# Sampling Techniques for Improved Algorithmic Efficiency in Electromagnetic Sensing

Kyle R. Krueger, James H. McClellan, Waymond R. Scott Jr.
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332–0250

***Abstract*—Ground-penetrating radar (GPR) and electromagnetic induction (EMI) sensors are used to image and detect subterranean objects; for example, in landmine detection. Compressive sampling at the sensors is important for reducing the complexity of the acquisition process. However, there is a second form of sampling done in the imaging-detection algorithms where a parametric forward model of the EM wavefield is used to invert the measurements. This parametric model includes all the features that need to be extracted from the object; for subterranean targets this includes but is not limited to type, 3D location, and 3D orientation. As parameters are added to the model, the dimensionality increases. Current sparse recovery algorithms employ a dictionary created by sampling the entire parameter space of the model. If uniform sampling is done over the high-dimensional parameter space, the size of the dictionary and the complexity of the inversion algorithms grow rapidly, exceeding the capability of real-time processors. This paper shows that strategic sampling practices can be exploited in both the parameter space, and the acquisition process to dramatically improve the efficiency and scalability of the these EM sensor systems.**

## I. Introduction

Parameter estimation of unknown objects through the use of wavefield sensors is a well researched area. An increasingly popular solution to these types of problems comes from the advancements in compressive sensing (CS) and sparse recovery [1]. These inversion algorithms rely on the fact that a highly accurate forward model of the data could be created to describe the dependence of the physical sensor data (i.e., the measurements) on the interesting parameters of the objects being imaged. This approach highlights an issue with CS. The inherent need for a random sensing matrix does not always lend itself easily to practical data acquisition from sensors. On the other hand, creating a comprehensive target model, oftentimes called a dictionary, and referred to in the CS world as a sparsifying transform, can quickly become too large and too computationally intensive for real-time computers. The data collection and imaging flow is shown in Fig. 1.

The key sampling issue is creating a dictionary of manageable size, even when it is desirable to add more parameters to the model. A $d$-parameter, $m$-measurment model leads to a dictionary of size $\mathcal{O}(N^{d+m})$, assuming equal sampling $(N)$ of each variable. This paper will show, through the use of strategic parameter-space sampling, that the dimensionality of the dictionary can be reduced in two different acquisition environments. Thus the computational complexity of these parameter detection problems can be drastically reduced for these sensor systems.

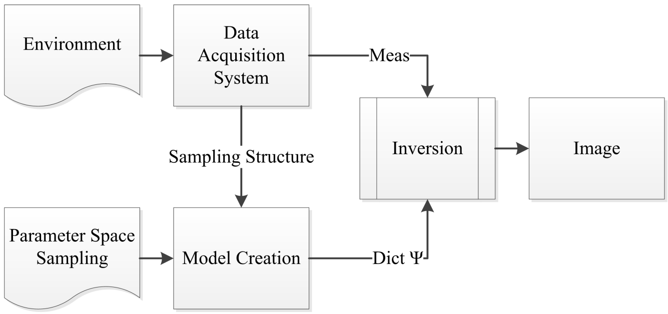


Fig. 1. Imaging algorithm flow.

There are two different acquisition systems that are discussed in this paper. The first system is three-dimensional (3D) imaging of subterranean targets using a ground-penetrating radar (GPR). The typical acquisition sampling pattern in GPR allows for reduced data acquisition time through the use of a random sensing pattern. The simplification of sampling the parametric forward model comes from exploiting the translationally-invariant nature of the physical model. The second system uses electromagnetic induction (EMI) sensors to detect and classify underground metallic targets. In this case, the strategic sampling of the parameter space comes from adopting an efficient tensor model to describe the orientation and magnetic polarizability of the target. This can be extracted using a rank-minimization detection algorithm. This model represents orientation space continuously with very few samples, instead of requiring the entire 3D angle space to be enumerated. This dramatically reduces computational complexity and also increases accuracy by eliminating the off-grid parameter sampling problem with regards to orientation.

## II. Ground-penetrating Radar

The GPR system considered here is a stepped-frequency system that has been previously described in detail [2]. The forward model is a point-target model, and the detection algorithm is based on sparse recovery (CS). The remainder of this section explains how acquisition sampling and model-parameter sampling together lead to a translational-invariance property that can achieve the computational complexity reductions in the detection algorithm that were shown in [3].

### A. Model

The point-target model used for the detection algorithm is

$$\psi(f, \boldsymbol{l}_s, \boldsymbol{l}_t) = \frac{g(f)e^{-2j\pi f\tau(\boldsymbol{l}_s,\boldsymbol{l}_t)}}{S(\boldsymbol{l}_s, \boldsymbol{l}_t)}, \quad (1)$$

which is a function of the stepped frequencies, $f$; the sensor positions, $\boldsymbol{l}_s = (l_s(x), l_s(y), 0)$; the target locations, $\boldsymbol{l}_t = (l_t(x), l_t(y), l_t(z))$; and the spreading parameter, $S(\boldsymbol{l}_s, \boldsymbol{l}_t)$. This forward model, $\psi(f, \boldsymbol{l}_s, \boldsymbol{l}_t)$, is used as the dictionary, or sparsifying transform. If $\psi$ is discretized and enumerated for all possible frequencies, sensor locations and target locations, the resulting model $\boldsymbol{\Psi}$ is 6D. The storage requirements are $\mathcal{O}(N^6)$ for equal discretization of all parameters, [3]. Our objective is to use properties inherent in the model and the acquisition system to reduce this storage and computational burden.

### B. Special Properties

A special property that can be used for increased efficiency is the fact that the model above can be translationally invariant. A translationally-invariant model can be applied using the Fast Fourier Transform (FFT), which eliminates the storage requirements for each dimension having this property. The translational-invariance property is true when the parameter space and the measurement space are evenly sampled in the same direction. In other words, when the target and the sensor are moved an equal distance in a horizontal dimension, $x$ or $y$, the radar response will remain the same. Also, to use the FFT to garner the complexity reduction, the stepped frequencies at each sensor position, $\boldsymbol{l}_s$, must be the same. This runs counter to the usual random sampling approach in CS, but it is a very important constraint when trying to exploit this special property even in a CS environment.

### C. Compressive Sensing Detection Algorithm

Now that the special properties in the model are identified, the detection algorithm itself can reduce the time needed for computation and data acquisition, if the sampling is done properly. The idea behind CS is that if the model parameters can be sparse, then projecting the model onto a known random subspace with much lower dimensionality than the original can still enable an accurate inversion [1]. Often the projections are done with a random sampling matrix $\boldsymbol{\Phi}$ applied to the model, $\boldsymbol{\Psi}$. For good results, $\boldsymbol{\Phi}$ should be independent and identically distributed (IID) random. There are a few techniques that will reduce the computational complexity of this general matrix multiplication, but they do not allow for any reduction in acquisition time for this particular GPR acquisition system [4], [5].

To get a mix of computational complexity reduction and data acquisition time reduction while staying within the CS framework, a strategic $\boldsymbol{\Phi}$ should be designed. An in-depth analysis of the trade-offs in designing $\boldsymbol{\Phi}$ for this GPR acquisition system were studied by Gurbuz et al. [6]. The basic trade-off is that the more structured $\boldsymbol{\Phi}$ becomes, the higher the coherence of the dictionary, and thus the higher the number of samples required for reconstruction, but this can also allow for reduction in data acquisition time. For this problem, some additional structure is needed in $\boldsymbol{\Phi}$ to exploit the translational invariance. To use the FFT across $x$ and $y$ dimensions, the full $x$ and $y$ must be sampled for a given $f$. This means that $\boldsymbol{\Phi}$ can be built to randomly select a small number of $f$ to get both a reduction in complexity for the CS algorithm, but also reduce the data acquisition time, and amount of data that needs to be collected.

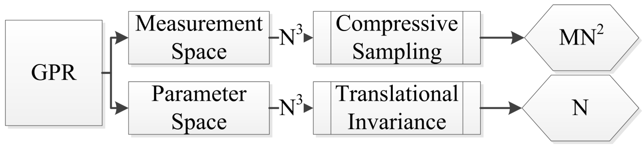


Fig. 2. Flow chart of GPR acquisition system, models, and complexity reducing properties

### D. Complexity Reduction

The complexity reduction for exploiting these model properties in this particular system are quite significant. In terms of storage space, the original fully discretized parameter-space model is 6D having a storage requirement of $\mathcal{O}(N^6)$. By using the FFT and CS, the storage requirement for the dictionary was reduced to $\mathcal{O}(MN^3)$, where $M < N$ is the number of random frequency measurements. In practical application of this method to laboratory measurements, the frequency requirements are reduced from 401 to 10 [6]. For an actual system, the data acquisition would take a fortieth of the time, as well as saving a factor of 40 in the amount of storage needed. The flow of the GPR acquisition system, the special properties, and their effect on complexity are summarized in Fig. 2.

There is also a rather significant reduction in algorithm time in using the translationally-invariant model over using a direct approach. Direct matrix multiplication for the 6D problem has a complexity of $\mathcal{O}(N^6)$, but the translationally-invariant model can be applied in $\mathcal{O}(N^4 \log_2(N))$ because the FFT can be applied along two of the parameter dimensions. A semilog plot of computation time versus problem size $(N)$ for both of these models in theory, and the FFT-based method in practice, can be seen in Fig. 3. For $N = 70$, the FFT-based method is more than 400 times faster. In fact, the direct method was not measured since it cannot be applied for $N = 70$ because the storage requirements are about 950 Gbytes, while the FFT-based method requires around 200 Mbytes.

## III. Electromagnetic Induction Sensor

A different acquisition system used for collecting target data is a multi-frequency (wideband) EMI sensor system. Multiple sensors are scanned in a down-track pattern, acquiring a sampled frequency response at uniformly spaced locations along the scan path. The forward model is a frequency domain model with many more parameters than the point target model [7]–[10]. The sparse recovery algorithm used is formulated as a combined least-squares and low-rank approximation problem.

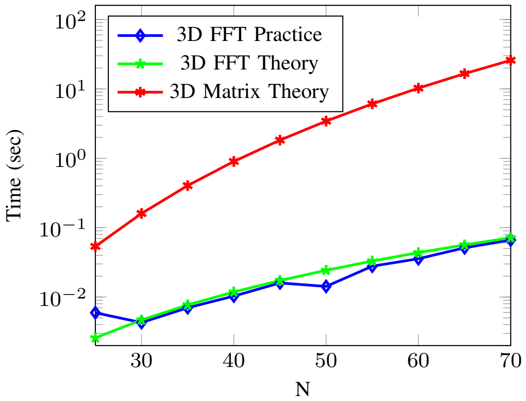


Fig. 3. Run time for different model applications

### *A. Model*

The basic model used for this system is one that is written in the frequency, $\omega$, domain, for a single target type, $\mu$,

$$r(\omega, \boldsymbol{l}_s, \boldsymbol{l}_t, \boldsymbol{o}_t, \mu) = \boldsymbol{g}^T(\boldsymbol{l})\boldsymbol{R}(\boldsymbol{o}_t)\boldsymbol{A}(\omega, \mu)\boldsymbol{R}^T(\boldsymbol{o}_t)\boldsymbol{f}(\boldsymbol{l}). \quad (2)$$

$\boldsymbol{l}_s = (l_s(x), 0, 0)$ is sensor position, $\boldsymbol{l}_t = (l_t(x), l_t(y), l_t(z))$ is 3D target location, $\boldsymbol{o}_t = (o_t(\alpha), o_t(\beta), o_t(\gamma))$ is target orientation, $\boldsymbol{A}(\omega, \mu)$ is a $3\times3$ matrix that defines the magnetic polarizability of the target, and $\boldsymbol{l} = \boldsymbol{l}_t - \boldsymbol{l}_s$ is the relative location vector for the target if the sensor was the origin. $\boldsymbol{g}(\boldsymbol{l})$ and $\boldsymbol{f}(\boldsymbol{l})$ are vectors that contain the spatial components of the magnetic field on the receive coil and the transmit coil respectively based on the relative location vector $\boldsymbol{l}$. $\boldsymbol{R}(\boldsymbol{o}_t)$ is a simple rotation matrix that rotates by angle $\boldsymbol{o}_t$. When all the measurements, $\{\omega, l_s(x)\}$, and parameters, $\{l_t(x), l_t(y), l_t(z), o_t(\alpha), o_t(\beta), o_t(\gamma), \mu\}$, are enumerated; the result is a data hyper-cube of 9D. The storage requirement is enormous when the parameter space is sampled finely enough.

An important change is to model the response as an expansion of magnetic dipoles, each with a frequency relaxation [9]. The coefficients of the expansion can be computed from the experimental data [11]. The coefficient for the term in the expansion with a relaxation frequency, $\zeta$, is

$$r^{\zeta}(\boldsymbol{l_s}, \boldsymbol{l_t}, \boldsymbol{o}_t, \boldsymbol{\Lambda}) = \boldsymbol{g}^T(\boldsymbol{l})\boldsymbol{R}(\boldsymbol{o}_t)\boldsymbol{\Lambda}\boldsymbol{R}^T(\boldsymbol{o}_t)\boldsymbol{f}(\boldsymbol{l}). \quad (3)$$

Each individual $\zeta$ can be imaged separately using the same model (3), regardless of type. Typically, the number of relaxation frequencies, $N_\zeta$, is between one and six. There are two significant benefits of the expansion. First, a specific frequency response for each target type is no longer needed. Second, $\boldsymbol{A}(\omega, \mu)$ changes to $\boldsymbol{\Lambda}$, which is a $3\times3$ diagonal, positive semidefinite, real matrix that does not depend on $\omega$ or $\mu$. These benefits greatly reduce the storage requirements. However, since each $\zeta$ must be imaged independently, the number of imaging steps increases from one to $N_\zeta$, even though the model itself does not depend on $\zeta$.

### *B. Special Properties*

This model (3) has two special properties. First, the model is separable into a product of functions. There are separate functions for location, orientation, and magnetic polarizibility that contribute to the product. This means that individual parameters can be isolated from one another. The second property comes from thinking of $\boldsymbol{R}(\boldsymbol{o}_t)\boldsymbol{\Lambda}\boldsymbol{R}^T(\boldsymbol{o}_t)$ as a "generalized amplitude" of the target. Usually, the response of a point target is a scalar that represents the strength of the target, but the matrix $\boldsymbol{R}(\boldsymbol{o}_t)\boldsymbol{\Lambda}\boldsymbol{R}^T(\boldsymbol{o}_t)$ encodes additional information about how the target strength depends on symmetry and orientation.

To build a dictionary, there needs to be an enumeration for every possible sample in the interesting parameter space, but it is undesirable to enumerate all possible entries of the matrix $\Lambda$ along with all possible orientation angles $\boldsymbol{\alpha}$. To avoid storing a large number of samples, a change can be made to the fundamentals of sampling this model. Instead of thinking about a point target response as having a scalar amplitude, it can be thought of as having a tensor amplitude by rewriting the model in (3) as

$$r^{\zeta}(\boldsymbol{l}_s, \boldsymbol{l}_t, \boldsymbol{o}_t, \boldsymbol{\Lambda}) = \boldsymbol{g}^T(\boldsymbol{l})\boldsymbol{T}(\boldsymbol{o}_t, \boldsymbol{\Lambda})\boldsymbol{f}(\boldsymbol{l}), \quad (4)$$

where $\boldsymbol{T}$ is a symmetric, positive semidefinite matrix that is only $3\times3$. This will be referred to as a "tensor amplitude." It has a great advantage over just the scalar amplitude. It contains the continuous orientation and the magnetic polarizability of the target in its eigenvectors and eigenvalues respectively. This gives a more accurate model, because it does not require sampling of the orientation parameter, so there is no modeling error associated with having targets whose orientations do not lie exactly on the sampled orientation space. Also, this reformulation reduces a 3D grid of angle samples to just six independent values in $\boldsymbol{T}(\boldsymbol{o}_t, \boldsymbol{\Lambda})$ which provides a large computational savings. Once $\boldsymbol{T}(\boldsymbol{o}_t, \boldsymbol{\Lambda})$ is found, an eigen-decomposition will yield $\boldsymbol{o}_t$ and $\boldsymbol{\Lambda}$.

### *C. Detection Algorithm*

The detection algorithm for the EMI acquisition system is a combination of least squares and a semidefinite programming (SDP) technique used to get a low-rank approximation. The solution to this problem is sparse in 3D, in just the same way the GPR system is sparse. In fact, in most cases it should be even more sparse, because the model (4) is much more sophisticated and is looking for magnetic dipoles, and not just a sum of point reflections.

The full problem can be solved using a block-tensor representation to simultaneously find the target location and the tensor amplitude through a convex relaxation to the rank-minimization algorithm [12],

$$\begin{aligned} \min \quad & \mathbf{tr}(\hat{\boldsymbol{T}}) \\ \text{s.t.} \quad & \hat{\boldsymbol{T}} \succeq 0, \| \mathbf{b} - \boldsymbol{\Psi}\mathbf{s} \| < \epsilon. \end{aligned} \quad (5)$$

$\hat{\boldsymbol{T}}$ is a block-diagonal tensor made up of $3\times3$ tensors $\boldsymbol{T}_t$, one for each possible target location. $\mathbf{b}$ is the collected measurement vector, $\boldsymbol{\Psi}$ is the dictionary enumerated from (4), and $\mathbf{s}$ is a sparse parameter vector that makes up the nonzero values in $\boldsymbol{T}$. This exploits the fact that the block-tensor structure will be extremely low rank. This is the case

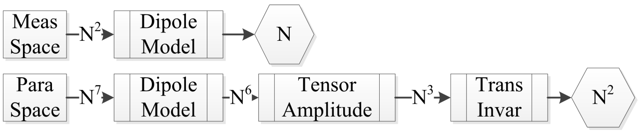


Fig. 4. Storage size for different versions of the sampled EMI acquisition and processing system. Complexity is reduced by exploiting special sampling properties of parametric models for EMI.

because the rank of the large block tensor structure, $\hat{\boldsymbol{T}}$, is just the sum of the rank of all tensor amplitudes, $\boldsymbol{T}_t$, of each present target. This work is in its initial stage of development because the full problem is computationally intense.

A shortcut can be used to break the algorithm into two steps to address the complexity of the full problem. Using an orthogonal matching pursuit type technique, a least-squares problem is solved to approximate the location of the target first [13]. Then (5) can be recast as a very small SDP,

$$\begin{aligned} \min \quad & \mathbf{tr}(\boldsymbol{T}_t) \\ \mathrm{s.t.} \quad & \boldsymbol{T}_t \succeq 0, \| \mathbf{b} - \boldsymbol{\Psi}_t \mathbf{s}_t \| < \epsilon, \end{aligned} \tag{6}$$

to get the tensor amplitude of the target at location, $\boldsymbol{l}_t$. The target response is then subtracted from the measurement and the process is repeated until the stopping criteria, a small enough residual, is met.

The EMI system also has the translationally-invariant property in the scanning dimension, $x$, just like the GPR acquisition system. However, the detection algorithm for this model setup is more complicated than direct matrix multiplication, so it will be more difficult to take advantage simultaneously of both the tensor representation property and the translational invariance in the large problem. Such a combined algorithm would be desirable, but it has not been implemented yet for a practical application.

### D. Complexity Reduction

The flow chart of measurement and parameter space simplifications of the EMI system in Fig. 4 summarizes the special properties exploited, and their resulting complexity reductions. Using the dipole model is a very important computation saving step, eliminating $N^2$ storage, going from a data hyper-cube of $N^9$ to $N^7$. Using the tensor amplitude representation, which changes the fundamentals of how the forward model is sampled, both increases the accuracy of the solution and garners an $N^3$ savings to drop the overall storage requirements to $N^4$. This is the result of tensor sampling (which needs six values) eliminating the need to finely sample the entire 3D orientation parameter. The EMI system also has the same translational invariance as the GPR system, and if it were exploited, there is another dimension of savings. Ultimately, the result of taking advantage of these special properties could obtain a savings of $N^6$.

## IV. Conclusion

This paper emphasizes the importance of the model representation. How the measurements are acquired, how the parameter space of the forward model is sampled, and how these two sampling operations can be adjusted to take advantage of special properties can all contribute to reducing the computational complexity. The tensor representation is also a different way to think about modeling data, and has been exploited in other applications such as seismic [14]. Using discrete values to provide continuous responses can allow for more accurate models while still harnessing the power of computers. A variation of this idea has been done in modeling continuous signals with Taylor series and cosine representations which allow for discrete values to be acquired [15]. The advantages of these sampling structures have been shown to drastically reduce computational complexity, increase accuracy, and reduce data acquisition times when combined with dictionary based detection algorithms.

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