Coding and sampling for compressive tomography

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***Abstract*—This paper discusses sampling system design for estimation of multidimensional objects from lower dimensional measurements. We consider examples in geometric, diffractive, coherence, spectral and temporal tomography. Compressive tomography reduces or eliminates conventional tradeoffs between temporal and spatial resolution.**

I. Introduction

Compressive measurement is generally defined as the estimation of N signal values from M measurements for $M < N$. While this definition has been highly useful and successful in many sensing and imaging applications, an alternative definition is of equal utlity in tomographic imaging. Tomography most commonly consists of imaging 3D objects from measurements distributed over 1D or 2D sensor arrays. Typical tomographic systems may be described by integral equations of the form

$$g(y) = \int f(x)h(x,y)dx \tag{1}$$

where $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. One may define "compressive tomography" as estimation of $f(x)$ from $g(y)$ in the case that $M < N$.

Tomographic systems typically use sensor arrays embedded on the boundary or surface of a volume under observation. In fan beam tomography, for example, a linear detector array measures attenuation of rays through a 2D object space. In cone beam tomography, a planar detector arrray measures rays projected through a 3D volume. Conventional tomography overcomes the dimensional mismatch between the object and measurement spaces by varying illumination and sensor geometry as a function of time, thereby increasing the dimension of the measurement space by 1. Thus in conventional systems $M = N - 1$ for measurements taken at a fixed time, but $M = N$ when time is taken into account.

The most unfortunate aspect of the conventional approach is that it requires that the object remain static as measurements are collected over time. Over the past several years, my group has applied compressive sampling theory to implement snapshot compressive tomography. For example, we have shown that 3D hyperspectral [1], diffraction [2] and x-ray scatter [3] images may be reconstructed from 2D data. We have also analyzed compressive sampling for reconstruction of 3D objects with conventional optics [4]. Most recently, we have shown that 3D video data cubes may be constructed from 2D frames [5], thus using compressive tomography to reconstruct time itself.

While the object distribution $f(x)$ is by definition distributed over continous space, measurements ultimately consist of discrete digital data. There is no fundamental requirement that discrete measurements be indexed by a continous variable. Standard compressive sampling models assume independent kernels for each measurement. Unfortunately, completely independent kernels are difficult or impossible to implement on measurements embedded in continuous physical space. Due in part to this challenge, Candes' early analysis of compressive tomography focused on discrete subsampling of the temporal portion of Radon space with continuous sampling in each snapshot [6].

My group's initial theoretical studies of compressive tomography focused on the use of multidimensional reference structures to enable random or decorrelated measurement over a continuous space [7]. However, most subsequent efforts to implement practical compressive tomography may be described in the context of three basic coding strategies

1) **Measurement space coding.** The standard model for increasing measurement dimensionality with time involves varying the the measurement kernel to obtain

$$g(y,t) = \int f(x)h(x,y,t)dx \tag{2}$$

Measurement space coding multiplexes diverse kernels in a snapshot to obtain

$$\tilde{g}(y) = \int g(y,t)C(y,t)dt = \int C(y,t)f(x)h(x,y,t)dxdt \tag{3}$$

where $C(y,t)$ is a code applied to each time slice in measurement space. $C(y,t)$ is designed to allow "code division multiple access" (CDMA) such that $g(y,t)$ can be isolated from $\tilde{g}(y)$.

2) **Object space coding** modulates the object density prior to measurement to obtain the forward model

$$g(y) = \int f(x)C(x)h(x,y)dx \tag{4}$$

Again, $C(x)$ enables the use of CDMA to increase the effective dimensionality of the measurements.

3) **Transform subsampling** expands the subsampling strategy of [6] to optimize which portions of the transform space measured.

CDMA is, of course, most commonly understood in the context of multiuser communications. CDMA considers the case that a set of relatively low frequency signals $f_i(t)$ must communicate over the same channel. Multiplication of each signal with an independent high frequency code $C_i(t)$ enables one to isolate each signal even when the overall transmitted data is $g(t) = \sum_i f_i(t)C_i(t)$. This is achieved by assuming that the codes are orthogonal over short time windows such that

$$\int_{t-T}^{t} g(t')C_j(t')dt' = \sum_i f_i(t) \int C_i(t')C_j(t')dt' = f_j(t) \tag{5}$$

In effect, coding turns the 1D measurement over time into a 2D measurement over time and transmitter index. In the same way, coding in tomography systems effectively increases the dimensionality of the measurements. The question "What is the maximum bandwidth of $f(t)$ relative to the bandwidth of $C(t)$ such that this dimensionality increase can be achieved is a central issue in compressive sampling theory.

The goal of the remainder of this paper is to relate these abstract coding strategies to practical tomographic imagers. Tomographic system design is inherently an integrated sensing and processing challenge by which physical and geometric constraints must be matched to mathematical conditioning and algorithms. The next section reviews the basic physical structure of tomographic imagers and dicusses how coding strategies 1-3 are implemented in these systems.

II. Field Models and Coding

While Eqn. (1) might describe many different measurement systems, the underlying concept that measurements and objects are distributed over continous spaces linked by a continous kernel uniquely describes remote sensing systems. The transformation from object to measurement is mediated by radiation fields propagating between the two spaces. While "tomographic imaging" in its most general sense refers to systems as diverse as MRI and electron microscopy, most analyses of computed tomographic imaging focus specifically on imaging using radiating fields [8].

Radiation fields are commonly described by (1) geometric models, under which the fields propagate as nondiffracting rays, (2) diffraction models, under which the fields propagate as waves and (3) coherence models, which generalize wave models to account for quantum noise and measurement characteristics [9]. Each field model is most applicable in specific contexts, corresponds to specific measureable features and is amenable to specific coding strategies.

For geometric tomography, attenuation or scatter of rays is the basic measureable quantity. Specifically, one measures

$$g(y, \theta) = \int f(y + \alpha\theta)d\alpha \tag{6}$$

where $y \in \mathbb{S}^{N-1}$ is a point on a boundary enclosing the object and $\theta \in \mathbb{S}^{N-1}$ is the direction vector for a ray passsing through y. For $N > 2$, the dimensionality of the potential ray measurement space, $M = 2N - 2$, is greater than N and inversion is over constrained. The challenge of geometric tomography is that it is not possible to simultaneously discriminate all rays passing through y. Typical detectors have no mechanism for discriminating rays and simple intetrate the total irradiance over all rays passing through the detector point. Conventional tomographic imagers overcome this problem by ensuring that only 1 ray passes through each measurement point in each measurement time. This is most often achieved by illuminating with a collimated pencil, fan or cone beam source. Under this scenario, θ is a single valued function of y and the measurement is

$$g(y, t) = \int f\left(y + \alpha\theta(y, t)\right) d\alpha \tag{7}$$

for $y \in \mathbb{S}^{N-1}$ and $t \in \mathbb{R}$. A dimensional match between measurements and the object is achieved by changing $\theta(y, t)$ as a function of time.

Each of coding strategies 1-3 may be implemented in geometric tomography. Measurement space coding is applied in x-ray scatter imaging by placing a coded aperture between the scattering target and the measurement plane. Where conventional scatter imaging scans a colimator as function of time, coding allows distinct range, cross range and momentum slices to be multiplexed and reconstructed from a single time step [3], [10]. Measurement space coding may also be applied using a coded aperture with multiple illumination sources. Illumination angle-based code shifts allow disambiguation of the sources and simultaneous aquisition of multiple source data [11]. Multisource coding in combination with scatter imaging may also be understood as object space coding. Rather than using coded aperture shadows to disambiguate scatter sources, one may use structured illumination to code scatter position of distributed targets. Finally, as noted above, subsampling of multiple source data is an example of transform subsampling. While in [6], this subsampling takes the form of discontinous selection of continuous subspaces of the Radon transformation, more effective compression is obtained by combining mulitsource illumination with coded apertures, reference structures or collimation filters to more randomly sample Radon space. As suggested by this survey of practical strategies, detailed analysis of the coding strategy depends both on physical feasibility, object priors and mathematical structure.

Despite all the complexity of wave mechanics, the most immediate difference between geometric tomography and diffraction tomography is that the diffraction sample surface integrals rather than via line integrals. More substantive differences arise from object field-interaction models, typical object priors and the use of time to measure space. Diffraction tomography, including radar, millimeter wave and terahertz imaging, ultrasound and optical holography, most often considers scattered radiation rather than attenuation or primary sources. Under the Born approximation, the scattered field for single plane wave illumination samples a spherical shell in the Fourier space of the object density [12]. A single measurement corresponds to a point in the Fourier space in this case.

From a practical perspective, the use of phase delay or time of flight to measure range is the most unique and powerful aspect of diffraction tomography. This technique enables optical coherence tomography (OCT), which measures spatial range with resolution proportional to spectral bandwidth rather than aperture size. Compressive OCT has been considered in several studies using transform subsampling [13]. Time of flight from a monostatic transceiver integrates the object density on a sphere surrounding the transceiver with a range proportional to the observation time. Measurement of a family of spheres obtained by translating the transceiver obtains a Randon-like transformation of the volume. Bistatic or multistatic systems sample integrals over hyperbolic surfaces between emitter and receiver positions.

As with geometric tomography, strategies 1-3 may be applied in compressive diffraction tomography. While I am not aware of any examples of measurement space coding with coherent waves, the use of a metamaterial transceiver to create structured illumination [14] is an example of compressive tomography using object space coding. 3D object estimation from Fourier space manifolds in [15], [16] is an example of tranform subsampling, although disjoint or randomized subsampling as described for 2D images in [17] may be considered more sample efficient. Accounting for the unique physical priors arising from diffuse and specular reflection of coherent radiation is the most challenging aspect of diffraction tomography, however.

The scattered field on the surface of a diffuse reflector is a complex Gaussian random variable. Since the mean of the field is 0, estimation of the mean is an ineffective imaging strategy. The magnitude of the field is exponentially distributed, estimation of the magnitude lead to speckled images. Given that the field is random and uncorrelated in each pixel, the field over a 2D image is not generally compressible. Compressive tomography is therefore best implemented by building a forward model on the nonnegative object scattering density, which corresponds to the variance of the Gaussian random process [18]. Whether the scatter is diffuse or specular, however, one notes that diffraction tomography tends to be most useful in imaging interfaces and surfaces rather than continuous volumes. While the reason for this may be simply that volume imaging is too noisy and random to allow imaging to occur, design of compressive diffraction tomography systems would most effectively build on the assumption that the object consists exclusively of surfaces. This prior should enable highly compressive and super-resolved estimation of even diffuse scatters and is thus a worthy area for ongoing research.

Optical coherence functions, most typically consisting of the cross spectral density, describe fields radiated by random natural sources. While one may apply interferometric methods to directly sample the cross spectral density for transform subsampling based compressive tomography [19], such methods are ill-conditioned for complex sources. Focal imaging is the only mathematically well conditioned strategy for measurement of random sources but is incapable of mapping volume distributions onto measurement planes [9]. Object space modulation [4] and focal stacking (sweeping focal parameters during exposure) may be used to overcome this limitation. Compressive tomography of random volume sources is much more challenging that geometric or diffraction tomography, however, and remains an active research challenge.

III. Conclusion

The reader may be surprised to complete an entire article on tomographic imaging without encountering a single image. To my knowledge, however, this is the first print article to explicitly consider compressive tomography as defined in Eqn. (1). As such I hope that the reader will find the intellectual exercise of mapping this definition onto essentially the complete gamut of remote sensing systems sufficiently fascinating as to agree that a few simple images of traditional phantoms would only be a distraction. The conventional concept of an image as a 2D object that can be captured on a focal plane and displayed in an article is an artifact of analog image processing. In the modern world of computational and compressive imaging, all images are multidimensional and all imaging systems are tomographic.

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