Taylor and rank-1 lattice based nonequispaced fast Fourier transform

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***Abstract*—The nonequispaced fast Fourier transform (NFFT) allows the fast approximate evaluation of trigonometric polynomials with frequencies supported on full box-shaped grids at arbitrary sampling nodes. Due to the curse of dimensionality, the total number of frequencies and thus, the total arithmetic complexity can already be very large for small refinements at medium dimensions. In this paper, we present an approach for the fast approximate evaluation of trigonometric polynomials with frequencies supported on symmetric hyperbolic cross index sets at arbitrary sampling nodes. This approach is based on Taylor expansion and rank-1 lattice methods. We prove error estimates for the approximation and present numerical results.**

I. Introduction

We consider the evaluation of trigonometric polynomials $f\colon \mathbb{T}^d := [0,1)^d \to \mathbb{C}$,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{l}\in\mathcal{I}_N} \hat{f}_{\boldsymbol{l}}\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}\boldsymbol{x}},\ \hat{f}_{\boldsymbol{l}} \in \mathbb{C},\ \mathcal{I}_N \subset \mathbb{Z}^d \cap [-N,N]^d, \quad (1)$$

at arbitrary sampling nodes $\boldsymbol{y}_\ell \in \mathbb{T}^d$, $\ell = 0,\ldots,L-1$. For given Fourier coefficients $\hat{f}_{\boldsymbol{l}}$, the direct evaluation of the trigonometric sums $f(\boldsymbol{y}_\ell)$, $\ell=0,\ldots,L-1$, takes $\mathcal{O}(L\,|\mathcal{I}_N|)$ arithmetic operations. Various fast methods for the approximate evaluation of the trigonometric sums $f(\boldsymbol{y}_\ell)$ were developed.

In the case, when the frequency index set $\mathcal{I}_N$ is a full grid, $\mathcal{I}_N = G_N^d := \mathbb{Z}^d \cap [-N,N)^d$, the nonequispaced fast Fourier transform (NFFT, see [1] and references therein) allows the fast approximate evaluation of the trigonometric polynomial f at arbitrary sampling nodes $\boldsymbol{y}_\ell$, $\ell=0,\ldots,L-1$, in $\mathcal{O}(|\log\epsilon|^d L + |G_N^d| \log|G_N^d|)$ arithmetic operations, where ϵ is the approximation error. Furthermore, there exist Taylor based versions (cf. [2], [3]) with an arithmetic complexity of $\mathcal{O}(|\log\epsilon|^d (L + |G_N^d|\log|G_N^d|))$, which use fast Fourier transforms (FFT) for evaluating the trigonometric polynomial f as well as its derivatives at equispaced nodes and approximate the trigonometric sum $f(\boldsymbol{y}_\ell)$ by a Taylor expansion at the closest equispaced node. However, since the cardinality of the full grid G_N^d is $|G_N^d| = (2N)^d$, the total number of arithmetic operations can already be very large for small refinements N at medium dimensionality (e.g. $d=3,4,5$).

For dyadic hyperbolic crosses $\tilde{H}_n^d := \cup_{\boldsymbol{j}\in\mathbb{N}_0^d, \|\boldsymbol{j}\|_1 = n} \tilde{G}_{\boldsymbol{j}}$, $\tilde{G}_{\boldsymbol{j}} := \mathbb{Z}^d \cap \times_{t=1}^d (-2^{j_t-1}, 2^{j_t-1}]$, $\|\boldsymbol{j}\|_1 = |j_1| + \ldots + |j_d|$, the nonequispaced hyperbolic cross fast Fourier transform (NHCFFT) [4] allows the fast approximate evaluation of trigonometric polynomials with frequencies supported on the index set $\mathcal{I}_N = \tilde{H}_n^d$ at arbitrary sampling nodes $\boldsymbol{y}_\ell$, $\ell=0,\ldots,L-1$. The NHCFFT is based on the hyperbolic cross FFT (cf. [5], [6]) and has an arithmetic complexity of $\mathcal{O}(|\log\epsilon|^d\ L\ \log|\tilde{H}_n^d| + |\log\epsilon|\ |\tilde{H}_n^d| + |\tilde{H}_n^d|\log|\tilde{H}_n^d|)$, where $|\tilde{H}_n^d| \le C\, n^{d-1}\, 2^n$ with a constant $C>0$ depending only on d. In [7], the stability of the hyperbolic cross discrete Fourier transform was studied.

For symmetric hyperbolic cross index sets $\mathcal{I}_N = H_N^d := \{\boldsymbol{j}\in\mathbb{Z}^d\colon r(\boldsymbol{j}) \le N\}$ in frequency domain with refinement $N\in\mathbb{N}$, $r(\boldsymbol{j}) := \prod_{t=1}^d \max(1,|j_t|)$, we present an approach for the fast approximate evaluation at arbitrary sampling nodes $\boldsymbol{y}_\ell$. This method uses one-dimensional FFTs for evaluating the trigonometric polynomial f and its derivatives at nodes of a rank-1 lattice. Then, for each sampling node $\boldsymbol{y}_\ell$, a Taylor expansion of degree $m-1$, $m\in\mathbb{N}$, at a closest rank-1 lattice node is performed. This results in a total arithmetic complexity of $\mathcal{O}(m^d\,(L + M\log M + |H_N^d|))$, where $M\in\mathbb{N}$ is the size of the rank-1 lattice. We show error estimates for the approximation error of the presented method. Note, that we have the inclusion $\tilde{H}_n^d \subset H_{2^{n-1}}^d \subset \tilde{H}_{n-1+2d}^d$, see [8, Lemma 2.1].

In Section II, we give a short overview over Taylor expansion of trigonometric polynomials and define rank-1 lattices. We show that trigonometric polynomials can be evaluated at rank-1 lattice nodes using a one-dimensional FFT. The proposed method is presented in Section III as well as error estimates for symmetric hyperbolic cross index sets H_N^d. Results of numerical tests are presented in Section IV. Finally, we summarize the results in Section V.

II. Prerequisite

A. Taylor expansion

We approximate a function $f:\mathbb{T}^d\to\mathbb{C}$ by

$$f(\boldsymbol{x}) \approx s_m(\boldsymbol{x}) := \sum\nolimits_{0\le|\boldsymbol{s}|<m} \frac{D^{\boldsymbol{s}} f(\boldsymbol{a})}{\boldsymbol{s}!} (\boldsymbol{x}-\boldsymbol{a})^{\boldsymbol{s}},$$

where $m\in\mathbb{N}$, $D^{\boldsymbol{s}} f := \frac{\partial^{s_1}}{\partial {x_1}^{s_1}} \cdots \frac{\partial^{s_d}}{\partial {x_d}^{s_d}}$, $\boldsymbol{x} := (x_1,\ldots,x_d)^\top$, $\boldsymbol{s} := (s_1,\ldots,s_d)\in\mathbb{N}_0^d$, $|\boldsymbol{s}| := |s_1| + \ldots + |s_d|$, $D^{\boldsymbol{0}} f := f$, $\boldsymbol{s}! := s_1!\cdot\ldots\cdot s_d!$, $\boldsymbol{x}^{\boldsymbol{s}} := {x_1}^{s_1}\cdot\ldots\cdot {x_d}^{s_d}$.

For a trigonometric polynomial f from (1), we have $D^{\boldsymbol{s}} f(\boldsymbol{x}) = \sum_{\boldsymbol{l}\in\mathcal{I}_N} (-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}}\, \hat{f}_{\boldsymbol{l}}\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}\boldsymbol{x}}$ and thus,

$$s_m(\boldsymbol{x}) = \sum_{0\le|\boldsymbol{s}|<m} \frac{(\boldsymbol{x}-\boldsymbol{a})^{\boldsymbol{s}}}{\boldsymbol{s}!} \sum_{\boldsymbol{l}\in\mathcal{I}_N} (-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}}\, \hat{f}_{\boldsymbol{l}}\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}\boldsymbol{a}}. \quad (2)$$

B. Rank-1 lattice

Definition II.1 (rank-1 lattice). *Let $M\in\mathbb{N}$, $\boldsymbol{z}\in\mathbb{Z}^d$. We define the rank-1 lattice $\Lambda(\boldsymbol{z},M)\subset\mathbb{T}^d$ of size M with generating vector $\boldsymbol{z}\in\mathbb{Z}^d$ by $\Lambda(\boldsymbol{z},M) := \{\boldsymbol{x}_k := ((k\boldsymbol{z}) \mod M)/M\}_{k=0}^{M-1}$.* □

Definition II.2 (mesh norm). *Let the metric $\mu(\boldsymbol{x},\boldsymbol{y}) := \min_{\boldsymbol{k}\in\mathbb{Z}^d} \|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{k}\|_\infty$ be given for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{T}^d$. We define the mesh norm δ of an arbitrary point set $\mathcal{X} := \{\boldsymbol{x}_k\}_{k=0}^{M-1}\subset\mathbb{T}^d$ by $\delta := 2\max_{\boldsymbol{x}\in\mathbb{T}^d}\min_{\boldsymbol{x}_k\in\mathcal{X}} \mu(\boldsymbol{x}_k,\boldsymbol{x})$.* □

For an arbitrary point set $\mathcal{X}\subset\mathbb{T}^d$ of size $|\mathcal{X}| = M$, we have $\delta \ge 1/\sqrt[d]{M}$, see e.g. [9, Lemma 3.1]. The following Lemma shows the existence of a rank-1 lattice $\Lambda(\boldsymbol{z},M)$ of size M, such that the mesh norm $\delta \le C_d/\sqrt[d]{M}$, where $C_d > 1$ is a constant depending only on d, i.e., we have $\delta \sim 1/\sqrt[d]{M}$.

Lemma II.3. *Let $b\in\mathbb{N}$, $b\ge 3$. Then, there exists a rank-1 lattice $\Lambda(\boldsymbol{z},M)$ of size $M = b(b+1)$ for $d=2$ and $b^d\cdot 2^{\frac{d(d-1)}{2}-1} < M \le b^d\cdot 2^{d(d-2)}$ for $d\ge 3$ with generating vector $\boldsymbol{z}\in\mathbb{Z}^d$, such that the mesh norm $\delta\le C_d/\sqrt[d]{M}$, where $C_d>1$ is a constant depending only on d.*

Proof: In the case $d=2$, we choose the rank-1 lattice size $M := b\cdot(b+1)$ and the generating vector $\boldsymbol{z} := (b,b+1)^\top$. Since b and $b+1$ are relatively prime to each other, there exists a bijective mapping between the rank-1 lattice nodes $\boldsymbol{x}_k := (k\boldsymbol{z} \mod M)/M$, $k=0,\ldots,M-1$, and the grid $(j_1/(b+1), j_2/b)^\top$, $j_1 = 0,\ldots,b$ and $j_2 = 0,\ldots,b-1$, cf. [10]. Obviously, the mesh norm $\delta = 1/b \le \frac{2}{\sqrt{3}}/\sqrt{M}$.

In the case $d=3$, we set $v_1 := 2b+1$ and $v_2 := 2b$. Due to Bertrand's postulate there exists a prime number $p_3\in\mathbb{N}$, $b\le p_3 < 2b$. We choose $v_3\in\{p_3,\ldots,v_2-1\}$, such that v_3 is relatively prime to v_1 and v_2. We set the rank-1 lattice size $M := v_1\cdot v_2\cdot v_3$ and the generating vector $\boldsymbol{z} := (M/v_1, M/v_2, M/v_3)^\top$. Then, the mesh norm $\delta\le 1/v_3\le 1/b\le 2/\sqrt[3]{M}$ and the rank-1 lattice size $M = (2b+1)\cdot 2b\cdot v_3 \ge (2b+1)\cdot 2b\cdot b > b^3\cdot 2^2$.

In the case $d\ge 4$, we set $v_1 := b\cdot 2^{d-2}+1$ and $v_2 := b\cdot 2^{d-2}$. We apply Bertrand's postulate $d-2$ times and choose $v_3,\ldots,v_d$, such that $v_1,\ldots,v_d$ are relatively prime to each other and $v_3 > \ldots > v_d \ge b$. We choose the rank-1 lattice size $M := \prod_{t=1}^d v_t$ and the generating vector $\boldsymbol{z} := (M/v_1,\ldots,M/v_d)^\top$. This yields that the mesh norm $\delta\le 1/v_d\le 1/b\le 2^{d-2}/\sqrt[d]{M}$ and the rank-1 lattice size $M \ge (2^{d-2}b+1)\cdot 2^{d-2}b\cdot\prod_{t=3}^d (2^{d-t}b) > b^d\cdot 2^{\frac{d(d-1)}{2}-1}$. ■

The following Lemma shows that rank-1 lattices exist where the constant C_d is arbitrarily close to 1 for constant d and increasing rank-1 lattice size M.

Lemma II.4. *For each constant $C_d>1$, there exists a parameter $M^*\in\mathbb{N}$, such that for all $M'\ge M^*$ we can construct a rank-1 lattice $\Lambda(\boldsymbol{z},M)$ of size $M\in\left(M',(C_d)^d M'\right]$ with mesh norm $\delta < C_d/\sqrt[d]{M}$.*

Proof: Let $R_{c,d}$ be the dth c-Ramanujan prime [11], i.e., the smallest integer such that there are at least d primes in the interval $(cx,x]$ for all $x\ge R_{c,d}$, where $c\in(0,1)$. For arbitrary constant $C_d>1$, we set $c := (C_d)^{-1}$, $M^* := \left((C_d)^{-1}R_{(C_d)^{-1},d}\right)^d$ and $x := C_d\sqrt[d]{M'}$, $M'\ge 1$. Then, there are at least d primes $v_1,\ldots,v_d$ in the interval $(\sqrt[d]{M'}, C_d\sqrt[d]{M'}]$ for all $M'\ge M^*$. We choose the rank-1 lattice size $M := \prod_{t=1}^d v_t$ and the generating vector $\boldsymbol{z} := (M/v_1,\ldots,M/v_d)^\top$. Consequently, we have $M' < M \le (C_d)^d M'$ and $\delta < 1/\sqrt[d]{M'} \le C_d/\sqrt[d]{M}$. ■

C. Evaluation at rank-1 lattice nodes (rank-1 lattice FFT)

We consider the evaluation of a trigonometric polynomial $g:\mathbb{T}^d\to\mathbb{C}$ supported on the frequency index set $\mathcal{I}_N\subset\mathbb{Z}^d\cap[-N,N]^d$, $g(\boldsymbol{x}) := \sum_{\boldsymbol{l}\in\mathcal{I}_N} \hat{g}_{\boldsymbol{l}}\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}\boldsymbol{x}}$, $\hat{g}_{\boldsymbol{l}}\in\mathbb{C}$, at rank-1 lattice nodes $\boldsymbol{x}_k\in\Lambda(\boldsymbol{z},M)$. As presented in [8], we have

$$g(\boldsymbol{x}_k) = g(k\boldsymbol{z}/M) = \sum_{j=0}^{M-1}\left(\sum_{\substack{\boldsymbol{l}\in\mathcal{I}_N\\ \boldsymbol{l}\boldsymbol{z}\equiv j(\mathrm{mod} M)}} \hat{g}_{\boldsymbol{l}}\right)\mathrm{e}^{-2\pi\mathrm{i}\frac{kj}{M}}$$

and the outer sum is a one-dimensional discrete Fourier transform of length M. Using a one-dimensional FFT, the trigonometric polynomial g can be evaluated at all rank-1 lattice nodes in $\mathcal{O}(M\log M + |\mathcal{I}_N|)$ arithmetic operations.

Setting the Fourier coefficients $\hat{g}_{\boldsymbol{l}} := (-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}}\hat{f}_{\boldsymbol{l}}$, where $\hat{f}_{\boldsymbol{l}}$ are the Fourier coefficients of a trigonometric polynomial f from (1), yields $g(\boldsymbol{x}_k) = D^{\boldsymbol{s}}f(\boldsymbol{x}_k)$. Thus, for fixed $\boldsymbol{s}\in\mathbb{N}_0^d$, the mixed derivatives $D^{\boldsymbol{s}}f(\boldsymbol{x})$ of the trigonometric polynomial f can be evaluated at all rank-1 lattice nodes $\boldsymbol{x}_k$, $k=0,\ldots,M-1$, in $\mathcal{O}(M\log M + |\mathcal{I}_N|)$ arithmetic operations.

III. NFFT based on Taylor expansion and rank-1 lattice FFT

A. Method

Let a frequency index set $\mathcal{I}_N\subset\mathbb{Z}^d\cap[-N,N]^d$ and a rank-1 lattice $\Lambda(\boldsymbol{z},M)$ of size M be given. We replace the expansion point $\boldsymbol{a}$ in (2) by a closest rank-1 lattice node $\boldsymbol{x}_{k'} = \arg\min_{\boldsymbol{x}_k\in\Lambda(\boldsymbol{z},M)} \mu(\boldsymbol{x},\boldsymbol{x}_k)$, and obtain the Taylor expansion

$$s_m(\boldsymbol{x}) = \sum_{0\le|\boldsymbol{s}|<m} \frac{(\boldsymbol{x}-\boldsymbol{x}_{k'})^{\boldsymbol{s}}}{\boldsymbol{s}!} \sum_{\boldsymbol{l}\in\mathcal{I}_N} (-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}}\, \hat{f}_{\boldsymbol{l}}\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}\boldsymbol{x}_{k'}}. \quad (3)$$

Assuming that a closest rank-1 lattice node $\boldsymbol{x}_{k'}$ is known for each sampling node $\boldsymbol{y}_\ell$, the Taylor expansion s_m in (3) can be calculated in $\mathcal{O}\left(m^d(L + M\log M + |\mathcal{I}_N|)\right)$ arithmetic operations for all sampling nodes $\boldsymbol{y}_\ell$, $\ell = 0,\ldots,L-1$.

For symmetric hyperbolic cross index sets $\mathcal{I}_N = H_N^d$, $N\in\mathbb{N}$, $N\ge 2$, we have $|H_N^d| \le C_{\mathrm{H}} N\log^{d-1}N$ for $N\ge 2$ with a constant $C_{\mathrm{H}}>0$, see e.g. [12]. Choosing the rank-1 lattice size $M\sim|H_N^d|$, we obtain an arithmetic complexity of $\mathcal{O}\left(m^d(L + N\log^d N)\right)$.

B. Error estimates for symmetric hyperbolic cross index sets

Theorem III.1. *Let a trigonometric polynomial $f\colon \mathbb{T}^d \to \mathbb{C}$ supported on the symmetric hyperbolic cross index set $\mathcal{I}_N = H_N^d$, $f(\boldsymbol{x}) = \sum_{\boldsymbol{l}\in H_N^d} \hat{f}_{\boldsymbol{l}}\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}\boldsymbol{x}}$, $\hat{f}_{\boldsymbol{l}} \in \mathbb{C}$, $N \in \mathbb{N}$, be given. Furthermore, let $\Lambda(\boldsymbol{z}, M)$ be a rank-1 lattice with mesh norm δ. Then, for the approximation of the trigonometric polynomial f by a truncated Taylor series $s_m(\boldsymbol{x}) := \sum_{|\boldsymbol{s}|=0}^{m-1} \frac{D^{\boldsymbol{s}} f(\boldsymbol{x}_{k'})}{\boldsymbol{s}!} (\boldsymbol{x} - \boldsymbol{x}_{k'})^{\boldsymbol{s}}$ of degree $m-1$ from (3), where $m \in \mathbb{N}$ and $\boldsymbol{x}_{k'} = \arg\min_{\boldsymbol{x}_k \in \Lambda(\boldsymbol{z},M)} \mu(\boldsymbol{x}, \boldsymbol{x}_k)$, the remainder $R_m(\boldsymbol{x}) := f(\boldsymbol{x}) - s_m(\boldsymbol{x})$ is bounded by*

$$|R_m(\boldsymbol{x})| \le \frac{d^m \pi^m}{m!}\, \delta^m N^{m-\alpha} \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha,$$

where $\alpha \in [0, m]$ is the smoothness parameter.

Proof: Let $\boldsymbol{\xi}(t) := \boldsymbol{x}_{k'} + t(\boldsymbol{x} - \boldsymbol{x}_{k'})$, $t \in [0,1]$. The remainder $R_m(\boldsymbol{x})$ can be written (cf. [13, Ch. 1]) in the form

$$R_m(\boldsymbol{x}) = m \int_0^1 (1-t)^{m-1} \sum_{|\boldsymbol{s}|=m} D^{\boldsymbol{s}} f(\boldsymbol{\xi}(t)) \frac{(\boldsymbol{x} - \boldsymbol{x}_{k'})^{\boldsymbol{s}}}{\boldsymbol{s}!} \mathrm{d}t.$$

Then,

$$\begin{aligned}
&|R_m(\boldsymbol{x})| \\
\le\;& m \int_0^1 (1-t)^{m-1} \sum_{|\boldsymbol{s}|=m} |D^{\boldsymbol{s}} f(\boldsymbol{\xi}(t))| \frac{|(\boldsymbol{x} - \boldsymbol{x}_{k'})^{\boldsymbol{s}}|}{\boldsymbol{s}!} \mathrm{d}t \\
\le\;& \max_{t\in[0,1]} \sum_{|\boldsymbol{s}|=m} \left| \sum_{\boldsymbol{l}\in H_N^d} (-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}} \hat{f}_{\boldsymbol{l}}\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{l}(\boldsymbol{\xi}(t))} \right| \frac{|(\boldsymbol{x} - \boldsymbol{x}_{k'})^{\boldsymbol{s}}|}{\boldsymbol{s}!} \\
\le\;& \sum_{|\boldsymbol{s}|=m} \frac{|(\boldsymbol{x} - \boldsymbol{x}_{k'})^{\boldsymbol{s}}|}{\boldsymbol{s}!} \sum_{\boldsymbol{l}\in H_N^d} |(-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}}| |\hat{f}_{\boldsymbol{l}}|.
\end{aligned}$$

Since $\mu(\boldsymbol{x}, \boldsymbol{x}_{k'}) \le \delta/2$ and by applying the multinomial theorem, we get

$$\begin{aligned}
|R_m(\boldsymbol{x})| &\le \sum_{|\boldsymbol{s}|=m} \frac{\left(\frac{\delta}{2}\right)^{|\boldsymbol{s}|}}{\boldsymbol{s}!} \sum_{\boldsymbol{l}\in H_N^d} |(-2\pi\mathrm{i}\boldsymbol{l})^{\boldsymbol{s}}|\, |\hat{f}_{\boldsymbol{l}}| \\
&\le \pi^m \delta^m \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}| \sum_{|\boldsymbol{s}|=m} \frac{|l_1|^{s_1} \cdot \ldots \cdot |l_d|^{s_d}}{\boldsymbol{s}!} \\
&\le \pi^m \delta^m \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}| \frac{\|\boldsymbol{l}\|_1^m}{m!}.
\end{aligned}$$

Introducing weights $r(\boldsymbol{l})^\alpha$, $0 \le \alpha \le m$, we obtain

$$\begin{aligned}
|R_m(\boldsymbol{x})| &\le \pi^m \delta^m \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha\, \frac{\|\boldsymbol{l}\|_1^m}{r(\boldsymbol{l})^\alpha\, m!} \\
&\le \frac{\pi^m \delta^m}{m!} \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha\, \frac{d^m\, r(\boldsymbol{l})^m}{r(\boldsymbol{l})^\alpha} \\
&\le \frac{d^m \pi^m \delta^m}{m!} \left(\sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha \right) \max_{\boldsymbol{l}\in H_N^d} r(\boldsymbol{l})^{m-\alpha} \\
&= \frac{d^m \pi^m}{m!}\, \delta^m N^{m-\alpha} \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha.
\end{aligned}$$

■

Corollary III.2. *Let a hyperbolic cross index set $\mathcal{I}_N = H_N^d$, $N \in \mathbb{N}$, $N \ge 2$, and a rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ of size $M := C_{\mathrm{L}} N \log^{d-1} N \sim |H_N^d|$ for some constant $C_{\mathrm{L}} \ge 1$ be given, where the generating vector $\boldsymbol{z}$ is chosen as in the proof of Lemma II.3. Then,*

$$\begin{aligned}
|R_m(\boldsymbol{x})| &\le \frac{d^m \pi^m}{m!} (C_d)^m M^{-m/d}\, N^{m-\alpha} \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha \\
&= \frac{d^m \pi^m}{m!} (C_d)^m \frac{N^{m-\alpha}}{(C_{\mathrm{L}} N \log^{d-1} N)^{\frac{m}{d}}} \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha
\end{aligned}$$

is valid for all smoothness parameters $\alpha \in [0, m]$, where $C_d > 1$ is the constant from Lemma II.3.

Proof: From Lemma II.3, we obtain that the mesh norm $\delta \le C_d\, M^{-1/d}$. Applying Theorem III.1 yields the result. ■

Remark III.3. *If we choose the smoothness parameter $\alpha \in [\frac{d-1}{d}\, m, m]$, Corollary III.2 guarantees a decreasing relative error $|R_m(\boldsymbol{x})| / \left(\sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha \right)$ for increasing refinement N. Setting the smoothness parameter $\alpha := m$ yields $|R_m(\boldsymbol{x})| \le \frac{d^m \pi^m}{m!} (C_d)^m\, (C_{\mathrm{L}} N \log^{d-1} N)^{-\frac{m}{d}} \sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^m$.*

Remark III.4. *The presented method can also be used for the approximate evaluation of trigonometric polynomials f supported on other frequency index sets. For instance, consider the case of l_1 balls, $\mathcal{I}_N = \{\boldsymbol{j} \in \mathbb{Z}^d \colon \|\boldsymbol{j}\|_1 \le N\}$. In the proof of Theorem III.1, we introduce weights $\|\boldsymbol{l}\|_1^\alpha$ instead of $r(\boldsymbol{l})^\alpha$. Then, we obtain $|R_m(\boldsymbol{x})| \le \frac{\pi^m}{m!}\, \delta^m\, N^{m-\alpha} \sum_{\boldsymbol{l}\in\mathcal{I}_N} |\hat{f}_{\boldsymbol{l}}|\, \|\boldsymbol{l}\|_1^\alpha$.*

IV. Numerical results

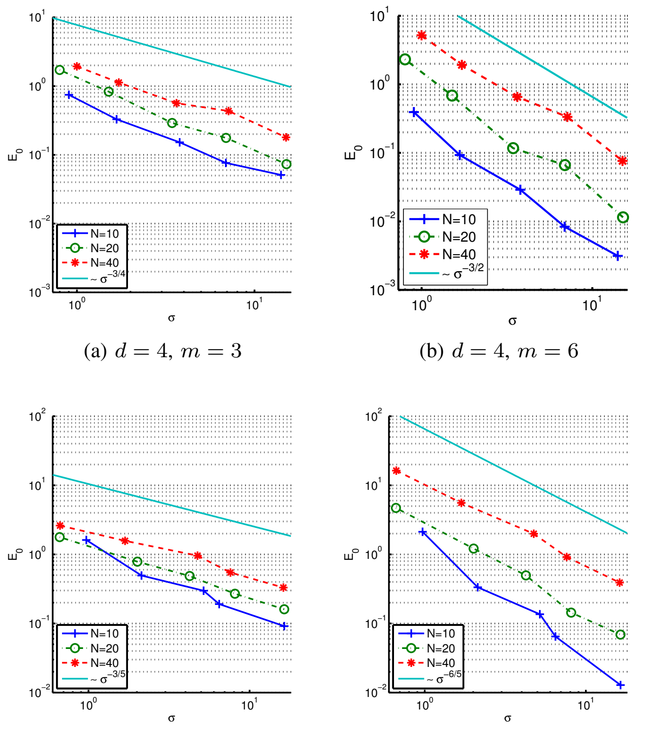
The Taylor expansion s_m in (3) was implemented in MATLAB for trigonometric polynomials f from (1) as described in Section III-A.

For symmetric hyperbolic cross index sets $\mathcal{I}_N = H_N^d$, numerical tests were performed. The generating vector $\boldsymbol{z}$ of each rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ was chosen as in the proof of Lemma II.3. The maximum relative approximation error $E_\alpha := \max_{\boldsymbol{y}_\ell \in \mathcal{Y}} |R_m(\boldsymbol{y}_\ell)| / \left(\sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha \right)$ was determined using $L = 100\,000$ uniformly random sampling nodes $\boldsymbol{y}_\ell \in \mathbb{T}^d$, $\mathcal{Y} := \{\boldsymbol{y}_\ell\}_{l=0}^{L-1}$.

A. Decreasing error E_α for increasing rank-1 lattice size M

In this test case, we uniformly randomly chose the Fourier coefficients $\hat{f}_{\boldsymbol{l}} \in (0,1]/r(\boldsymbol{l})^\alpha$, $\boldsymbol{l} \in \mathcal{I}_N = H_N^d$. All tests were repeated five times using different Fourier coefficients $\hat{f}_{\boldsymbol{l}}$ and sampling nodes $\boldsymbol{y}_\ell$. Then, the average error of these five test runs was used.

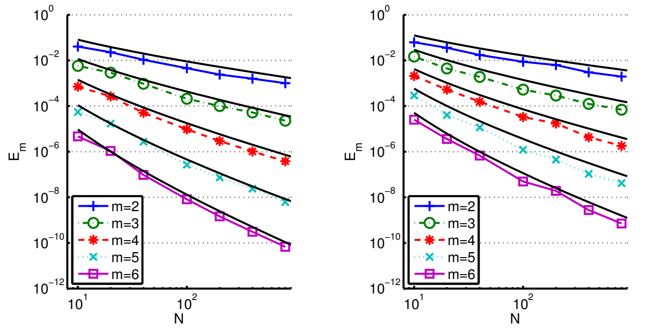
We set the rank-1 lattice size $M := \sigma \cdot 2\,|H_N^d|$ with a factor $\sigma \ge \frac{1}{2}$. Due to Corollary III.2, the error E_α should decrease at least like $\sim \sigma^{-m/d}$ for increasing factor σ. In tests performed for the cases $d = 2, \ldots, 5$ and $m = 2, \ldots, 6$, this behaviour could be observed. Figure 1 shows the error E_0 for increasing values of factor σ for refinements $N = 10, 20, 40$ and $m = 3, 6$ in the four- and five-dimensional case as well as the lines $\sim \sigma^{-m/d}$.



(a) $d=4$, $m=3$ (b) $d=4$, $m=6$

(c) $d=5$, $m=3$ (d) $d=5$, $m=6$

Fig. 1. Approximation error E_0 for increasing values of factor σ with rank-1 lattice size $M=\sigma\, 2|H_N^d|$ for Taylor expansions s_m of degree $m-1$, $m=3,6$, in the cases $d=4,5$.



(a) $d=4$ (b) $d=5$

Fig. 2. Approximation error E_m for increasing hyperbolic cross refinements N with rank-1 lattice size $M \approx 2|H_N^d|$ for Taylor expansions s_m of degree $m-1$, $m=2,\ldots,6$, and theoretical bounds $\sim (N\log^{d-1} N)^{-m/d}$ (solid lines without symbols) in the cases $d=4,5$.

B. Decreasing error E_m for increasing refinement N of the symmetric hyperbolic cross index set $\mathcal{I}_N = H_N^d$

In order to obtain a large error E_m, the Fourier coefficients $\hat{f}_{\boldsymbol{l}}$, $\boldsymbol{l} \in H_N^d$, were set to zero except $\hat{f}_{(\pm 1,0,\ldots,0)^\top} = 1$, $\hat{f}_{(0,\pm 1,0,\ldots,0)^\top} = 1$, $\ldots$, $\hat{f}_{(0,\ldots,0,\pm 1)^\top} = 1$ and $\hat{f}_{(\pm N,0,\ldots,0)^\top} = 1/N^m$, $\hat{f}_{(0,\pm N,0,\ldots,0)^\top} = 1/N^m$, $\ldots$, $\hat{f}_{(0,\ldots,0,\pm N)^\top} = 1/N^m$. We set the rank-1 lattice size $M \approx 2|H_N^d|$. Test cases included Taylor expansion degrees $m-1$, $m=2,\ldots,6$, and refinements up to $N=10^4$ for $d=2$, up to $N=10^3$ for $d=3$ and up to $N=800$ for $d=4,5$. Remark III.3 states, that the error E_m should decrease at least like $\sim (N\log^{d-1} N)^{-\frac{m}{d}}$. In the results of the performed tests, a decrease of $\sim (N\log^{d-1} N)^{-\frac{m}{d}}$ could be observed. Figure 2 shows the results for the cases $d=4,5$.

V. Conclusion

Based on rank-1 lattice methods and Taylor expansion, we presented a method for the fast approximate evaluation of trigonometric polynomials f with frequencies supported on symmetric hyperbolic cross index sets $\mathcal{I}_N = H_N^d$ with refinement N at arbitrary sampling nodes $\boldsymbol{y}_\ell \in \mathbb{T}^d$, $\ell = 0,\ldots,L-1$. We showed conditions which guarantee a decreasing approximation error $|R_m(\boldsymbol{x})|/\left(\sum_{\boldsymbol{l}\in H_N^d} |\hat{f}_{\boldsymbol{l}}|\, r(\boldsymbol{l})^\alpha\right)$ for increasing refinement N. In particular for smoothness parameter $\alpha = m$, a rank-1 lattice $\Lambda(\boldsymbol{z},M)$ of size $M \sim |H_N^d|$ exists, such that the approximation error decreases at least like $\sim (N\log^{d-1} N)^{-m/d}$ for increasing refinement N. For such a rank-1 lattice of size $M \sim |H_N^d|$, the total arithmetic complexity of the presented method is $\mathcal{O}(m^d L + m^d\, N\log^d N)$. The results of the numerical tests confirmed the theoretical upper bounds.

Acknowledgment

The author thanks Daniel Potts, Stefan Kunis and Lutz Kämmerer for numerous valuable discussions on the presented subject as well as the referees for their valuable suggestions. Moreover, he gratefully acknowledges support by German Research Foundation within the project PO 711/10-2.

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