Phase retrieval using time and Fourier magnitude measurements

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***Abstract*—We discuss the reconstruction of a finite-dimensional signal from the absolute values of its Fourier coefficients. In many optical experiments the signal magnitude in time is also available. We combine time and frequency magnitude measurements to obtain closed reconstruction formulas. Random measurements are discussed to reduce the number of measurements.**

I. Introduction

Phase retrieval, within a discrete model, deals with the problem of reconstructing a signal $z \in \mathbb{C}^d$ from a collection of magnitudes $\{|\langle z, x_j\rangle|^2\}_{j=1}^n$, where $\{x_j\}_{j=1}^n \subset \mathbb{C}^d$ are measurement vectors. The signal z, of course, can be determined up to a global phase factor at best.

Standard algorithms are based on Gerchberg/Saxton [14] and Fienup [13] and usually involve some iterative alternate projection scheme. As phase retrieval is such a long-standing problem, it appears impossible to give a complete list of references, so let us simply refer to [7], [16], [17], [22] and references therein.

Algebraic conditions on measurement vectors have led to closed reconstruction formulas of zz^* [6], so that a singular value decomposition enables the extraction of z up to its global phase. However, such conditions can only be satisfied when the number of measurements n scales at least like d^2. Currently, reducing this number to scale linearly in d is an active field of research, see [1] for the use of graph theory. Random measurement vectors and signal recovery with high probability has been considered in [8], [9], [10]. There, the reconstruction formula is replaced with an optimization procedure based on semidefinite programming, and the number of random measurement vectors n then scales linearly in d.

Both approaches though suffer from limitations. The deterministic reconstruction formula in [6] does not apply to Fourier measurements, which arise in many optical measurement processes and appear to be the largest application field of phase retrieval. The underlying probability measure of the random measurement vectors in [8], [9], [10] has full support on the unit sphere. Thus, although only linearly many measurements have to be performed in physical experiments, any point on the sphere is a potential measurement vector and is not allowed to be excluded a-priori. It is desirable to decrease the set of potential measurement vectors to better reflect the physical constraints in actual experiments.

In this short note we shall discuss approaches to overcome the aforementioned problems and limitations. Many physical experiments additionally provide the signal power in time, i.e., $\{|z_k|\}_{k=1}^d$. By using a generalization of the algebraic condition in [6], developed in [3], we observe that certain Fourier measurements combined with the signal power in time lead to a closed reconstruction formula for zz^*. Building upon such results, we also propose specific Fourier type probability measures that may allow for signal reconstruction within the random setting. For the latter, we do not provide rigorous proofs here but collect some indications.

II. Unstructured measurements

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. The aim of the present note is to discuss some ideas about the reconstruction of an unknown vector $x \in \mathbb{K}^d$ from a collection of magnitude measurement $\{|\langle z, x_j\rangle|^2\}_{j=1}^n$, where $\{x_j\}_{j=1}^n \subset \mathbb{K}^d$ are some measuring vectors chosen a-priori.

A. Lower bounds on the number of measurements for $\mathbb{K} = \mathbb{R}$

Although we shall concentrate on $\mathbb{K} = \mathbb{C}$ later, let us consider $\mathbb{K} = \mathbb{R}$ for a moment. If we assume that the first entry of x is nonzero, then $\{e_k\}_{k=1}^d \cup \{e_1 + e_k\}_{k=2}^d \subset \mathbb{R}^d$ is a collection of $n = 2d - 1$ measurement vectors that allow to recover x. Indeed, the first d measurements yield the absolute values of the entries of x and the following measurements enable us to check if the signs change from one coordinate to the other. The reconstruction algorithm is simple but note that we assumed the first entry of x to be nonzero. Similarly, a stable algorithm requiring $\mathcal{O}(d \log d)$ measurements, starting from $\{e_k\}_{k=1}^d$, and then determining relative phases between entries of z has been proposed in [24], [1]. If we can perform adaptive measurements, then the application of $\{e_k\}_{k=1}^d$ would tell us the location of the nonzero entries of z, say $k_1, \ldots, k_\ell$. So the additional measurement vectors $\{e_{k_1} + e_{k_j}\}_{j=2}^\ell$ would enable us to recover $\pm z$ from a total of $d+\ell-1$ measurements. Without any adaptivity and knowledge on z, we shall see next that $2d - 1$ measurements are sufficient to reconstruct z up to its sign, but there may not be any efficient reconstruction algorithms. Let us deal with the collection of matrices $\mathcal{M} := \{zz^* : z \in \mathbb{R}^d\}$ and define the map $\mathcal{F}_n : \mathcal{M} \to \mathbb{R}^n$ as

$$\mathcal{F}_n(zz^*) := \big(\operatorname{trace}(zz^* x_j x_j^*)\big)_{j=1}^n = \big(|\langle z, x_j\rangle|^2\big)_{j=1}^n. \quad (1)$$

There are $n = 2d-1$ measuring vectors $\{x_j\}_{j=1}^n$ necessary (and generically sufficient) to ensure injectivity of $\mathcal{F}_n$, cf. [6]. If we are willing to remove a set of measure zero, then the lower bound can be relaxed: There is a set $\Omega \subset \mathbb{R}^d$ of measure zero, such that any $z \in \mathbb{R}^d \setminus \Omega$ is uniquely determined up to its sign by measuring with the $d+1$ vectors $\{e_k\}_{k=1}^d \cup \{e_1 + \ldots + e_d\}$.

B. Deterministic measurements

From here on we suppose $\mathbb{K} = \mathbb{C}$ and denote $S^{d-1} = \{x \in \mathbb{K}^d : \|x\| = 1\}$. A collection $\{x_j\}_{j=1}^n \subset S^{d-1}$ with weights $\{\omega_j\}_{j=1}^n \subset \mathbb{R}_+$ is called a projective cubature of strength 2 if $\sum_{j=1}^n \omega_j = 1$ and

$$\sum_{j=1}^n \omega_j |\langle z, x_j\rangle|^4 = \frac{2}{d(d+1)}\|z\|^4, \quad \text{for all } z \in \mathbb{C}^d. \quad (2)$$

Given such a projective cubature of strength 2, the results in [3] yield that

$$zz^* = d\sum_{j=1}^n \omega_j |\langle z, x_j\rangle|^2 \big((d+1)x_j x_j^* - I\big). \quad (3)$$

Equation (3) was derived in [5] for constant weights. Therefore, any matrix zz^*, for $z \in \mathbb{C}^d$, can be reconstructed from its measurements $\{|\langle z, x_j\rangle|^2\}_{j=1}^n$.

C. Random measurements

It is well-known that any projective cubature of strength 2 must have cardinality $n \geq d^2$, cf. [2], [21]. To reduce the number of measurements, semidefinite programming and random measuring vectors were used in [9], [10], [11] to reconstruct zz^* with high probability. Indeed, let $\mathscr{H}$ denote the collection of hermitian matrices in $\mathbb{C}^{d\times d}$. For $\{x_j\}_{j=1}^n \subset \mathbb{C}^d$, we extend the operator in (1) and define

$$\mathcal{F}_n : \mathscr{H} \to \mathbb{R}^n, \quad H \mapsto \big(\operatorname{trace}(H x_j x_j^*)\big)_{j=1}^n. \quad (4)$$

Given $b := \mathcal{F}_n(zz^*)$ and excluding the pathological case $b = 0$, we see that zz^* is a solution to

$$\min_{H\in\mathscr{H}} (\operatorname{rank}(H)), \quad \text{subject to} \quad \mathcal{F}_n(H) = b,\ H \succeq 0, \quad (5)$$

where $H \succeq 0$ stands for H being positive semidefinite. The general affine rank minimization problem is NP-hard, see for instance [19], [20], and commonly replaced by

$$\min_{H\in\mathscr{H}} (\operatorname{trace}(H)), \quad \text{subject to} \quad \mathcal{F}_n(H) = b,\ H \succeq 0, \quad (6)$$

a semidefinite program, for which efficient solvers such as interior point methods are available.Let us assume that $\{x_j\}_{j=1}^n \subset S^{d-1}$ are an independent sample from the uniform distribution on S^{d-1}. According to [9], [10], there are two constants $c, \gamma > 0$, such that, for all $n \geq cd$, the minimizer of (6) is unique and given by zz^* with probability at least $1 - e^{-\gamma n}$. The same statement holds if the entries of $\{x_j\}_{j=1}^n \subset \mathbb{C}^d$ are chosen independently from the standard Gaussian distribution.

The proof in [9], [10], see also [3], for the uniform distribution on the sphere is based on the probabilistic reconstruction formula

$$zz^* = d\,\mathbb{E}|\langle z, X\rangle|^2\big((d+1)XX^* - I\big), \quad \text{(PRF-1)}$$

for all $z \in \mathbb{C}^d$, where $X \in \mathbb{C}^d$ denotes a random vector uniformly distributed on S^{d-1}. In view of (2), we observe that (PRF-1) is equivalent to

$$\mathbb{E}|\langle z, X\rangle|^4 = \frac{2}{d(d+1)}\|z\|^4, \quad \text{for all } z \in \mathbb{C}^d, \quad (7)$$

which implies

$$d\,\mathbb{E}|\langle z, X\rangle|^2 = \|z\|^2, \quad \text{for all } z \in \mathbb{C}^d, \quad (8)$$

cf. [4]. The condition (8) is equivalent to

$$d\,\mathbb{E}XX^* = I, \quad (9)$$

so that (PRF-1) implies (9). The proof of the equivalence between (5) and (6) is based on (PRF-1) and (9), and, besides some technical ingredients, then turns the expectation in both conditions into suitable statements on the sample mean by using tail bound estimates, cf. [3], [10].

III. Time-frequency structured measurements

A. Fourier measurements

The measuring vectors in the previous section were either unstructured or chosen from the uniform distribution on the sphere. In optical experiments, Fourier type measurements are performed. Naturally, we consider the random Fourier vector

$$X = \tfrac{1}{\sqrt{d}}(e^{2\pi i\lambda_1 t}, \ldots, e^{2\pi i\lambda_d t})^\top, \quad (10)$$

where $\{\lambda_i\}_{i=1}^d$ are real numbers and t is a random variable uniformly distributed on $[0,1)$. Of course, the measurements

$$\langle z, x_j\rangle = \tfrac{1}{\sqrt{d}}\sum_{k=1}^d z_k e^{-2\pi i\lambda_k t_j}$$

consist of a randomly sampled trigonometric polynomial, which brings all sorts of nonequispaced fast Fourier transforms into play. Unfortunately and to no surprise, the vector X is not uniformly distributed on S^{d-1}. Nevertheless, we could check if (7) holds, and if so, then there might be a good chance that the trace minimization works out numerically although not stringently proven mathematically yet. Unfortunately, Fourier magnitude measurements alone are not sufficient to resolve time translates. For instance, the canonical basis vectors $e_1, \ldots, e_d$ cannot be distinguished by the absolute values of their Fourier coefficients. Thus, (PRF-1) is violated:

Proposition III.1. *Let $\{\lambda_k\}_{k=1}^d$ be a sequence of real numbers. If $t : \Omega \to [0,1)$ is a random variable, then the Fourier random vector* (10) *does not satisfy* (PRF-1).

The same, of course, holds for the deterministic setting:

Proposition III.2. *If* $\{\lambda_k\}_{k=1}^d$ *and* $\{t_j\}_{j=1}^n$ *are sequences of real numbers, then there are no weights* $\{\omega_j\}_{j=1}^n$ *such that the Fourier vectors*

$$\{x_j\}_{j=1}^n = \{\tfrac{1}{\sqrt{d}}(e^{2\pi i\lambda_1 t_j},\dots,e^{2\pi i\lambda_d t_j})^\top\}_{j=1}^n$$

satisfy (3).

B. Additional time measurements

To resolve time translates we must perform additional measurements beyond the Fourier spectrum. In optical experiments the magnitudes in time are often available as well. The latter results in additional measurement vectors $\{e_k\}_{k=1}^d$. We shall discuss three scenarios:

1) Deterministic time-frequency measurements: First, we combine special Fourier vectors with time measurements, inspired by ideas in [15, Proposition 4] and [23, Section 2.1.2]. Let q be a prime and let $d = q^r + 1$ for some $r \in \mathbb{N}$. For $m = d^2 - d + 1$, there exist integers $0 \le \lambda_1 < \dots < \lambda_d < m$ such that all numbers $1,\dots,m-1$ occur as residues $\mathrm{mod}\ m$ of the $d(d-1)$ differences $(\lambda_k - \lambda_\ell)$, for $k \neq \ell$, cf. [15]. For $j = 1,\dots,m$ we define the Fourier vectors

$$x_j = \frac{1}{\sqrt{d}}(e^{2\pi i\lambda_1 j/m},\dots,e^{2\pi i\lambda_d j/m})^\top \in \mathbb{C}^d. \qquad (11)$$

To add time measurements, we form the set $\mathcal{X} = \{x_j\}_{j=1}^m \cup \{e_k\}_{k=1}^d$ with weights $\mathcal{W} = \{\frac{d}{d^3+1}\}_{j=1}^m \cup \{\frac{1}{d(d+1)}\}_{i=1}^d$, respectively. Note that $\mathcal{X}$ is a projective cubature of strength 2, cf. [15], and, therefore, satisfies (3). Its cardinality is $n = d^2 + 1$, the weights split into two groups of constants, and they become almost equal for large ambient dimensions d. The set $\mathcal{X}$ models special Fourier and time measurements, hence, forms a highly structured collection of measurement vectors. In contrast to a naive evaluation of the reconstruction formula (3) in $\mathcal{O}(d^4)$, this allows for a computation in only $\mathcal{O}(d^3 \log d)$ arithmetic operations.

Example III.3. For $d = 4 = 3^1 + 1$, we can select $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (0,3,5,12)$ for the above scheme which yields a projective cubature of strength 2 whose cardinality is $n = 17$.

2) Random time-frequency measurements: Let μ_1 denote the discrete probability measure with support $\mathcal{X}$ and mass distribution according to the weights $\mathcal{W}$. Any random vector $X_1 \sim \mu_1$ satisfies (PRF-1). Therefore, (9) is also satisfied. As a first step for the proof about equivalence of the optimization problems (5) and (6), we can turn (9) into a suitable statement on the sample mean. Indeed, let $\{x_j\}_{j=1}^n$ be independent and identical distributed according to μ_1 and $0 < s < 1$ arbitrary. The Chernoff's matrix inequalities yield that there exist constants $c, C > 0$ such that, for all $n \ge cd\log(d)$,

$$\|\frac{d}{n}\sum_{j=1}^n x_j x_j^* - I\| \le s \qquad (12)$$

holds with probability at least $1 - e^{-Cn/d}$, cf. [12]. The estimate (12) turns the identity about the population mean (9) into an estimate on the deviation of the sample mean measured by the operator norm. It is just a first step, and to derive a complete mathematical proof of the equivalence between (5) and (6), we additionally need a suitable sample mean version of (PRF-1) and few more technical ingredients that would go beyond the present note. Here, we understand the above observations as an indication that the proof can be completed.

3) Deterministic time and random frequency measurements: Switching from the deterministic to the random setting avoids the requirement of d^2 many measurements. We are still consistent with this objective when choosing d measurements in a deterministic fashion and on the order of d many additional random measurements. Matching experimental setups, we propose to keep the time measurements $\{e_k\}_{k=1}^d$ as deterministic information and randomly select samples from the random vector

$$X = \tfrac{1}{\sqrt{d}}(e^{2\pi i\lambda_1 t},\dots,e^{2\pi i\lambda_d t})^\top,$$

where t is uniformly distributed on $[0,1)$ and $\{\lambda_j\}_{j=1}^d$ is a Golomb ruler, i.e, a set of integers whose pairwise differences $\lambda_k - \lambda_\ell$, $k \neq \ell$ are all distinct. Then one can verify that, for all $z \in \mathbb{C}^d$,

$$zz^* = d^2\mathbb{E}|\langle z, X\rangle|^2 XX^* + \sum_{k=1}^d |\langle z, e_k\rangle|^2 (e_k e_k^* - I) \quad \text{(PRF-2)}$$

holds. Note that (PRF-2) is the analogue of (PRF-1). The requirements on $\{\lambda_k\}_{k=1}^d$ can be satisfied for special values d as above and for any d, for instance, by choosing $\lambda_k := d(k-1)^2 + k - 1$, $k = 1,\dots,d$, cf. [18] and references therein. Thus, the maximal frequency (the length of the Golomb ruler) can be chosen smaller than d^3. A simple counting argument yields that it must be bigger than $\frac{1}{2}d(d-1)$, and it is conjectured that, for any $d > 0$, one can find a Golomb ruler with length less than d^2.

Using the semidefinite program (6) for the last two scenarios in particular asks for the iterated evaluation of $\mathcal{F}_n(H)$. Assuming moreover that only $n = \mathcal{O}(d\log d)$ measurement vectors $x_j \in \mathbb{C}^d$ suffice for reconstruction with high probability, a naive evaluation of $\mathcal{F}_n(H) = (\mathrm{trace}(Hx_jx_j^*))_{j=1}^n$ requires $\mathcal{O}(d^3 \log d)$ floating point operations. Applying fast Fourier transforms, tailored to the indices $\lambda_k \in \mathbb{Z}$, $k = 1,\dots,d$, we expect a reduction to preferably $\mathcal{O}(d^2 \log^2 d)$ floating point operations for one application of the map $\mathcal{F}_n$.

IV. Discussion and Conclusion

The deterministic time-frequency measurements yield a closed reconstruction formula. However, this formula is only available in certain dimensions and the number of measurements is $d^2 + 1$. This number can be reduced by switching to the random setting, in which we proposed to select time-frequency measurements through a discrete probability measure with mass distributed according to the proposed deterministic measurement process. There is still the restriction to certain special dimensions. To overcome such limitations, we propose a hybrid model in which Fourier measurements are performed randomly and time measurements are added in a deterministic fashion. The latter may also better match the experimental measurement setting. When the associated Fourier vectors are based on Golomb rulers, then the key ingredient (PRF-2) for a proof that the semidefinite program recovers the correct signal is satisfied. Therefore, we have strong indication that the rigorous mathematical proof can be derived.

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