity in the complex case (Theorem 4). Next, we make a rather surprising identification: that intensity measurements are injective in the complex case precisely when the corresponding phase-only measurements are injective in some sense (Theorem 5). We then use this identification to establish the necessity of the complement property for injectivity (Theorem 7). Later, we conjecture that $4M - 4$ intensity measurements are necessary and sufficient for injectivity in the complex case, which we validate in the cases where $M = 2, 3$ (Theorems 10 and 12). We also introduce a new test for injectivity, which we then use to verify the injectivity of a certain quantum-mechanics-inspired measurement ensemble; with this ensemble, we conclude by suggesting a refinement of Wright's conjecture from [12] (see Conjecture 13). The proofs of the presented results are provided in [4].

Before we begin, let $\Phi = \{\varphi_n\}_{n=1}^N$ in $V = \mathbb{R}^M$ or $\mathbb{C}^M$ be a given collection of measurement vectors and consider the intensity measurement process defined by

$$(\mathcal{A}(x))(n) := |\langle x, \varphi_n \rangle|^2.$$

Note that $\mathcal{A}(x) = \mathcal{A}(y)$ whenever $y = cx$ for some scalar c of unit modulus. Thus, the mapping $\mathcal{A}\colon V \to \mathbb{R}^N$ is necessarily not injective. To resolve this issue, we consider sets of the form V/S, where S is a multiplicative subgroup of the field of scalars. This notation means to identify vectors $x, y \in V$ which satisfy $y = cx$ for some scalar $c \in S$, and we write $y \equiv x \bmod S$ to convey this identification. Most of the time, V/S is either $\mathbb{R}^M/\{\pm 1\}$ or $\mathbb{C}^M/\mathbb{T}$ (here, $\mathbb{T}$ is the complex unit circle), and the intensity measurement process is viewed as a mapping $\mathcal{A}\colon V/S \to \mathbb{R}^N$. Injectivity of the measurement process is considered with respect to this mapping.

II. Injectivity and the complement property

Phase retrieval is impossible without injective intensity measurements. Balan, Casazza and Edidin [3] first analyzed injectivity by introducing the *complement property*, which we define in the following:

Definition 2. *We say $\Phi = \{\varphi_n\}_{n=1}^N$ in $\mathbb{R}^M$ ($\mathbb{C}^M$) satisfies the complement property (CP) if for every $S \subseteq \{1,\ldots,N\}$, either $\{\varphi_n\}_{n\in S}$ or $\{\varphi_n\}_{n\in S^c}$ spans $\mathbb{R}^M$ ($\mathbb{C}^M$).*

The complement property is characteristic of injectivity in the real case:

Theorem 3. *Consider $\Phi = \{\varphi_n\}_{n=1}^N \subseteq \mathbb{R}^M$ and the mapping $\mathcal{A}\colon \mathbb{R}^M/\{\pm 1\} \to \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \varphi_n\rangle|^2$. Then $\mathcal{A}$ is injective if and only if Φ satisfies the complement property.*

This result was demonstrated in [3]. However, it was also erroneously used as justification for the necessity of CP for injectivity in the complex case. Although this statement is indeed true, the proof of Theorem 3 overlooks the peculiarity of equivalence modulo $\mathbb{T}$ and so cannot be used in the complex setting. We will address this issue, but first we characterize injectivity in the complex case:

Theorem 4. *Consider $\Phi = \{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and the mapping $\mathcal{A}\colon \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \varphi_n\rangle|^2$. Viewing $\{\varphi_n\varphi_n^* u\}_{n=1}^N$ as vectors in $\mathbb{R}^{2M}$, denote $S(u) := \operatorname{span}_\mathbb{R}\{\varphi_n\varphi_n^* u\}_{n=1}^N$. Then the following are equivalent:*

- (a) *$\mathcal{A}$ is injective.*
- (b) *$\dim S(u) \geq 2M-1$ for every $u \in \mathbb{C}^M \setminus \{0\}$.*
- (c) *$S(u) = \operatorname{span}_\mathbb{R}\{\mathrm{i}u\}^\perp$ for every $u \in \mathbb{C}^M \setminus \{0\}$.*

Note that unlike in the real case, it is not clear whether this characterization can be tested in finite time; instead of being a statement about all (finitely many) partitions of $\{1,\ldots,N\}$, it is a statement about all nonzero vectors $u \in \mathbb{C}^M$. We can, however, view this characterization as an analog to the real case, in which the complement property is equivalent to having $\operatorname{span}\{\varphi_n\varphi_n^* u\}_{n=1}^N = \mathbb{R}^M$ for all nonzero $u \in \mathbb{R}^M$. The fact that more information is lost with phase in the complex case is what causes $\{\varphi_n\varphi_n^* u\}_{n=1}^N$ to fail to span all of $\mathbb{R}^{2M}$. As a result, it is still not intuitively apparent what it takes for an ensemble of complex vectors to yield injective intensity measurements. The following bizarre characterization was established while working toward a clearer understanding:

Theorem 5. *Consider $\Phi = \{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and the mapping $\mathcal{A}\colon \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \varphi_n\rangle|^2$. Then $\mathcal{A}$ is injective if and only if the following statement holds: If for every $n = 1,\ldots,N$, either $\arg(\langle x, \varphi_n\rangle^2) = \arg(\langle y, \varphi_n\rangle^2)$ or one of the sides is not well-defined, then $x = 0$, $y = 0$, or $y \equiv x \bmod \mathbb{R}\setminus\{0\}$.*

Theorem 5 is a consequence of a more general statement about the geometric properties of complex numbers: For $a, b \in \mathbb{C}$, $\operatorname{Im} a\bar{b} = 0$ if and only if $\arg(a^2) = \arg(b^2)$, $a = 0$, or $b = 0$. The proof leverages this fact within a restatement of part (c) of Theorem 4. This seemingly unrelated result is actually useful in correctly establishing the necessity of CP for injectivity in the complex case. Specifically, Theorem 5, leads to the following lemma, which in turn is used to prove necessity (Theorem 7).

Lemma 6. *Consider $\Phi = \{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and the mapping $\mathcal{A}\colon \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \varphi_n\rangle|^2$. If $\mathcal{A}$ is injective, then the mapping $\mathcal{B}\colon \mathbb{C}^M/\{\pm 1\} \to \mathbb{R}^N$ defined by $(\mathcal{B}(x))(n) := \langle x, \varphi_n\rangle^2$ is also injective.*

Theorem 7. *Consider $\Phi = \{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and the mapping $\mathcal{A}\colon \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \varphi_n\rangle|^2$. If $\mathcal{A}$ is injective, then Φ satisfies the complement property.*

The problem alluded to earlier concerning the proof of Theorem 3 is the reason that Theorem 7 is stated separately. This issue is resolved by using the injectivity of $\mathcal{B}$ modulo $\{\pm 1\}$. The proof is eerily similar to that of the necessity of CP for injectivity in Theorem 3, only using $\mathcal{B}$ in place of $\mathcal{A}$.

We emphasize here that the complement property is necessary but not sufficient for injectivity in the complex case. To see this, consider the ensemble $(1,0)$, $(0,1)$ and $(1,1)$. These certainly satisfy the complement property, but $\mathcal{A}((1,\mathrm{i})) = (1,1,2) = \mathcal{A}((1,-\mathrm{i}))$, despite the fact that $(1,\mathrm{i}) \not\equiv (1,-\mathrm{i}) \bmod \mathbb{T}$; in general, real frames fail to lend injective intensity measurements for the complex case. Indeed, a sufficient condition for injectivity in the complex case has yet to be found. As an analogy for what we really want, consider the notion of *full spark*: An ensemble $\{\varphi_n\}_{n=1}^N \subseteq \mathbb{R}^M$ is said to be full spark if every subcollection of M vectors spans $\mathbb{R}^M$. Full spark ensembles with $N \geq 2M-1$ necessarily satisfy the complement property, and the notion of full spark is simple enough to admit deterministic constructions [2], [11]. Because such constructions are particularly desirable for the complex case, finding a good sufficient condition for injectivity is an important problem that remains open.

III. Introducing the $4M-4$ Conjecture

Thinking of a matrix Φ as being built one column at a time, the rank-nullity theorem states that each column contributes to either the column space or the null space. If these columns are then used as linear measurement vectors, then the subspace that is actually sampled is described by the column space of Φ, while the null space captures the algebraic nature of redundancy in the measurements. An efficient sampling of an entire vector space would therefore apply a matrix Φ having a small null space and large column space. Although we are not dealing with linear measurements in our case, we would like to build our ensemble of intensity measurements so as to sample as much of the space as possible. More precisely, we are faced with the following question:

Problem 8. *For any dimension M, what is the smallest number $N^*(M)$ of injective intensity measurements, and how do we design such measurement vectors?*

To be clear, this problem was completely solved in the real case by Balan, Casazza and Edidin [3]. Indeed, Theorem 3 immediately implies that $2M-2$ intensity measurements are necessarily not injective, and furthermore that $2M-1$ measurements are injective if and only if the measurement vectors are full spark.

In the complex case, Problem 8 has some history in the quantum mechanics literature. For example, [12] presents *Wright's conjecture* that any pure state is uniquely determined by three observables. In other words, the conjecture states that there exist unitary matrices U_1, U_2 and U_3 such that $\Phi = [U_1\ U_2\ U_3]$ lends injective intensity measurements. Note that Wright's conjecture actually implies that $N^*(M) \leq 3M-2$, since U_1 determines the norm of the signal, rendering the last column of both U_2 and U_3 unnecessary. Finkelstein [8] later proved that $N^*(M) \geq 3M-2$ which, combined with Wright's conjecture, has led many to believe that $N^*(M) = 3M-2$. However, this was recently disproved by Heinosaari, Mazzarella and Wolf [10], who used embedding theorems from differential geometry to prove that $N^*(M) \geq 4M - 2\alpha(M-1) - 3$, where $\alpha(M-1) \leq \log_2(M)$ is the number of 1's in the binary representation of $M-1$. Combined with Balan, Casazza and Edidin's result that $N^*(M) \leq 4M-2$, we at least have the asymptotic expression $N^*(M) = (4+o(1))M$.

The lemma that follows will help to refine our intuition for $N^*(M)$. Before stating the result, however, we must first define the *super analysis operator* $\mathbf{A}\colon \mathbb{H}^{M\times M} \to \mathbb{R}^N$. Given an ensemble of measurement vectors $\{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$, this operator acts on the real M^2-dimensional vector space of self-adjoint $M\times M$ matrices, $\mathbb{H}^{M\times M}$, and is defined by $(\mathbf{A}H)(n) = \langle H, \varphi_n\varphi_n^*\rangle_{\mathrm{HS}}$, where $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ denotes the Hilbert-Schmidt inner product. Note that the super analysis operator is a linear operator which satisfies

$$(\mathbf{A}xx^*)(n) = \langle xx^*, \varphi_n\varphi_n^*\rangle_{\mathrm{HS}} = |\langle x, \varphi_n\rangle|^2 = (\mathcal{A}(x))(n).$$

To clarify, $x \bmod \mathbb{T}$ can be "lifted" to xx^*, a process which linearizes the intensity measurement process at the price of squaring the dimension of the vector space. This identification is not new, and as the following lemma shows, it can also be used to characterize injectivity:

Lemma 9. *$\mathcal{A}$ is not injective if and only if there exists a matrix of rank 1 or 2 in the null space of $\mathbf{A}$.*

Lemma 9 indicates that we want the null space of $\mathbf{A}$ to avoid nonzero matrices of rank ≤ 2. This is easier when the "dimension" of this set of matrices is small. As an exercise in intuition, we count real degrees of freedom to get an idea of this dimension: By the spectral theorem, almost every matrix in $\mathbb{H}^{M\times M}$ of rank ≤ 2 can be uniquely expressed in the form $\lambda_1 u_1u_1^* + \lambda_2 u_2u_2^*$. The pair of coefficients (λ_1, λ_2) introduces two degrees of freedom, while the vector u_1, which can be any vector in $\mathbb{C}^M$ of unit norm and is unique up to global phase, has a total of $2M-2$ real degrees of freedom. Finally, u_2 has the same norm and phase constraints as u_1, with the additional requirement that it must be orthogonal to u_1, (i.e., $\mathrm{Re}\langle u_2, u_1\rangle = \mathrm{Im}\langle u_2, u_1\rangle = 0$). Thus, u_2 has $2M-4$ real degrees of freedom. In this way we expect the set of matrices in question to have $2+(2M-2)+(2M-4) = 4M-4$ real dimensions.

If the set S of matrices of rank ≤ 2 formed a subspace of $\mathbb{H}^{M\times M}$, then we could expect it to have a nontrivial intersection with the null space of $\mathbf{A}$ whenever

$$\dim\mathrm{null}(\mathbf{A}) + (4M-4) > \dim(\mathbb{H}^{M\times M}) = M^2.$$

By the rank-nullity theorem, this would indicate that injectivity requires

$$N \geq \mathrm{rank}(\mathbf{A}) = M^2 - \dim\mathrm{null}(\mathbf{A}) \geq 4M-4.$$

Of course, this logic is not valid since S is not a subspace of $\mathbb{H}^{M\times M}$. It is, however, a special kind of set: a real projective variety (a real algebraic variety which is closed under scalar multiplication). If S were a projective variety over an *algebraically closed* field, then the projective dimension theorem (Theorem 7.2 of [9]) would imply that it intersects $\mathrm{null}(\mathbf{A})$ nontrivially whenever the dimensions are large enough: $\dim\mathrm{null}(\mathbf{A}) + \dim S > \dim\mathbb{H}^{M\times M}$, and so injectivity would require $N \geq 4M-4$. Unfortunately, this theorem is not valid when the field is $\mathbb{R}$; for example, the cone defined by $x^2+y^2-z^2=0$ in $\mathbb{R}^3$ is a projective variety of dimension 2, but its intersection with the 2-dimensional xy-plane is trivial, despite the fact that $2+2>3$.

In the absence of a proof, we pose the natural conjecture:

The $4M-4$ Conjecture. *Consider $\Phi = \{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and the mapping $\mathcal{A}\colon \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \varphi_n\rangle|^2$. If $M \geq 2$, then the following statements hold:*

(a) *If $N < 4M-4$, then $\mathcal{A}$ is not injective.*
(b) *If $N \geq 4M-4$, then $\mathcal{A}$ is injective for generic Φ.*

For the sake of clarity, we state what is meant by the word "generic." A real algebraic variety is the set of common zeros of a finite set of polynomials with real coefficients. Taking all such varieties in $\mathbb{R}^n$ to be closed sets then defines the *Zariski topology* on $\mathbb{R}^n$. If we view Φ as a member of $\mathbb{R}^{2MN}$, we then say a *generic* Φ is any member of some nonempty Zariski-open subset of $\mathbb{R}^{2MN}$. Since Zariski-open sets are either empty or dense with full measure, genericity is a particularly strong property. As such, another way to state part (b) of the $4M-4$ conjecture is "If $N \geq 4M-4$, then there exists a real algebraic variety $V \subseteq \mathbb{R}^{2MN}$ such that $\mathcal{A}$ is injective for every $\Phi \notin V$." The work of Balan, Casazza and Edidin [3] already proves this for $N \geq 4M-2$. Furthermore, Bodmann and Hammen [5] establish that whenever $N \geq 4M-4$, there exists Φ such that $\mathcal{A}$ is injective, so for (b), it only remains to show that generic Φ make $\mathcal{A}$ injective.

The following results are given in the interest of resolving the $4M-4$ conjecture:

Theorem 10. *The $4M-4$ Conjecture is true when $M=2$.*

Since in this case injectivity is equivalent to having a full-rank super analysis operator (see Lemma 9), Theorem 10 can be established by defining the real algebraic variety

Algorithm 1 The HMW test for injectivity when $M = 3$

Input: Measurement vectors $\{\varphi_n\}_{n=1}^N \subseteq \mathbb{C}^3$
Output: Whether $\mathcal{A}$ is injective
Define $\mathbf{A}\colon \mathbb{H}^{3\times 3} \to \mathbb{R}^N$ such that $\mathbf{A}H = \{\langle H, \varphi_n\varphi_n^*\rangle_{\text{HS}}\}_{n=1}^N$ {assemble the super analysis operator}
if $\dim \operatorname{null}(\mathbf{A}) = 0$ **then**
"INJECTIVE" {if $\mathbf{A}$ is injective, then $\mathcal{A}$ is injective}
else
Pick $H \in \operatorname{null}(\mathbf{A})$, $H \neq 0$
if $\dim \operatorname{null}(\mathbf{A}) = 1$ and $\det(H) \neq 0$ **then**
"INJECTIVE" {if $\mathbf{A}$ only maps nonsingular matrices to zero, then $\mathcal{A}$ is injective}
else
"NOT INJECTIVE" {in the remaining case, $\mathbf{A}$ maps differences of rank-1 matrices to zero}
end if
end if

$V = \{\mathbf{A} : \operatorname{Re}\det\mathbf{A} = \operatorname{Im}\det\mathbf{A} = 0\}$ and showing that V^c is nonempty, and therefore dense with full measure. Before stating the analogous result for $M = 3$, we introduce the *HMW test* for injectivity (see Algorithm 1); we name it after Heinosaari, Mazarell and Wolf, who implicitly introduce this algorithm in their paper [10].

Theorem 11. *When $M = 3$, the HMW test correctly determines whether $\mathcal{A}$ is injective.*

The proof of Theorem 11 relies heavily on Lemma 9. For the case of $\dim \operatorname{null}(\mathbf{A}) = 2$, an application of the intermediate value theorem shows that a singular matrix of rank 1 or 2 can always be constructed from matrices in the null space of $\mathbf{A}$.

Theorem 12. *The $4M - 4$ Conjecture is true when $M = 3$.*

The proof of Theorem 12 first constructs the real algebraic variety V of matrices U, each gotten by a generalized cross product of a basis for the range of the adjoint of some $\mathbf{A}$, and further satisfying $\det U = 0$; the first part ensures that U spans the null space of $\mathbf{A}$, while at the same time being defined using polynomials of the entries of the matrix representation of $\mathbf{A}$. The HMW test is then used to show that V^c is nonempty.

Note that the HMW test can be used to test for injectivity in three dimensions regardless of the number of measurement vectors. Thus, it can be used to evaluate ensembles of 3×3 unitary matrices for quantum mechanics. For example, consider the 3×3 fractional discrete Fourier transform, defined in [6] using discrete Hermite-Gaussian functions:

$$
\begin{aligned}
F^\alpha =& \frac{1}{6}\begin{bmatrix} 3+\sqrt{3} & \sqrt{3} & \sqrt{3} \\ \sqrt{3} & \frac{3-\sqrt{3}}{2} & \frac{3-\sqrt{3}}{2} \\ \sqrt{3} & \frac{3-\sqrt{3}}{2} & \frac{3-\sqrt{3}}{2} \end{bmatrix} \\
&+ \frac{e^{\alpha i\pi}}{6}\begin{bmatrix} 3-\sqrt{3} & -\sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & \frac{3+\sqrt{3}}{2} & \frac{3+\sqrt{3}}{2} \\ -\sqrt{3} & \frac{3+\sqrt{3}}{2} & \frac{3+\sqrt{3}}{2} \end{bmatrix} \\
&+ \frac{e^{\alpha i\pi/2}}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

It can be shown by the HMW test that $\Phi = [I\ F^{1/2}\ F\ F^{3/2}]$ lends injective intensity measurements. This leads to the following refinement of Wright's conjecture:

Conjecture 13. *Let F denote the $M \times M$ discrete fractional Fourier transform defined in [6]. Then for every $M \geq 3$, $\Phi = [I\ F^{1/2}\ F\ F^{3/2}]$ lends injective intensity measurements.*

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