

Balayage and short time Fourier transform frames

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Abstract—Using his formulation of the potential theoretic notion of balayage and his deep results about this idea, Beurling gave sufficient conditions for Fourier frames in terms of balayage. The analysis makes use of spectral synthesis, due to Wiener and Beurling, as well as properties of strict multiplicity, whose origins go back to Riemann. In this setting and with this technology, we formulate and prove non-uniform sampling formulas in the context of the short time Fourier transform (STFT).

I. INTRODUCTION

A. Background and theme

Frames provide a natural tool for dealing with signal reconstruction in the presence of noise in the setting of overcomplete sets of atoms, and with the goals of numerical stability and robust signal representation. Fourier frames were originally studied in the context of non-harmonic Fourier series by Duffin and Schaeffer [1], with a history going back to Paley and Wiener [2] (1934) and farther, and with significant activity in the 1930s and 1940s, e.g., see [3]. Since [1], there have been significant contributions by Beurling (unpublished 1959-1960 lectures), [4], [5], Beurling and Malliavin [6], [7], Kahane [8], Landau [9], Jaffard [10], and Seip [11], [12].

Definition I.1. (Frame) Let H be a separable Hilbert space. A sequence $\{x_n\}_{n \in \mathbb{Z}} \subseteq H$ is a *frame* for H if there are positive constants A and B such that

$$\forall f \in H, \quad A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, x_n \rangle|^2 \leq B\|f\|^2.$$

The constants A and B are lower and upper frame bounds, respectively.

Our overall goal is to formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M_b(\mathbb{R}^d)$ of bounded Radon measures, see [13]. This formulation provides a natural way to generalize non-uniform sampling to the setting of short time Fourier transforms (STFTs) [14], Gabor theory [15], [16], [17], and pseudo-differential operators [14], [18]. The techniques are based on Beurling's methods from 1959-1960, [5], [4], which incorporate balayage, spectral synthesis, and strict multiplicity. In this short paper, we show how to achieve this goal for STFTs.

B. Definitions

We define the Fourier transform $\mathcal{F}(f)$ of $f \in L^2(\mathbb{R}^d)$ and its inverse Fourier transform $\mathcal{F}^{-1}(f)$ by

$$\mathcal{F}(f)(\gamma) = \hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \gamma} dx,$$

and

$$\mathcal{F}^{-1}(\hat{f})(x) = f(x) = \int_{\widehat{\mathbb{R}}^d} \hat{f}(\gamma) e^{2\pi i x \cdot \gamma} d\gamma.$$

$\widehat{\mathbb{R}}^d$ denotes \mathbb{R}^d considered as the spectral domain. We write $F^\vee(x) = \int_{\widehat{\mathbb{R}}^d} F(\gamma) e^{2\pi i x \cdot \gamma} d\gamma$. The notation “ \int ” designates integration over \mathbb{R}^d or $\widehat{\mathbb{R}}^d$. When f is a bounded continuous function, its Fourier transform is defined in the sense of distributions. If $X \subseteq \mathbb{R}^d$, where X is closed, then $M_b(X)$ is the space of bounded Radon measures μ with the support of μ contained in X . $C_b(\mathbb{R}^d)$ denotes the space of complex-valued bounded continuous functions on \mathbb{R}^d .

Definition I.2. (Fourier frame) Let $E \subseteq \mathbb{R}^d$ be a sequence and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact set. Notationally, let $e_x(\gamma) = e^{2\pi i x \cdot \gamma}$. The sequence $\mathcal{E}(E) = \{e_{-x} : x \in E\}$ is a *Fourier frame* for $L^2(\Lambda)$ if there are positive constants A and B such that

$$\begin{aligned} \forall F \in L^2(\Lambda), \\ A\|F\|_{L^2(\Lambda)}^2 \leq \sum_{x \in E} |\langle F, e_{-x} \rangle|^2 \leq B\|F\|_{L^2(\Lambda)}^2. \end{aligned}$$

Define the *Paley-Wiener space*,

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \text{supp}(\hat{f}) \subseteq \Lambda\}.$$

Clearly, $\mathcal{E}(E)$ is a Fourier frame for $L^2(\Lambda)$ if and only if the sequence,

$$\{(e_{-x} \mathbb{1}_\Lambda)^\vee : x \in E\} \subseteq PW_\Lambda,$$

is a frame for PW_Λ , in which case it is called a *Fourier frame* for PW_Λ . Note that $\langle F, e_{-x} \rangle = f(x)$ for $f \in PW_\Lambda$, where $\hat{f} = F \in L^2(\widehat{\mathbb{R}}^d)$ can be considered an element of $L^2(\Lambda)$.

Beurling introduced the following definition in his 1959-1960 lectures.

Definition I.3. (Balayage) Let $E \subseteq \mathbb{R}^d$ and $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be closed sets. *Balayage* is possible for $(E, \Lambda) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if

$$\forall \mu \in M_b(\mathbb{R}^d), \exists \nu \in M_b(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

Balayage originated in potential theory, where it was introduced by Christoffel (early 1870s) and by Poincaré (1890). Kahane formulated balayage for the harmonic analysis of restriction algebras. The set, Λ , of group characters (in this case \mathbb{R}^d) is the analogue of the original role of Λ in balayage as a set of potential theoretic kernels.

Let $\mathcal{C}(\Lambda) = \{f \in C_b(\mathbb{R}^d) : \text{supp}(\hat{f}) \subseteq \Lambda\}$.

Definition I.4. (Spectral synthesis) A closed set $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a set of *spectral synthesis* (*S-set*) if

$$\begin{aligned} \forall f \in \mathcal{C}(\Lambda) \text{ and } \forall \mu \in M_b(\mathbb{R}^d), \\ \widehat{\mu} = 0 \text{ on } \Lambda \Rightarrow \int f d\mu = 0, \end{aligned}$$

see [19].

Closely related to spectral synthesis is the ideal structure of L^1 , which can be thought of as the Nullstellensatz of harmonic analysis. As examples of sets of spectral synthesis, polyhedra are S-sets, and the middle-third Cantor set is an S-set which contains non-S-sets. Laurent Schwartz (1947) showed that $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set; and, more generally, Malliavin (1959) proved that every non-discrete locally compact abelian group contains non-S sets. See [19] for a unified treatment of this material.

Definition I.5. (Strict multiplicity) A closed set $\Gamma \subseteq \widehat{\mathbb{R}}^d$ is a set of *strict multiplicity* if

$$\exists \mu \in M_b(\Gamma) \setminus \{0\} \text{ such that } \lim_{\|x\| \rightarrow \infty} |\mu^\vee(x)| = 0.$$

The notion of strict multiplicity was motivated by Riemann's study of sets of uniqueness for trigonometric series. Menchov (1906) showed that there exists a closed set $\Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z}$ and $\mu \in M(\Gamma) \setminus \{0\}$, such that $|\Gamma| = 0$ and $\mu^\vee(n) = O((\log |n|)^{-1/2})$, $|n| \rightarrow \infty$. There have been intricate refinements of Menchov's result by Bary (1927), Littlewood (1936), Beurling, et al., see [19].

The above concepts are used in the deep proof of the following theorem.

Theorem I.6. Assume that Λ is an S-set of strict multiplicity, and that balayage is possible for (E, Λ) . Let $\Lambda_\epsilon = \{\gamma \in \widehat{\mathbb{R}}^d : \text{dist}(\gamma, \Lambda) \leq \epsilon\}$. There is $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then balayage is possible for (E, Λ_ϵ) .

Definition I.7. A sequence $E \subseteq \mathbb{R}^d$ is *separated* if

$$\exists r > 0 \text{ such that } \inf\{\|x - y\| : x, y \in E \text{ and } x \neq y\} \geq r.$$

The following theorem, due to Beurling, gives a sufficient condition for Fourier frames in terms of balayage. Its history and structure are analyzed in [13] as part of a more general program. Theorem I.6 is used in its proof.

Theorem I.8. Assume that $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is an S-set of strict multiplicity and that $E \subseteq \mathbb{R}^d$ is a separated sequence. If balayage is possible for (E, Λ) , then $\mathcal{E}(E)$ is a Fourier frame for $L^2(\Lambda)$, i.e., $\{(e_{-x} \mathbb{1}_\Lambda)^\vee : x \in E\}$ is a Fourier frame for PW_Λ .

See [9], [20], [21] (SampTA 1999), and [22].

II. SHORT TIME FOURIER TRANSFORM (STFT) FRAME INEQUALITIES

Definition II.1. Let $f, g \in L^2(\mathbb{R}^d)$. The *short time Fourier transform* (STFT) of f with respect to g is the function $V_g f$

on \mathbb{R}^{2d} defined as

$$V_g f(x, \omega) = \int f(t) \overline{g(t-x)} e^{-2\pi i t \cdot \omega} dt,$$

see [14], [18] (chapter 8).

The STFT is uniformly continuous on \mathbb{R}^{2d} . Further, for a fixed “window” $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$, we can recover the original function $f \in L^2(\mathbb{R}^d)$ from its STFT $V_g f$ by means of the vector-valued integral inversion formula,

$$f = \int \int V_g f(x, \omega) e_{\omega} \tau_x g d\omega dx,$$

where $(\tau_x g)(t) = g(t-x)$.

Theorem II.2. Let $E = \{x_n\} \subseteq \mathbb{R}^d$ be a separated sequence, that is symmetric about $0 \in \mathbb{R}^d$; and let $\Lambda \subseteq \mathbb{R}^d$ be an S-set of strict multiplicity that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d$. Assume balayage is possible for (E, Λ) . Further, let $g \in L^2(\mathbb{R}^d)$, $\widehat{g} = G$, have the property that $\|g\|_2 = 1$.

a. We have that

$$\exists A > 0, \text{ such that } \forall f \in PW_\Lambda \setminus \{0\}, \widehat{f} = F,$$

$$\begin{aligned} A \|f\|_2^2 &\leq \sum_{x \in E} \int |V_G F(\omega, x)|^2 d\omega \\ &= \sum_{x \in E} \int |V_g f(x, \omega)|^2 d\omega. \end{aligned}$$

b. Let $G_0(\lambda) = 2^{d/4} e^{-\pi \|\lambda\|^2}$ so that $\|G_0\|_2 = 1$; and assume $\|V_{G_0} G\|_1 < \infty$. We have that

$$\exists B > 0, \text{ such that } \forall f \in PW_\Lambda \setminus \{0\}, \widehat{f} = F,$$

$$\begin{aligned} \sum_{x \in E} \int |V_g f(x, \omega)|^2 d\omega &= \sum_{x \in E} \int |V_G F(\omega, -x)|^2 d\omega \\ &\leq B \|f\|_2^2, \end{aligned}$$

where B can be taken as $C \|V_{G_0} G\|_1$ and where

$$C = \sup_{y, \gamma} \left\{ \sum_{x \in E} \int |V_{G_0} G_0(\gamma + \omega, y + x)| d\omega \right\}.$$

The technique of using G_0 goes back to Feichtinger and Zimmermann [23] (Lemma 3.2.15) for a related type of problem, see also [16] (Lemma 3.2).

We next consider balayage being possible for (E, Λ) , where $E = \{(s_m, t_n)\} \subseteq \mathbb{R}^{2d}$ and $\Lambda \subseteq \widehat{\mathbb{R}}^{2d}$. This allows us to express the STFT $V_g f$ of f as

$$V_g f(y, \omega) = \sum_m \sum_n a_{mn}(y, \omega) h(s_m - y, t_n - \omega) V_g f(s_m, t_n),$$

where

$$\sum_m \sum_n |a_{mn}(y, \omega)| < \infty.$$

The following result and others like it, including Theorem II.2, can be formulated in terms of (X, μ) frames, [24], [25]. [26].

Theorem II.3. Assume balayage is possible for (E, Λ) , where $E = \{(s_m, t_n)\} \subseteq \mathbb{R}^{2d}$ is separated, and $\Lambda \subseteq \widehat{\mathbb{R}}^{2d}$ is an S-set

that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^{2d}$. Fix a window function $g \in L^2(\mathbb{R}^d)$ such that $\|g\|_2 = 1$. There are constants $A, B > 0$, such that if $f \in L^2(\mathbb{R}^d)$ satisfies the conditions,

- (1) $V_g f \in L^1(\mathbb{R}^{2d})$ and
 - (2) $\mathcal{F}(V_g f)(\zeta_1, \zeta_2)$ has support $\subseteq \Lambda \subseteq \widehat{\mathbb{R}}^{2d}$,
- then

$$A \int |f(x)|^2 dx \leq \sum_m \sum_n |V_g f(s_m, t_n)|^2 \leq B \int |f(x)|^2 dx.$$

The hypothesis that $V_g f \in L^1(\mathbb{R}^{2d})$ means that f belongs to the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^d)$. It is the smallest Banach space that is invariant under translations and modulations. There are other equivalent characterizations of $\mathcal{S}_0(\mathbb{R}^d)$, see [27], [23]. Fix a function $\mathcal{S}_0(\mathbb{R}^d)$ and define the vector space \mathcal{M}_1^1 of all non-uniform Gabor expansions

$$f = \sum_{n=1}^{\infty} c_n \tau_{x_n} e_{\omega_n} g,$$

where $\{(x_n, \omega_n) \in \mathbb{R}^{2d}, n \in \mathbb{N}\}$ is an arbitrary countable set of numbers and $\sum_{n=1}^{\infty} |c_n| < \infty$. For this space, the norm is taken to be $\inf \sum_{n=1}^{\infty} |c_n|$, where the infimum is taken over all possible representations. Then the vector space \mathcal{M}_1^1 coincides with $\mathcal{S}_0(\mathbb{R}^d)$. For functions in $\mathcal{S}_0(\mathbb{R}^d)$, Theorem II.3 should be compared to the following theorem of Gröchenig [15], [14] (Chapter 12):

Theorem II.4. *Given any $g \in \mathcal{S}_0(\mathbb{R}^d)$. There is $r = r(g) > 0$ such that if $E = \{(s_n, \sigma_n)\} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a separated sequence with the property that*

$$\bigcup_{n=1}^{\infty} \overline{B((s_n, \sigma_n), r(g))} = \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

then the frame operator, $S = S_{g,E}$, defined by

$$S_{g,E} f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle \tau_{s_n} e_{\sigma_n} g,$$

is invertible on $\mathcal{S}_0(\mathbb{R}^d)$.

Moreover, every $f \in \mathcal{S}_0(\mathbb{R}^d)$ has a non-uniform Gabor expansion,

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{x_n} e_{\omega_n} g \rangle S_{g,E}^{-1}(\tau_{x_n} e_{\omega_n} g),$$

where the series converges unconditionally in $\mathcal{S}_0(\mathbb{R}^d)$. (E depends on g .)

A critical, thorough comparison of Theorems II.3 and II.4 is given in [13].

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REFERENCES

- [1] R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.*, vol. 72, pp. 341–366, 1952.
- [2] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, ser. Amer. Math. Society Colloquium Publications. Providence, RI: American Mathematical Society, 1934, vol. XIX.
- [3] N. Levinson, *Gap and Density Theorems*, ser. Amer. Math. Society Colloquium Publications. Providence, RI: American Mathematical Society, 1940, vol. XXVI.
- [4] A. Beurling, *The Collected Works of Arne Beurling. Vol. 2. Harmonic Analysis*. Boston: Birkhäuser, 1989.
- [5] —, "Local harmonic analysis with some applications to differential operators," *Some Recent Advances in the Basic Sciences, Vol. 1 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1962–1964)*, pp. 109–125, 1966.
- [6] A. Beurling and P. Malliavin, "On Fourier transforms of measures with compact support," *Acta Mathematica*, vol. 107, pp. 291–309. [Online]. Available: <http://dx.doi.org/10.1007/BF02545792>
- [7] —, "On the closure of characters and the zeros of entire functions," *Acta Mathematica*, vol. 118, pp. 79–93. [Online]. Available: <http://dx.doi.org/10.1007/BF02392477>
- [8] J.-P. Kahane, "Sur certaines classes de séries de Fourier absolument convergentes," *J. Math. Pures Appl. (9)*, vol. 35, pp. 249–259, 1956.
- [9] H. J. Landau, "Necessary density conditions for sampling and interpolation of certain entire functions," *Acta Mathematica*, vol. 117, pp. 37–52, 1967. [Online]. Available: <http://dx.doi.org/10.1007/BF02395039>
- [10] S. Jaffard, "A density criterion for frames of complex exponentials," *Michigan Math. J.*, vol. 38, pp. 339–348, 1991.
- [11] K. Seip, "On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$," *J. Funct. Anal.*, vol. 130, pp. 131–160, 1995.
- [12] J. Ortega-Cerdà and K. Seip, "Fourier frames," *Ann. of Math.*, vol. 155, no. 3, pp. 789–806, 2002.
- [13] J. J. Benedetto and E. Au-Yeung, "Generalized Fourier frames in terms of balayage," *forthcoming*.
- [14] K. Gröchenig, *Foundations of Time-Frequency Analysis*, ser. Applied and Numerical Harmonic Analysis. Boston, MA: Birkhäuser Boston Inc., 2001.
- [15] —, "Describing functions: atomic decompositions versus frames," *Monatsh. Math.*, vol. 112, pp. 1–42, 1991.
- [16] H. G. Feichtinger and W. Sun, "Stability of Gabor frames with arbitrary sampling points," *Acta Mathematica Hungarica*, vol. 113, pp. 187–212, 2006. [Online]. Available: <http://dx.doi.org/10.1007/s10474-006-0099-4>
- [17] D. Labate, G. Weiss, and E. Wilson, "An Approach to the Study of Wave Packet Systems," *Contemporary Mathematics, Wavelets, Frames, and Operator Theory*, vol. 345, pp. 215–235, 2004.
- [18] K. Gröchenig, "A pedestrian's approach to pseudodifferential operators," in *Harmonic Analysis and Applications*, ser. Appl. Numer. Harmon. Anal. Boston: Birkhäuser, 2006, pp. 139–169.
- [19] J. J. Benedetto, *Spectral Synthesis*. New York-London: Academic Press, Inc., 1975.
- [20] J. J. Benedetto and H.-C. Wu, "A multidimensional irregular sampling algorithm and applications," *IEEE-ICASSP*, 1999.
- [21] J. J. Benedetto and H. Wu, "A Beurling covering theorem and multidimensional irregular sampling," in *SampTA*, Loen, 1999.
- [22] J. J. Benedetto and H.-C. Wu, "Non-uniform sampling and spiral MRI reconstruction," *SPIE*, 2000.

- [23] H. G. Feichtinger and G. Zimmermann, "A Banach space of test functions for Gabor analysis." in *Gabor analysis and algorithms*, ser. Appl. Numer. Harmon. Anal. Boston: Birkhäuser, 1998, pp. 123–170.
- [24] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent states, wavelets and their generalizations*, ser. Graduate Text in Contemporary Physics. New York: Springer-Verlag, 2000.
- [25] J.-P. Gabardo and D. Han, "Frames associated with measurable spaces. frames." *Adv. Comput. Math.*, vol. 18, no. 2-4, pp. 127–147, 2003.
- [26] M. Fornasier and H. Rauhut, "Continuous frames, function spaces, and the discretization problem," *J. Fourier Anal. Appl.*, vol. 11, no. 3, pp. 245–287, 2005.
- [27] H. G. Feichtinger, "On a new Segal algebra," *Monatsh. Math.*, vol. 92, pp. 269–289, 1981.