Recovery of Bandlimited Signal Based on Nonuniform Derivative Sampling

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***Abstract*—The paper focuses on the perfect recovery of bandlimited signals from nonuniform samples of the signal and its derivatives. The main motivation to address signal recovery using nonuniform derivative sampling is a reduction of mean sampling frequency under Nyquist rate which is a critical issue in event-based signal processing chains with wireless link. In particular, we introduce a set of reconstructing functions for nonuniform derivative sampling as an extension of relevant set of reconstructing functions derived by Linden and Abramson for uniform derivative sampling. An example of signal recovery using the first derivative is finally reported.**

I. Introduction

Modern sensor and control systems require advanced irregular sampling algorithms to represent continuous-time signals by a sequence of discrete-time samples taken at right time. A special class of irregular observations is constituted by the event-based sampling schemes. This class is characterized by the functional relationship between sampling instants and temporal signal behavior so the samples are captured when it is required. Event-based sampling, known since the 50s [15] experiences increasing interest because processing of events defined as a *significant change* of a selected signal parameter is the objective of various signal processing or monitoring and control systems [1], [9], [15], [19], [20], [26], [27]. All the event-based sampling schemes produce samples irregularly in time according to temporal signal variations.

Several signal-dependent sampling criteria have been proposed and investigated in recent years [15], [16], [25], [17], [11], [27] including the algorithms based on controlling the linear intersampling error [1], [5], [6], [11], [15], [21], integral error [16], [18], and the energy of intersampling error [17]. The most natural signal-dependent sampling scheme is based on the send-on-delta principle and consists in keeping the linear intersampling error bounded [1], [5], [6], [11], [15], [21].

The send-on-delta scheme is known in the literature also as Lebesgue sampling in the context of control systems theory [1], [6] and derivations from Lebesgue integral [6], or level-crossing sampling especially in the context of signal conversion and processing [5], [12], [20].

In [25], the send-on-delta/level-crossing sampling with prediction as an enhanced version of the pure send-on-delta/level-crossing principle has been introduced. The send-on-delta/level-crossing scheme with prediction is a sampling algorithm that employs the prediction to approximate the sampled signal between sampling instants.

The prediction is based on a belief that the sampled signal will vary according to the first-order (linear) or second-order (quadratic) approximation by the truncated Taylor series expanded at the instant of the most recent sample. The next sample is captured when the predicted signal value deviates from the real signal value by an interval of confidence [25], [23]. In particular, in the send-on-delta/level-crossing sampling with prediction, either signal or its time-derivatives are sampled and transmitted irregularly in time via communication channel for possible processing and/or reconstruction.

The present paper deals with involving signal derivatives to nonuniform sampling. More specifically, we examine the problem of recovery of original signal based on non-uniform discrete-time representation of the signal and its derivatives.

The present paper deals with involving signal derivatives to nonuniform sampling. More specifically, we examine the problem of recovery of original signal based on non-uniform discrete-time representation of the signal and its derivatives. The primary goal of adopting derivative sampling to irregular discrete-time signal representation is a desire to reduce the sampling rate below the Nyquist rate. Decreasing the mean rate of data records is an issue of primary importance in signal processing systems with wireless links since wireless communication is a major source of energy consumption. In the paper, we provide a procedure for perfect recovery of bandlimited signals for non-uniform derivative sampling. The original contribution of the paper is a formulation of a set of reconstructing functions for non-uniform derivative sampling as the extension of relevant set of reconstructing functions derived by Linden and Abramson for uniform derivative sampling in their classical paper on generalized sampling theorem [14]. Finally, we illustrate the reconstruction procedure on the example of signal recovery using the first derivative.

II. Problem formulation

Let us assume that a signal $x(t)$ of finite energy is bandlimited, i.e. $X(\omega) = 0$ for $\omega \notin (-\Omega, \Omega)$. Suppose that the signal $x(t)$ and its first $(m-1)$ time-derivatives are sampled irregularly in time which results in producing a set of samples $\{x^{(0)}(t_n), x^{(1)}(t_n), \ldots, x^{(m-1)}(t_n)\}$ taken at the instants t_n,

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$n \in \mathbb{Z}$. The aim of the present study is to recover the original signal $x(t)$ using the given samples.

A. Recovery of signal from nonuniform samples

Recovering bandlimited signal from its samples taken at nonuniform time instants is possible on the basis of theory of frames and non-harmonic Fourier series. Both concepts were introduced by Duffin and Schaeffer in [2]. The frame $\{g_n\}$ generalize the idea of a basis in a Hilbert space H in the sense that it allows representing an arbitrary element $x \in H$ as long as there exist the *frame bounds* $A, B > 0$ such that the following *frame condition* is fulfilled

$$0 < A\|x\|^2 < \sum_{n=-\infty}^{+\infty} |\langle g_n, x\rangle|^2 < B\|x\|^2 < \infty \tag{1}$$

If the set of functions $\{g_n(t)\}$ is a frame or a basis, there exists a set of coefficients c_n which allows to represent a function $x(t)$ as

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n g_n(t) \tag{2}$$

The Shannon uniform sampling theory uses the basis of functions $g_n(t) = \text{sinc}(\Omega(t - nT))$ and samples $x(nT)$ as coefficients c_n where $\text{sinc}(t) := \sin(t)/t$. Frame theory allows obtaining coefficients c_n for frame composed of functions $g_n(t) = \text{sinc}(\Omega(t - t_n))$, where t_n are sampling instants. The set t_n is not arbitrary and must obey certain conditions: $|t_n - n| < 1/4$ [10] and for finite subset of t_n it is allowed that $|t_n - n| = \mathcal{O}(n^{-\gamma}), n \to \infty, \gamma > 1$ [3]. To obtain values of coefficients c_n, we insert the known time instants t_n as t (we mark them by t_l to avoid confusion) in (2), getting

$$x(t_l) = \sum_{n=-\infty}^{+\infty} c_n g_n(t_l) \tag{3}$$

where $g_n(t_l) = \text{sinc}(\Omega(t_l - t_n))$. This may be written in matrix form

$$\mathbf{x} = \mathbf{G}\mathbf{c} \tag{4}$$

where $\mathbf{x^T} = [\ldots, x(t_{n-1}), x(t_n), x(t_{n+1}), \ldots]$, and $[\mathbf{G}]_{i,j} = g_i(t_j)$. The matrix $\mathbf{G}$ is infinite dimensional, so to apply practical recovery algorithm a truncated matrix is used [24]. The values of c_n can be then calculated on the basis of computation of the pseudo-inverse matrix

$$\mathbf{c} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{x} \tag{5}$$

The recovery method stated above has been used for recovery of amplitude information from time-encoded signals and proposed for applications in neurocomputing and time-mode signal processing systems [13].

B. Derivative sampling

The problem of reconstructing bandlimited signal from the samples of the signal and its time derivative(s) has been studied in the context of uniform sampling for decades [8]. The significant benefit of including samples of time derivatives to procedure of signal recovery is a reduction of the sampling frequency. In particular, a possibility to recover the signal based on knowledge of the samples of the signal and its first time-derivative was mentioned by Shannon in one of his milestone papers [22]. This idea was further developed by Jagerman and Vogel [4], [7] for first derivative. The derivative sampling theorem was generalized for arbitrary number of derivatives by Linden [14]. Sampling of m derivatives at once (in our notation signal itself is zero-order derivative) allows for m-fold decrease of sampling frequency, so the sampling period becomes $T_m = m\pi/\Omega$. The reconstruction formula [14] is given by

$$\begin{aligned} x(t) = \sum_{n=-\infty}^{+\infty} & x^{(0)}(nT_m)g_0(t - nT_m)+ \\ & + x^{(1)}(nT_m)g_1(t - nT_m)+ \\ & + \ldots + \\ & + x^{(m-1)}(nT_m)g_{m-1}(t - nT_m) \end{aligned} \tag{6}$$

with reconstruction functions

$$g_k(t) = \frac{t^k}{k!}\,\text{sinc}^m\left(\frac{\Omega}{m}t\right) \tag{7}$$

for $k \in (0, \ldots, m-1)$. Note that $g_k(t)$ is a function corresponding not to a single sample $x(kT)$ but to the infinite set of samples $x^{(k)}(nT)$.

C. Nonuniform derivative sampling

Since reconstruction formulas (2) and (6) are both based on convergence of Fourier series, then we can write also the nonuniform analogue of (6)

$$\begin{aligned} x(t) = \sum_{n=-\infty}^{+\infty} & c_{0,n}g_0(t - t_n)+ \\ & + c_{1,n}g_1(t - t_n)+ \\ & + \ldots + \\ & + c_{m-1,n}g_{m-1}(t - t_n) \end{aligned} \tag{8}$$

assuming that all derivatives are sampled nonuniformly, at the same instants t_n. Using a matrix notation (4) with $[\mathbf{G}^k]_{i,j} = g_k(t_i - t_j)$ for $k \in (0, \ldots, m-1)$ we obtain

$$\mathbf{x} = \mathbf{G}_0\mathbf{c}_0 + \mathbf{G}_1\mathbf{c}_1 + \ldots + \mathbf{G}_{m-1}\mathbf{c}_{m-1} \tag{9}$$

The vector $\mathbf{x}$ contains samples of $x(t)$ taken with frequency m-times lower than Nyquist frequency. For this reason the formula (9) is not sufficient to obtain coefficients $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{m-1}\}$. To make use of derivatives samples, we differentiate r-times both sides of (8), which yields

$$x^{(r)}(t) = \sum_{n=-\infty}^{+\infty} c_{0,n} g_0^{(r)}(t-t_n) + \\ + c_{1,n} g_1^{(r)}(t-t_n) + \\ + \ldots + \\ + c_{m-1,n} g_{m-1}^{(r)}(t-t_n) \tag{10}$$

$$g_k^{(r)}(t) = \frac{d^r}{dt^r}\left(\frac{t^k}{k!}\operatorname{sinc}^m\left(\frac{\Omega}{m}t\right)\right) \tag{11}$$

In particular, for $m = 0$, the set of reconstructing functions given by (11) is reduced to (7) which represents the classical nonuniform signal recovery from based on (2) without the use of derivative sampling. The equation (10) can be arranged for each $r \in (0, \ldots, m-1)$ into system of equations, which can be also written in block-matrix form

$$\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \\ \vdots \\ \mathbf{x}^{(m-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_0^{(0)} & \mathbf{G}_1^{(0)} & \ldots & \mathbf{G}_{m-1}^{(0)} \\ \mathbf{G}_0^{(1)} & \mathbf{G}_1^{(1)} & \ldots & \mathbf{G}_{m-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_0^{(m-1)} & \mathbf{G}_1^{(m-1)} & \ldots & \mathbf{G}_{m-1}^{(m-1)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_{m-1} \end{bmatrix} \tag{12}$$

Solving this system gives coefficients $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{m-1}$ for reconstruction with (8). Therefore computational complexity of the reconstruction procedure corresponds to classic matrix inversion complexity $\mathcal{O}(n^3)$ and it is dependent on the number of samples used. Summing up, the proposed procedure of signal recovery is based on computation of the coefficients $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{m-1}$ to the set of reconstructing functions defined by (7). The number of reconstructing functions depends on the number of derivatives used for signal recovery.

III. Simulations

As an example illustrating the procedure of signal recovery based on nonuniform derivative sampling, we present a reconstruction from samples of the signal and its first derivative, i.e. for $m = 2$. In this case we have the following reconstructing functions on the basis of (11)

$$g_0^{(0)}(t) = \frac{4\sin^2(\Omega t/2)}{\Omega^2 t^2}$$

$$g_1^{(0)}(t) = \frac{4\sin^2(\Omega t/2)}{\Omega^2 t}$$

$$g_0^{(1)}(t) = \frac{2\left(-2 + 2\cos(\Omega t) + \Omega t \sin(\Omega t)\right)}{\Omega^2 t^3}$$

$$g_1^{(1)}(t) = \frac{2\left(-1 + \cos(\Omega t) + \Omega t \sin(\Omega t)\right)}{\Omega^2 t^2}$$

The coefficients $\mathbf{c}_0, \mathbf{c}_1$, are computed on the basis of the following reconstruction equation:

$$\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_0^{(0)} & \mathbf{G}_1^{(0)} \\ \mathbf{G}_0^{(1)} & \mathbf{G}_1^{(1)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \end{bmatrix}$$

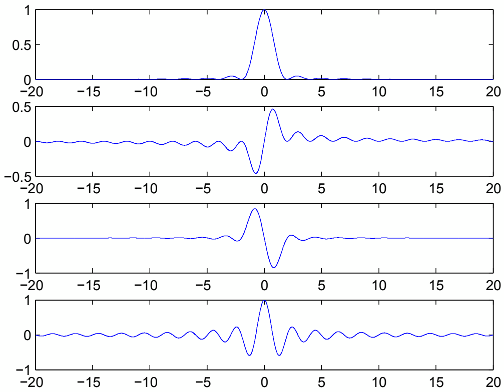

Fig. 1. Reconstruction functions $g_0^{(0)}$, $g_1^{(0)}$, $g_0^{(1)}$, $g_1^{(1)}$, when signal is recovered from the samples of $x(t)$ and its derivative $x'(t)$

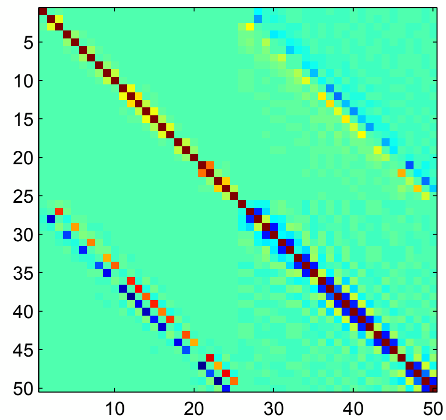

Fig. 2. Reconstruction matrix, composed of matrices $\mathbf{G}_0^{(0)}$, $\mathbf{G}_1^{(0)}$, $\mathbf{G}_0^{(1)}$, $\mathbf{G}_1^{(1)}$ corresponding to respective reconstructon functions $g_0^{(0)}$, $g_1^{(0)}$, $g_0^{(1)}$, $g_1^{(1)}$.

The signal $x(t)$ used for exemplified recovery was generated using Shannon-Whittaker reconstruction formula

$$x(t) = \sum_{n=1}^{40} x_n \frac{\sin(\pi(t-n))}{\pi(t-n)} \tag{13}$$

where $x_1, \ldots, x_{40}$ were selected as independent realizations of random variable with normal distribution. Thus, the signal $x(t)$ is bandlimited to $\Omega = \pi$. The signal derivative $x'(t)$ was computed using differentiated $\operatorname{sinc}(\cdot)$ function. The signal $x(t)$ has been sampled using send-on-delta sampling scheme with linear prediction [23], [25], resulting in 25 samples of the signal and 25 samples of its derivative. The reconstruction block-matrix $\mathbf{G}$ is depicted in the Fig. 2, where red corresponds to higher, and blue to the lower values. The number of each type of samples required for reconstruction is 20 so signal is slightly oversampled. As stated in Introduction, in the send-on-delta scheme with linear prediction, the sampling operation is triggered when the predicted signal based on linear prediction deviates from the real signal value by an interval of confidence [23]. The original and the reconstructed signal are depicted in the Fig. 3. The absolute linear error of reconstruction is presented in the Fig. 4.

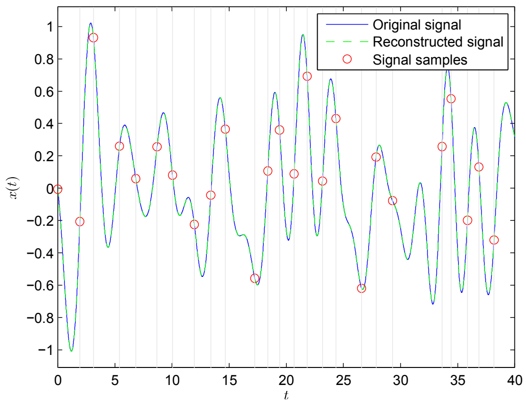

Fig. 3. Original bandlimited signal $x(t)$ and its reconstruction $\tilde{x}(t)$ from the nonuniform samples of $x(t)$ and its first derivative $x'(t)$.

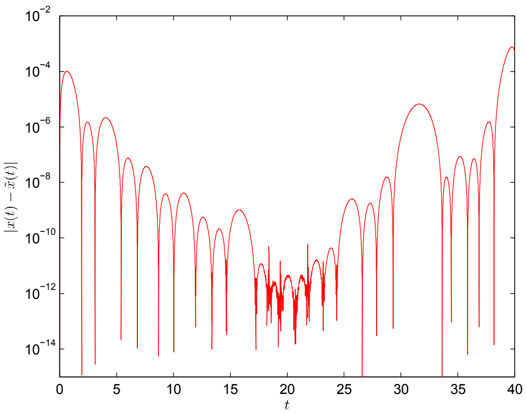

Fig. 4. Error of the reconstruction of the signal from the nonuniform samples of $x(t)$ and its derivative $x'(t)$. The lowest values occure at sampling instants t_n (Compare Fig. 3).

IV. Conclusions

The paper focuses on the perfect reconstruction of band-limited signal from the nonuniformly spaced samples of the signal and its derivatives. The principal benefit of signal recovery using nonuniform derivative sampling is a reduction of mean sampling frequency under Nyquist rate which is a critical issue in signal processing chains with wireless link based on event-based sampling. The computational complexity of the proposed recovery procedure is connected with the matrix inversion needed to calculate the coefficients $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{m-1}$.

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