Sampling and Reconstruction of Bandlimited BMO-Functions

Holger Boche*
Technische Universität München
Lehrstuhl für Theoretische Informationstechnik
E-mail: boche@tum.de

Ullrich J. Mönich†
Massachusetts Institute of Technology
Research Laboratory of Electronics
E-mail: moenich@mit.edu

***Abstract*—Functions of bounded mean oscillation (BMO) play an important role in complex function theory and harmonic analysis. In this paper a sampling theorem for bandlimited BMO-functions is derived for sampling points that are the zero sequence of some sine-type function. The class of sine-type functions is large and, in particular, contains the sine function, which corresponds to the special case of equidistant sampling. It is shown that the sampling series is locally uniformly convergent if oversampling is used. Without oversampling, the local approximation error is bounded.**

I. Notation

Let $\hat{f}$ denote the Fourier transform of a function f. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ is the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. For $0 < \sigma < \infty$ let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leq C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ.

II. Introduction and Motivation

A well-known result in sampling theory is Brown's theorem, which states that the Shannon sampling series

$$\sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

is locally uniformly convergent for all functions in the Paley–Wiener space $\mathcal{PW}_\pi^1$. $\mathcal{PW}_\pi^1$ is the space of all functions f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)\,\mathrm{e}^{iz\omega}\,\mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^1[-\pi,\pi]$. This sampling theorem has been extended in various directions, for example, larger signal spaces and non-equidistant sampling patterns [1].

In this paper we consider the sampling series

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t), \tag{1}$$

where the sampling points $\{t_k\}_{k\in\mathbb{Z}}$ are the zero sequence of some sine-type function and the functions $\{\phi_k\}_{k\in\mathbb{Z}}$ are certain reconstruction functions, and analyze its convergence behavior for bandlimited BMO$(\mathbb{R})$-functions.

Definition 1. A function $f : \mathbb{R} \to \mathbb{C}$ is said to belong to BMO$(\mathbb{R})$, provided that it is locally in $L^1(\mathbb{R})$ and $\frac{1}{|I|}\int_I |f(t) - m_I(f)|\,\mathrm{d}t \leq C_1$ for all bounded intervals I, where $m_I(f) := \frac{1}{|I|}\int_I f(t)\,\mathrm{d}t$ and the constant C_1 is independent of I. $|I|$ denotes the Lebesgue measure of the set I. We further define

$$\|f\|_{\mathrm{BMO}(\mathbb{R})} := \sup_I \frac{1}{|I|}\int_I |f(t) - m_I(f)|\,\mathrm{d}t,$$

where the supremum is over all bounded intervals I. By BMO$_\pi$ we denote the space of all functions in $\mathcal{B}_\pi$ that are in BMO$(\mathbb{R})$ when restricted to the real axis.

Note that $\|\cdot\|_{\mathrm{BMO}(\mathbb{R})}$ is actually a seminorm, because we have $\|C\|_{\mathrm{BMO}(\mathbb{R})} = 0$ for all constants $C \in \mathbb{C}$.

A consequence of the famous Fefferman–Stein theorem [2] is the fact that an arbitrary BMO$(\mathbb{R})$-function can be decomposed into a $L^\infty(\mathbb{R})$-function and the Hilbert transform of a $L^\infty(\mathbb{R})$-function [3, p. 248].

Theorem A (Fefferman–Stein)**.** *There exists a constant $C_2 > 0$ such that for all $f \in$ BMO$(\mathbb{R})$ there exist two functions $f_1, f_2 \in L^\infty(\mathbb{R})$ and a constant α such that $f = f_1 + \mathfrak{H} f_2 + \alpha$ and $\|f_1\|_\infty \leq C_2\|f\|_{BMO(\mathbb{R})}$, $\|f_2\|_\infty \leq C_2\|f\|_{BMO(\mathbb{R})}$.*

Theorem A is an important theoretical result [3], but it also has a high significance for applications where the Hilbert transform is used, for example the calculation of the analytic signal [4] and the analysis of signal properties [5], [6], [7]. Since the Hilbert transform of bounded functions is of particular interest, Theorem A is interesting because it essentially describes the range of the Hilbert transform for $L^\infty(\mathbb{R})$.

III. Sampling for BMO$_\pi$

Definition 2. An entire function f of exponential type π is said to be of sine type if the zeros of f are separated and simple, and there exist positive constants A, B, and H such that $A\,\mathrm{e}^{\pi|y|} \leq |f(x+iy)| \leq B\,\mathrm{e}^{\pi|y|}$ whenever x and y are real and $|y| \geq H$.

Without loss of generality, we assume that the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ is ordered strictly increasingly and

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that $t_0 = 0$. Then, it follows that the product

$$\phi(z) = z \lim_{N\to\infty} \prod_{\substack{|k|\le N\\ k\neq 0}} \left(1 - \frac{z}{t_k}\right) \tag{2}$$

converges uniformly on $|z| \le R$ for all $R < \infty$, and ϕ is an entire function of exponential type π [8]. It can bee seen from (2) that ϕ, which is often called generating function, has the zeros $\{t_k\}_{k\in\mathbb{Z}}$. Moreover, it follows that

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t-t_k)} \tag{3}$$

is the unique function in $\mathcal{B}_\pi^2$ that solves the interpolation problem $\phi_k(t_l) = \delta_{kl}$, where $\delta_{kl} = 1$ if $k = l$, and $\delta_{kl} = 0$ otherwise.

Sampling point sequences that are made of the zeros of functions of sine type are also complete interpolating sequences for $\mathcal{B}_\pi^2$ [9, p. 143]. This means that we restrict our analysis to a subclass of complete interpolating sequences. We conjecture that for arbitrary complete interpolating sequences a result like the one in this paper cannot be obtained even for smaller signal spaces. In particular, we conjecture that there exist complete interpolating sequences and functions in $\mathcal{PW}_\pi^1$ such that the sampling series is even locally divergent [10]. If this conjecture is true it shows the specialty of sine type function generated sampling patterns.

In [11] it was shown that for signals in $\mathcal{B}_{\beta\pi}^\infty$, $0 < \beta < 1$, the sampling series (1) is uniformly convergent on all compact subsets of $\mathbb{R}$. The proof in [11] makes use of certain essential properties of sine type functions.

Theorem C. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly. Furthermore, let ϕ_k be defined as in* (3) *and $0 < \beta < 1$. Then, for all $T > 0$ and all $f \in \mathcal{B}_{\beta\pi}^\infty$ we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

In the next theorem we provide a sampling theorem for BMO_π-functions, and thus extend Theorem C to a larger space.

Theorem 1. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly. Furthermore, let ϕ_k be defined as in* (3) *and $T > 0$.*

1) *We have*

$$\sup_{N\in\mathbb{N}} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| < \infty$$

for all $f \in BMO_\pi$.

2) *Let $0 < \beta < 1$. For all $f \in BMO_{\beta\pi}$ we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

Theorem 1 shows that without oversampling the local peak value of the approximation error is bounded. With oversampling the sampling series is uniformly convergent on all compact subsets of $\mathbb{R}$.

IV. Proof of Theorem 1

In this section we prove Theorem 1. For the proof we need several auxiliary results.

A. Basic Properties of BMO_π-Functions

For functions f in BMO_π, i.e., $\text{BMO}(\mathbb{R})$-functions that are additionally bandlimited, the Fefferman–Stein decomposition (Theorem A) is of course also possible because $\text{BMO}_\pi \subset \text{BMO}(\mathbb{R})$. The functions f_1 and f_2 in this decomposition are in $L^\infty(\mathbb{R})$. However, since the function f is additionally bandlimited, it is reasonable to ask whether the decomposition can be performed in a such a way that f_1 and f_2 are also bandlimited, i.e., in $\mathcal{B}_\pi^\infty$. The next theorem, which has been proved in [12], answers this question in the affirmative.

Theorem 2. *There exists a constant $C_3 > 0$ such that for all $f \in BMO_\pi$ there exist two functions $f_1, f_2 \in \mathcal{B}_\pi^\infty$ and a constant α such that $f = f_1 + \mathfrak{H} f_2 + \alpha$ and $\|f_1\|_\infty \le C_3\|f\|_{BMO(\mathbb{R})}$, $\|f_2\|_\infty \le C_3\|f\|_{BMO(\mathbb{R})}$.*

Theorem 2 has been stated in a form to have maximum similarity to the Fefferman–Stein theorem. However, the bandwidth of the function f_2 does not have to be π; it can be arbitrarily reduced. Hence, we have the next theorem [12]. It should be noted that a decrease of the bandwidth of the function f_2 comes in general with an increase of the L^∞-norm of f_1 and f_2.

Theorem 3. *For all $0 < \hat{\beta} \le 1$ there exists a constant C_4 such that for all $f \in BMO_\pi$ there exist two functions $f_3 \in \mathcal{B}_\pi^\infty$ and $f_4 \in BMO_{\hat{\beta}\pi}$ and a constant α such that $f = f_3 + f_4 + \alpha$ and $\|f_3\|_\infty \le C_4(\hat{\beta})\|f\|_{BMO(\mathbb{R})}$, $\|f_4\|_{BMO(\mathbb{R})} \le C_4(\hat{\beta})\|f\|_{BMO(\mathbb{R})}$.*

Finally, we need a theorem about the growth behavior of bandlimited $\text{BMO}(\mathbb{R})$-functions [12].

Theorem 4. *Let $f \in BMO_\sigma$, $0 < \sigma < \infty$. Then, for all $\gamma > \sigma$, there exists a constant C_5 such that $|f(z)| \le C_5\, \mathrm{e}^{\gamma|\mathrm{Im}(z)|}\log(2 + |\mathrm{Re}(z)|)$ for all $z \in \mathbb{C}$.*

B. Basic Properties of Sine-Type Functions

Two important properties of sine-type functions, which will be used in the proof, are stated in Lemmas 1 and 2.

Lemma 1. *Let f be a function of sine type, whose zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$ are ordered increasingly according to their real parts. Then we have*

$$\inf_{k\in\mathbb{Z}} |\lambda_{k+1} - \lambda_k| \ge \underline{\delta} > 0 \tag{4}$$

and

$$\sup_{k\in\mathbb{Z}} |\lambda_{k+1} - \lambda_k| \le \overline{\delta} < \infty \tag{5}$$

for some constants $\underline{\delta}$ and $\overline{\delta}$.

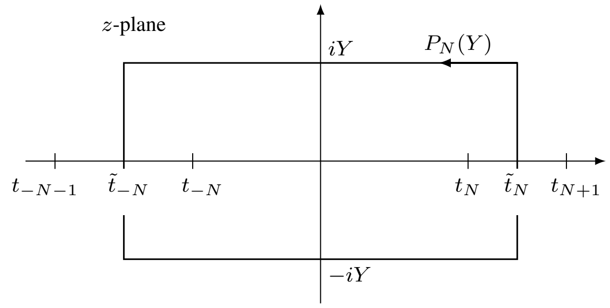

Fig. 1. Integration path $P_N(Y)$ in the complex plane.

Equation (4) follows directly from Definition 2 and the proof of (5) can be found in [8, p. 164].

Lemma 2. *Let f be a function of sine type. For each $\epsilon > 0$ there exists a number $C_6 > 0$ such that*

$$|f(x+iy)| \geq C_6 \, \mathrm{e}^{\pi|y|}$$

outside the circles of radius ϵ centered at the zeros of f.

A proof of Lemma 2 can be found in [9, p. 144]. For further information about sine-type functions see for example [8], [9].

C. Proof of Theorem 1

We first prove the second assertion of the theorem. To this end, we extend the proof technique from [13] and [11], which was developed to obtain results, similar to those in this paper, for $\mathcal{B}_\pi^\infty$.

Let ϕ be an arbitrary but fixed sine-type function, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly. Furthermore, let ϕ_k be defined as in (3), and let $0 < \beta < 1$, $f \in \mathrm{BMO}_{\beta\pi}$, and $T > 0$ be arbitrary but fixed. A key equation for the proof is the identity

$$f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi i} \oint_{P_N(Y)} \frac{\phi(t)}{(\zeta - t)} \frac{f(\zeta)}{\phi(\zeta)} \, \mathrm{d}\zeta, \quad (6)$$

which is valid for all $N \in \mathbb{N}$, $Y > 0$, and $t \in \mathbb{R}$ with $\tilde{t}_{-N} < t < \tilde{t}_N$, where

$$\tilde{t}_N = \begin{cases} (t_{N+1} + t_N)/2 & \text{for } N \geq 1 \\ (t_{N-1} + t_N)/2 & \text{for } N \leq -1. \end{cases} \quad (7)$$

The integration path $P_N(Y)$ is depicted in Figure 1. Equation (6) can be easily verified using the residue theorem.

Let

$$\underline{\delta} = \inf_{k\in\mathbb{Z}} |\lambda_{k+1} - \lambda_k|$$

and

$$\overline{\delta} = \sup_{k\in\mathbb{Z}} |\lambda_{k+1} - \lambda_k|.$$

According to Lemma 1, we have $\underline{\delta} > 0$ and $\overline{\delta} < \infty$. Further, let N_0 be the smallest natural number for which $N_0\underline{\delta} > T$. Since $\tilde{t}_N \geq N\underline{\delta}$ for all $N \in \mathbb{N}$, it follows that $\tilde{t}_{N_0} > T$. Furthermore, let $Y_N = N\overline{\delta}$. From (6) we see that

$$\begin{aligned} &\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \\ &\leq \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + iy)}{\phi(\tilde{t}_N + iy)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + iy - t|} \, \mathrm{d}y \\ &+ \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + iy)}{\phi(\tilde{t}_{-N} + iy)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + iy - t|} \, \mathrm{d}y \\ &+ \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left| \frac{f(x + iY_N)}{\phi(x + iY_N)} \right| \frac{|\phi(t)|}{|x + iY_N - t|} \, \mathrm{d}x \\ &+ \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left| \frac{f(x - iY_N)}{\phi(x - iY_N)} \right| \frac{|\phi(t)|}{|x - iY_N - t|} \, \mathrm{d}x \end{aligned} \quad (8)$$

for all $N \geq N_0$ and $t \in [-T, T]$. Next, we treat the integrals on the right hand side of (8) separately. Because of (4) and the definition of $\tilde{t}_N$, it follows that the distance between $\tilde{t}_N$ and the nearest zero of ϕ is at least $\underline{\delta}/2$. Hence, according to Lemma 2, there exists a constant $C_7 > 0$ such that $|\phi(\tilde{t}_N + iy)| \geq C_7 \, \mathrm{e}^{\pi|y|}$ for all $y \in \mathbb{R}$. Further, let γ satisfy $\beta\pi < \gamma < \pi$. Then we have

$$|f(\tilde{t}_N + iy)| \leq C_5 \, \mathrm{e}^{\gamma|y|} \log(2 + \tilde{t}_N)$$

for all $y \in \mathbb{R}$, according to Theorem 4. Therefore, for the first integral we obtain

$$\begin{aligned} &\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + iy)}{\phi(\tilde{t}_N + iy)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + iy - t|} \, \mathrm{d}y \\ &\leq \frac{C_5 \log(2 + \tilde{t}_N) \|\phi\|_\infty}{2\pi C_7} \int_{-Y_N}^{Y_N} \frac{\mathrm{e}^{-(\pi-\gamma)|y|}}{|\tilde{t}_N + iy - t|} \, \mathrm{d}y, \\ &\leq \frac{C_5 \log(2 + \tilde{t}_N) \|\phi\|_\infty}{\pi C_7 (N\underline{\delta} - T)} \frac{(1 - \mathrm{e}^{-(\pi-\gamma)Y_N})}{(\pi - \gamma)} \\ &\leq \frac{C_5 \log(2 + (N+1)\overline{\delta}) \|\phi\|_\infty}{\pi C_7 (N\underline{\delta} - T)(\pi - \gamma)} \end{aligned}$$

for all $N \geq N_0$ and $t \in [-T, T]$. It follows that

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + iy)}{\phi(\tilde{t}_N + iy)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + iy - t|} \, \mathrm{d}y = 0. \quad (9)$$

For the second integral in (8) we obtain by the same considerations that

$$\begin{aligned} &\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + iy)}{\phi(\tilde{t}_{-N} + iy)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + iy - t|} \, \mathrm{d}y \\ &\qquad \leq \frac{C_5 \log(2 + (N+1)\overline{\delta}) \|\phi\|_\infty}{\pi C_7 (N\underline{\delta} - T)(\pi - \gamma)} \end{aligned}$$

for all $N \geq N_0$ and $t \in [-T, T]$, and consequently

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + iy)}{\phi(\tilde{t}_{-N} + iy)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + iy - t|} \, \mathrm{d}y = 0. \quad (10)$$

Next we treat the third integral in (8). Since all zeros of ϕ are real and $Y_N = N\overline{\delta} \geq \overline{\delta}$, it follows from Lemma 2 that there exists a constant $C_8 > 0$ such that

$$|\phi(x+iY_N)| \geq C_8\,\mathrm{e}^{\pi Y_N}$$

for all $x \in \mathbb{R}$. Further, we have

$$|f(x+iY_N)| \leq C_5\,\mathrm{e}^{\gamma Y_N}\log(2+|x|)$$

for all $x \in \mathbb{R}$, according to Theorem 4. Thus, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x+iY_N)}{\phi(x+iY_N)}\right|\frac{|\phi(t)|}{|x+iY_N-t|}\,\mathrm{d}x \\
&\leq \frac{C_5\,\mathrm{e}^{\gamma Y_N}\|\phi\|_\infty}{2\pi C_8\,\mathrm{e}^{\pi Y_N}}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\frac{\log(2+|x|)}{|x+iY_N-t|}\,\mathrm{d}x \\
&\leq \frac{C_5\,\mathrm{e}^{-(\pi-\gamma)Y_N}\|\phi\|_\infty(2N+1)\overline{\delta}\log(2+(N+1)\overline{\delta})}{2\pi C_8 Y_N} \\
&= \frac{C_5\,\mathrm{e}^{-(\pi-\gamma)N\overline{\delta}}\|\phi\|_\infty(2N+1)\overline{\delta}\log(2+(N+1)\overline{\delta})}{2\pi C_8 N\overline{\delta}} \\
&\leq \frac{2C_5\,\mathrm{e}^{-(\pi-\gamma)N\overline{\delta}}\|\phi\|_\infty\log(2+(N+1)\overline{\delta})}{\pi C_8}
\end{aligned}$$

for all $N \geq N_0$ and $t \in [-T,T]$, and consequently

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x+iY_N)}{\phi(x+iY_N)}\right|\frac{|\phi(t)|}{|x+iY_N-t|}\,\mathrm{d}x = 0. \tag{11}$$

By the same considerations we obtain for the fourth integral in (8) that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x-iY_N)}{\phi(x-iY_N)}\right|\frac{|\phi(t)|}{|x-iY_N-t|}\,\mathrm{d}x \\
&\qquad\leq \frac{2C_5\,\mathrm{e}^{-(\pi-\gamma)N\overline{\delta}}\|\phi\|_\infty\log(2+(N+1)\overline{\delta})}{\pi C_8}
\end{aligned}$$

for all $N \geq N_0$ and $t \in [-T,T]$, and

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x-iY_N)}{\phi(x-iY_N)}\right|\frac{|\phi(t)|}{|x-iY_N-t|}\,\mathrm{d}x = 0. \tag{12}$$

Combining (8), (9), (10), (11), and (12) we see that

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\left|f(t)-\sum_{k=-N}^{N}f(t_k)\phi_k(t)\right| = 0,$$

which proves the second assertion of the theorem.

Next, we prove the first assertion of the theorem. Let $T > 0$ and $f \in \mathrm{BMO}_\pi$ be arbitrary but fixed, and choose some $\hat{\beta}$ with $0 < \hat{\beta} < 1$. According to Theorem 3 there exist two functions $f_3 \in \mathcal{B}_\pi^\infty$ and $f_4 \in \mathrm{BMO}_{\hat{\beta}\pi}$ and a constant α such that $f = f_3+f_4+\alpha$ and $\|f_3\|_\infty \leq C_4(\hat{\beta})\|f\|_{\mathrm{BMO}(\mathbb{R})}$. It follows that

$$\begin{aligned}
\left|f(t)-\sum_{k=-N}^{N}f(t_k)\phi_k(t)\right| &\leq \left|f_3(t)-\sum_{k=-N}^{N}f_3(t_k)\phi_k(t)\right| \\
&\quad+\left|f_4(t)-\sum_{k=-N}^{N}f_4(t_k)\phi_k(t)\right| \\
&\quad+\left|\alpha-\sum_{k=-N}^{N}\alpha\phi_k(t)\right|.
\end{aligned} \tag{13}$$

From Theorem 1 in [11] we know that there exists a constant C_9 such that

$$\sup_{N\in\mathbb{N}}\max_{t\in[-T,T]}\left|f_3(t)-\sum_{k=-N}^{N}f_3(t_k)\phi_k(t)\right| \leq C_9\|f_3\|_\infty$$

and

$$\sup_{N\in\mathbb{N}}\max_{t\in[-T,T]}\left|\alpha-\sum_{k=-N}^{N}\alpha\phi_k(t)\right| \leq C_9\alpha.$$

For the second term on the right hand side of (13) we have

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\left|f_4(t)-\sum_{k=-N}^{N}f_4(t_k)\phi_k(t)\right| = 0$$

according to the second assertion of the theorem. Thus, it follows that

$$\begin{aligned}
\sup_{N\in\mathbb{N}}\max_{t\in[-T,T]}&\left|f(t)-\sum_{k=-N}^{N}f(t_k)\phi_k(t)\right| \\
&\leq C_9(\|f_3\|_\infty+\alpha)+C_{10} \\
&< \infty,
\end{aligned}$$

which completes the proof of the first assertion.

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