

# Sampling aspects of approximately time-limited multiband and bandpass signals

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**Abstract**—We provide an overview of recent progress regarding the role of sampling in the study of signals that are in the image of a bandpass or multiband frequency limiting operation and have most of their energies concentrated in a given time interval. We finish considering a means to approximate essentially time-limited bandpass signals. In this case we present a new phase-locking metric that arises in the study of EEG signals.

## I. INTRODUCTION

We discuss relationships between time and band limiting and sampling, leading also to numerical computation of essentially time-limited multiband and bandpass signals. As an application we propose a method to analyze phase synchrony of bandpass projections of signals, illustrating a particular case of electroencephalographic (EEG) signals. In this introduction we briefly review basic elements of the theory of time and band limiting. In Section II we discuss connections between sampling and time and band limiting. In Section III we present a method to construct time- and multiband-limited signals from eigenfunctions for time and band limiting to separate bands and a numerical technique that takes advantage of sampling. In Section IV we provide a method to approximate essentially time-limited bandpass signals. We use this approach in Section V to provide a new method to study phase differences of bandpass projections of signals. The method is illustrated in the context of study of EEG signals. Relatively constant phase lag among two EEG channels can indicate recruitment of the corresponding cortical regions in distributed cognition.

### A. Time and band limiting

Set  $(Q_T)(f)(t) = \mathbb{1}_{[-T, T]}(t) f(t)$  where  $\mathbb{1}_S$  denotes the function equal to one on  $S \subset \mathbb{R}$  and zero outside  $S$ . Let  $Q = Q_1$ . Also let  $(P_\Sigma)(f)(t) = (\mathbb{1}_\Sigma \widehat{f})^\vee(t)$  where  $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t \xi} dt$ . We write  $P_\Omega = P_{[-\Omega/2, \Omega/2]}$  and  $P = P_1$ . The Paley–Wiener space  $\text{PW}_\Sigma$  is of the image of  $L^2(\mathbb{R})$  under the orthogonal projection  $P_\Sigma$ . We write  $\text{PW}_\Omega$  instead when  $\Sigma = [-\Omega/2, \Omega/2]$  and simply  $\text{PW}$  when  $\Omega = 1$ . For compact  $\Sigma$ , the operator  $P_\Sigma Q_T$  is compact and its trace is equal to  $2T|\Sigma|$  where  $|\Sigma|$  denotes the Lebesgue measure of  $\Sigma \subset \mathbb{R}$ . It is also self adjoint on  $\text{PW}_\Sigma$  while  $P_\Sigma Q_S P_\Sigma$  is self adjoint on  $L^2(\mathbb{R})$ . Since functions in  $\text{PW}_\Sigma$  are real analytic,  $P_\Sigma Q_S$  has no unit eigenfunctions and the discrete spectrum of  $P_\Sigma Q_T$  is contained in  $[0, 1)$ .

### B. Prolate functions and their properties

The operator  $P_{c/\pi} Q$  commutes with a certain self-adjoint second-order differential operators whose eigenfunctions, and hence those of  $P_{c/\pi} Q$ , are the *prolate spheroidal wave functions*, which form a complete orthogonal basis for  $\text{PW}_{c/\pi}$ . They are also eigenfunctions of the integral operator

$$(F_c f)(t) = \int_{-1}^1 e^{icst} f(s) ds = \widehat{Q}f(-ct/2\pi). \quad (1)$$

The eigenvalues  $P_{c/\pi} Q$  are non degenerate. Denote by  $\lambda_0(c) > \lambda_1(c) > \dots$  the  $n$ th eigenvalue of  $P_{c/\pi} Q$  and  $\varphi_n^c$  the corresponding prolate eigenfunction. That  $\varphi_n^c$  is an eigenfunction of (1) and other basic properties imply that

$$D_{c/\pi} \widehat{\varphi}_n^c = \frac{i^n}{\sqrt{\lambda_n}} Q \varphi_n^c \quad (2)$$

where  $D_a$  is the unitary dilation  $(D_a f)(t) = \sqrt{a} f(at)$ ,  $a > 0$ . When  $L^2(\mathbb{R})$ -normalized, the prolates  $\{\varphi_n^c\}$  form an orthonormal basis for  $\text{PW}_{c/\pi}$  as well as a complete, orthogonal set in  $L^2[-1, 1]$  with  $\lambda_n(c) = \int_{-1}^1 |\varphi_n^c|^2$ . As such, any  $f \in \text{PW}_{c/\pi}$  can be expanded in the form  $f = \sum_{n=0}^{\infty} \alpha_n \varphi_n^c$  with  $\|f\|_{L^2(\mathbb{R})}^2 = \sum \alpha_n^2$  and  $\int_{-1}^1 |f|^2 = \sum \lambda_n \alpha_n^2$ . The prolates are real valued and  $\varphi_n^c$  is even (odd) if  $n$  is even (odd). Further properties of prolates and justification of the facts just mentioned, which were established in the Bell Labs papers, especially [1], [2], can be found in [3].

### C. The $2\Omega T$ theorem

Suppose that  $\Sigma$  is a union of  $M$  pairwise disjoint frequency intervals of unit length so that the total time–bandwidth product corresponding to  $P_\Sigma Q_T$  is  $2MT$ . Denote by  $\mathcal{N}(2MT, \alpha)$  the number of eigenvalues of  $P_\Sigma Q_T$  larger than  $\alpha$ . The following is a special case of a version of the “ $2\Omega T$ ” theorem proved by Landau and Widom in [4].

*Theorem 1 (Landau–Widom, 1980):* As  $T \rightarrow \infty$  the number of eigenvalues of  $P_\Sigma Q_T$  exceeding  $\alpha \in (0, 1)$  satisfies

$$\mathcal{N}(2MT, \alpha) = 2MT + \frac{M}{\pi^2} \log 2T \log \left( \frac{\alpha}{1 - \alpha} \right) + o(\log 2MT).$$

## II. SAMPLING AND TIME AND BAND LIMITING

Walter and Shen [5] and Khare and George [6] observed

$$(PQ_T f)(t) = \sum_{n=0}^{\infty} \lambda_n \sum_{k=-\infty}^{\infty} f(k) \varphi_n(k) \varphi_n(t)$$

where  $\varphi_n$  are eigenfunctions of  $PQ_T$ . Oscillatory behavior of the prolates near the endpoints of  $[-T, T]$  prohibits an estimate  $\sum_{|k|>T} \varphi_n^2(k) \leq C(T)(1 - \lambda_n)$ . However, in [7] the estimate

$$\sum_{|k|>\pi^2 T(1+\log^{\gamma}(T))} \varphi_n^2(k) \leq C(1 - \lambda_n), \quad (3)$$

was proved for any  $\gamma > 1$ . It was conjectured in [7] that the log factor in the sum index is not necessary. The following consequence of (3) regarding approximation of  $Q_T f$  from samples of  $f$  near  $[-T, T]$  was also established in [7].

*Theorem 2:* Let  $f \in \text{span}\{\varphi_n\}_{n=0}^N$ , with  $\varphi_n$  the  $n$ th eigenfunction of  $PQ_T$ . Define  $\varphi_n^T = \sum_{|k|<M(T)} \varphi_n(k) \text{sinc}(t-k)$  with  $M(T)$  as in (3). Then

$$\|Q_T(f - \sum_{n=0}^N \langle f, \varphi_n^T \rangle \varphi_n^T)\| \leq C \|f\| \sum_{n=0}^N \lambda_n (1 - \lambda_n).$$

A method to obtain accurate numerical estimates of integer samples of prolates is outlined in Hogan et al., [7].

## III. TIME- AND MULTIBAND-LIMITED SIGNALS

This section reviews techniques underlying numerical computation of certain time- and multiband-limited signals. We start with a method to build eigenfunctions for the case  $\Sigma$  is a finite union of intervals from appropriately modulated prolates.

### A. Eigenfunctions for unions

If  $\Sigma$  is a finite union of pairwise disjoint intervals  $I_1, \dots, I_M$  then we can denote  $P_{\Sigma} = \sum_{k=1}^M P_{I_k}$ . Unlike  $PQ_T$ , the operator  $P_{\Sigma}Q_T$  does not commute with a finite order differential operator with polynomial coefficients when  $\Sigma$  is a union of two or more intervals. This important fact, established by Morrison in [8], bars us from using power series methods to compute eigenfunctions.

The following results were established in [9] in a more general setting. If  $J$  is a frequency interval of unit length then the orthogonal projection onto  $PW_J$ , the Paley–Wiener subspace of  $L^2(\mathbb{R})$  of functions frequency supported in  $J$ , has the form  $M_{m_J} P M_{-m_J}$  where, as before,  $P = P_{[-1/2, 1/2]}$  and  $(M_u f)(t) = e^{2\pi i t u} f(t)$  with  $m_{J_k}$  the midpoint of  $J_k$ . Suppose that one has  $M$  pairwise disjoint frequency intervals  $J_1, \dots, J_M$  each of unit length and set  $\Sigma = \cup_k J_k$ . Set  $m_k = m_{J_k}$ . Since the  $J$ -prolates  $\varphi_n^J = M_{m_J} \varphi_n$ , with  $\varphi_n$  the corresponding eigenfunction of  $PQ_T$ , form a complete family for  $PW_J$ , any function in  $PW_{\Sigma}$  has an orthogonal decomposition  $f = \sum_{k=1}^M \sum_{n=0}^{\infty} \langle f, M_{m_k} \varphi_n \rangle M_{m_k} \varphi_n$ . Consider now the problem of finding an eigenvalue–eigenfunction pair  $(\lambda, \psi)$  for  $P_{\Sigma}Q_T$ . Expanding  $\psi$  in terms of the modulated prolates  $M_{m_k} \varphi_n$  and applying  $P_{\Sigma}Q_T$  to these, one sees that one must identify the coefficients  $\Gamma_{nm}^{k,\ell} = \langle Q_T M_{m_k} \varphi_n, M_{m_{\ell}} \varphi_m \rangle$ . Note that  $\Gamma_{mn}^{\ell,k} = \overline{\Gamma_{nm}^{k,\ell}}$ , that is, if  $\Gamma^{k,\ell}$  is the matrix with entries  $\Gamma_{nm}^{k,\ell}$

then  $\Gamma^{\ell,k} = \overline{\Gamma^{k,\ell}}$ . The eigenvalue–eigenfunction pairs  $(\lambda, \psi)$  for  $P_{\Sigma}Q_T$  are produced as follows.

*Proposition 3:* Suppose that  $J_1, \dots, J_M$  are pairwise disjoint unit intervals with union  $\Sigma = \cup_{k=1}^M J_k$ . Let  $\Lambda$  denote the diagonal matrix with  $n$ th diagonal entry  $\lambda_n(PQ_T)$  and let  $\Gamma^{k,\ell}$  be the matrix with entries  $\gamma_{nm}^{k,\ell} = \langle Q_T M_{m_k - m_{\ell}} \varphi_n, \varphi_m \rangle$ ,  $k < \ell$ . Then any eigenvector–eigenvalue pair  $\psi$  and  $\lambda$  for  $P_{\Sigma}Q_T$  can be expressed as  $\psi = \sum_{k=1}^M \sum_{n=0}^{\infty} \alpha_n^k M_{m_k} \varphi_n$  where the vectors  $\alpha_k = \{\alpha_n^k\}$  together form a discrete eigenvector for the block matrix eigenvalue problem

$$\lambda \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{pmatrix} = \begin{pmatrix} \Lambda & \bar{\Gamma}^{12} & \dots & \bar{\Gamma}^{1M} \\ (\Gamma^{12})^T & \Lambda & \bar{\Gamma}^{23} & \dots \\ \vdots & \vdots & \ddots & \vdots \\ (\Gamma^{1M})^T & \dots & \dots & \Lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{pmatrix}.$$

In order to turn the method into a means to compute eigenvalues and eigenfunctions of  $P_{\Sigma}Q_T$  numerically, one needs to estimate the coefficients

$$\Gamma_{nm}^{k,\ell} = \int_{-T}^T e^{2\pi i(m_k - m_{\ell})t} \varphi_n(t) \varphi_m(t) dt$$

and to justify truncating the matrices  $\Lambda$  and  $\Gamma^{k,\ell}$ . The matrix truncations are justified by Theorem 1.

The corresponding  $\Gamma$ -matrix entries can be expressed as

$$\langle Q_T M_{m_I} \varphi_n, M_{m_J} \varphi_m \rangle = \sum_k \sum_{\ell} \varphi_n(k) \varphi_m(\ell) A(T; I, J)_{k\ell};$$

$$A(T; I, J)_{k\ell} = \int_{-T}^T e^{2\pi i(m_I - m_J)t} \text{sinc}(t-k) \text{sinc}(t-\ell) dt.$$

The inner products are computed using the following, see [9].

*Lemma 4:* As a bilinear form acting on the pair of sequences  $\{\varphi_n(k)\}, \{\varphi_m(\ell)\}$ , the matrix  $A(T; I, J)_{k\ell}$  coincides with  $i^{n+m} \sqrt{\lambda_m \lambda_n} \text{sinc}(2T(m_J - m_I) + k - \ell)$ .

An eigenfunction  $\psi$  of  $P_{\Sigma}Q_T$  will be called a *time- and multiband-limiting eigenfunction* (TMBLE). If  $\psi$  is a TMBLE with eigenvalue  $\lambda > 1/2$  then  $\psi$  should be, at least nearly, in the span of those eigenfunctions  $\varphi_n^I$ , where  $\Sigma = \cup I$ , corresponding to the eigenvalues of  $P_I Q_T$  larger than  $1/2$ , hence, of eigenfunctions  $\varphi_n^I$  corresponding to  $n \leq 2T$ . In this case,  $\varphi_n^I$  can be approximated accurately on  $[-T, T]$  by sinc interpolating its samples  $\varphi_n^I(k)$  where  $|k| \leq M(T)$  above.

### B. Numerical estimation of TMBLEs

Accurate numerical estimation of the TMBLEs is obtained via estimation of the entries of suitable truncations of the  $\Gamma$  matrices and eigenvectors of the corresponding truncation of the eigenproblem in Proposition 3. Details are given in [9]. Figure 1 illustrates the case with three frequency intervals. The corresponding eigenfunctions are plotted in Fig. 2.

## IV. TIME- AND BANDPASS-LIMITED SIGNALS

Given  $0 < c' < c$  denote by  $PW_{c',c}^{\pi}$  the orthogonal complement of  $PW_{c'/\pi}$  inside  $PW_{c/\pi}$ , that is, the closed subspace of  $L^2(\mathbb{R})$  of functions whose Fourier transforms  $\hat{f}(\xi)$  are supported in  $c'/\pi \leq |\xi| \leq c/\pi$ , and by  $P_{c',c}^{\pi}$  the orthogonal projection onto  $PW_{c',c}^{\pi}$ . The eigenfunctions of the operator

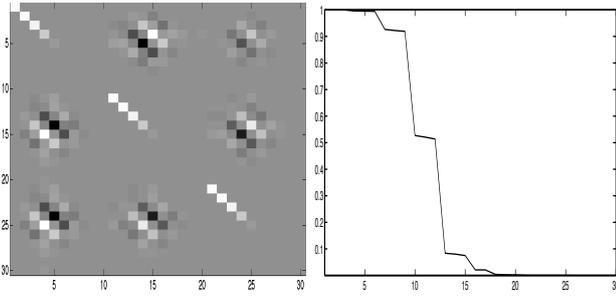


Fig. 1. Matrix in Proposition 3 for  $T = 2$ ,  $I = [-1/2, 1/2]$ ,  $J = [2, 3]$ , and  $K = [5, 6]$ . Intensity plot of the real part of the matrix in Proposition 3. Each  $\Gamma^{\mu\nu}$  term is truncated to size  $10 \times 10$ . On the right is a plot of the moduli of the eigenvalues of the same matrix.

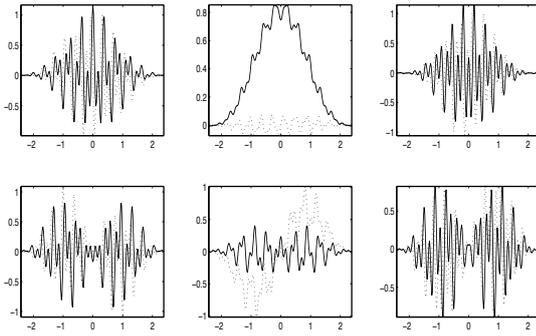


Fig. 2. TMBLEs for  $T = 2$ ,  $I = [-1/2, 1/2]$ ,  $J = [2, 3]$  and  $K = [5, 6]$ . Plotted are the TMBLMS corresponding to  $n = 0, 1, 2, 3, 4, 5$  respectively for three frequency bands. Real parts solid, imaginary parts dashed.

$P_{c'/c}^{\pi} Q$  corresponding to time truncation of a function in  $L^2(\mathbb{R})$  to a finite interval— $[-1, 1]$  in this work—followed by frequency limiting to frequencies  $|\omega| \in [c', c]/\pi$  will be called *bandpass prolates* here. Numerical approximation of the most time concentrated bandpass limited signals (called *bandpass prolates* here) was studied recently by SenGupta et al., [10] by expressing the kernel of the bandpass limiting operator in terms of Legendre polynomials, then identifying the bandpass prolates through their Legendre coefficients. Alternatively, Proposition 5 proved in [11], produces the coefficients of the bandpass prolates, expressed as superpositions of full-band prolates, from *partial inner products* of full-band prolates. As explained below, these partial inner products can be computed directly from pointwise values of  $\varphi_n^c$  and  $\varphi_n^{c'}$  where, as before,  $\varphi_n^c$  is the  $n$ th eigenfunction of  $P_{c/\pi} Q$ .

Denote by  $R = R(c', c)$  the matrix with entries  $R_{jk} = \frac{i^{k-j}}{\sqrt{\lambda_j \lambda_k}} \int_{-c'/c}^{c'/c} \varphi_k^c(\xi) \varphi_j^c(\xi) d\xi$ . The matrix  $R$  is real symmetric, a consequence of the parity properties of the  $\varphi_n^c$ . Let  $\Lambda = \Lambda(c)$  be the diagonal matrix with  $n$ th diagonal entry  $\lambda_n(c)$ .

**Proposition 5:** If  $\psi = \sum \alpha_n \varphi_n^c \in \text{PW}_{c/\pi}$  then

$$P_{c'/c}^{\pi} Q \psi = \sum_k \alpha_k \lambda_k \left( \varphi_k^c - \sum_j R_{jk} \varphi_j^c \right).$$

In particular, if  $\psi = \sum \alpha_n \varphi_n^c$  is an eigenfunction of  $P_{c'/c}^{\pi} Q$

with eigenvalue  $\lambda$  then, with  $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ ,

$$\lambda \alpha_n = \lambda_n \alpha_n - \sum_k \lambda_k \alpha_k R_{nk} \quad \text{i.e.} \quad \lambda \alpha = (I - R) \Lambda \alpha.$$

The discrete eigenvectors  $\alpha$  of the matrix  $(I - R) \Lambda$  thus give rise to eigenfunctions of  $P_{c'/c}^{\pi} Q$  and the eigenvalue  $\lambda$  measures the concentration of  $\psi$  in  $[-1, 1]$  just as in the case of standard prolates. The proof uses the identities (1) and (2).

The partial inner products can be calculated by virtue of the prolate differential equation and integration by parts. If  $n \neq m$  then, with  $\chi_n$  as in (??) and  $-1 \leq a \leq b \leq 1$ ,

$$(\chi_n - \chi_m) \int_a^b \varphi_n(t) \varphi_m(t) dt = \left[ (t^2 - 1) (\varphi_n' \varphi_m - \varphi_m' \varphi_n) (t) \right]_a^b.$$

Approximate bandpass prolates are obtained from finite size truncations of the eigenproblem in Proposition 5, see [11]. Khare [12] also considered the problem of numerical evaluation of bandpass prolates, focusing instead on the role of the interpolating function (sinc multiplied by a suitably dilated cosine) and establishing that the bandpass prolate samples form a discrete eigenvector of the matrix of partial integrals on  $[-1, 1]$  of shifts of the interpolating kernel, cf. also Hogan et al., [7]. Khare did not investigate dependence on  $c'/c$ .

## V. PHASE SYNCHRONY AND AN APPLICATION TO EEG

We discuss briefly an application of bandpass prolates to study phase synchrony—nearly constant average instantaneous phase difference—particularly of EEG signals. It is believed that communication between different regions of neural cortex in attention focusing tasks is manifest in phase synchrony of neural firing patterns, e.g., [13], [14], particularly in the *gamma band*, e.g., [15]. Measuring band specific synchrony between EEG channels requires (i) a means to associate instantaneous phase to a given frequency band and (ii) a method to measure temporal phase locking between a pair of signals in a given band by averaging instantaneous phase difference for enough oscillations that average phase difference makes sense—say three to five—but not so many that synchronous epochs are indistinguishable from asynchronous ones.

Proposed methods include filtered analytic signals and convolutions with modulated Gaussians [16], [17], and empirical mode decomposition methods [18], [19] among others. In each case, the instantaneous phase is defined as the log of the complex valued signal divided by its modulus. The instantaneous phase difference of two such signals is the log of the product of the first unimodular signal and the conjugate of the second. To quantify phase locking of two signals one takes a time average of the instantaneous phase difference over a period that amounts to several oscillations.

In the case of analytic extensions of signals filtered over a short duration, aliasing is a concern. In the case of convolution with modulated Gaussians, insufficiently many degrees of freedom are being employed. The empirical mode decomposition provides a data- and algorithmic-driven definition of phase. However, it can be impossible to physical from algorithmic factors underlying the measured phase.

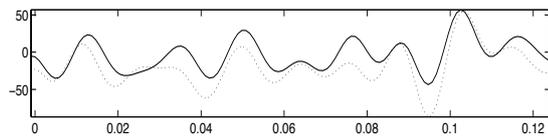


Fig. 3. EEG channel data 1/8 second record of two concurrent EEG channel measurements, digitally sampled at 1024 samples per second.

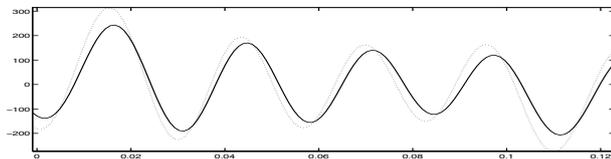


Fig. 4. Approximate  $\gamma$  projections. Projections of channel measurements onto the span of the six top eigenvectors of time-limiting to 1/8 second and bandpass limiting to 24–40 Hz.

We consider here a new phase-locking metric computed through the following steps. *Step 1*: define the duration and frequency band for which synchrony is to be measured. *Step 2*: define the projection onto the span of the bandpass prolates whose eigenvalues are close to one or, at least, not much smaller than one half. *Step 3*: compute the analytic signal for this projection, and divide by its amplitude to get its unimodular factor. *Step 4*: For a pair of such signals, multiply the unimodular part of one by the conjugate of that of the other, integrate over the given duration, and compute the modulus. This is the *phase locking value (PLV)*.

We implemented this algorithm as follows to produce Fig. 5. To analyze the gamma band of EEG signals, we chose the frequency range from 24 to 40 Hz. In order to compute the PLV over 3 to 5 oscillations of signals in this range, we took the duration of interest to be 1/8 second. The time bandwidth product in this case is  $2(40 - 24)/8 = 4$ . The corresponding time- and bandpass-limiting operator has six eigenvalues “not much smaller than 1/2.” We successively chose 1/8 second blocks of the EEG channels and computed the projections onto the span of the first six eigenfunctions. We then computed the analytic signal using the `matlab` builtin `hilbert`. A PLV was computed for each successive 1/8-second segment of the two EEG channels.

Fig. 5 shows PLVs of projections of 1/8-seconds of the two EEG channels onto the space generated by the six eigenfunctions of time limiting to 1/8-second duration and bandpass limiting to 24–40 Hz most concentrated to the given duration. The PLVs were computed for 1/8-second duration. In the data presented, a visual stimulus was shown to the subject after a half second. An initial interval of synchrony then presumably reflects response of the visual cortex. The subsequent interval of synchrony after “ $t = 0$ ” presumably then corresponds to the subject maintaining a mental representation of the stimulus.

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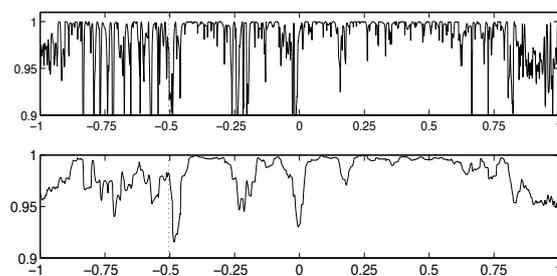


Fig. 5. Phase locking. Top shows PLVs of projections in Fig. 4 computed over the full two second record. Bottom shows PLVs time averaged over 20 consecutive time shifts.

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