Sampling by blocks of measurements in compressed sensing

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***Abstract*—Various acquisition devices impose sampling blocks of measurements. A typical example is parallel magnetic resonance imaging (MRI) where several radio-frequency coils simultaneously acquire a set of Fourier modulated coefficients. We study a new random sampling approach that consists in selecting a set of blocks that are predefined by the application of interest. We provide theoretical results on the number of blocks that are required for exact sparse signal reconstruction. We finish by illustrating these results on various examples, and discuss their connection to the literature on CS.**

Key-words : compressed sensing, blocks of measurements, sampling continuous trajectory, exact recovery, ℓ^1 minimization.

I. Introduction

In many applications, the sampling strategy imposes to acquire data in the form of blocks of measurements (see Fig. 1(b) for block-structured sampling), instead of isolated measurements (see Fig. 1(a)). For instance, in medical echography, images are sampled along lines in the space domain, while, in magnetic resonance imaging (MRI), acquiring data along radial lines or spiral trajectories is a popular sampling strategy. In compressed sensing (CS), various theoretical conditions have been proposed to guarantee the exact reconstruction of a sparse vector from a small number of isolated measurements that are randomly drawn, see [1], [2], [3], and [4] for a detailed review of the most recent results on this topic.

In a noise-free setting, the focus of the present paper is on studying the problem of exact recovery of a sparse signal in the case where the sampling strategy consists in randomly choosing blocks of measurements. Each block corresponds to a set of rows of an orthogonal sensing matrix. Our approach is more flexible than the angle chosen in [5], while we assert theoretical guarantees on the exact reconstruction of sparse signals from blocks of measurements. Moreover, we assume that physical acquisition devices impose block-structured measurements, whereas in [6], or in [7] the authors consider a block-sparse signal.

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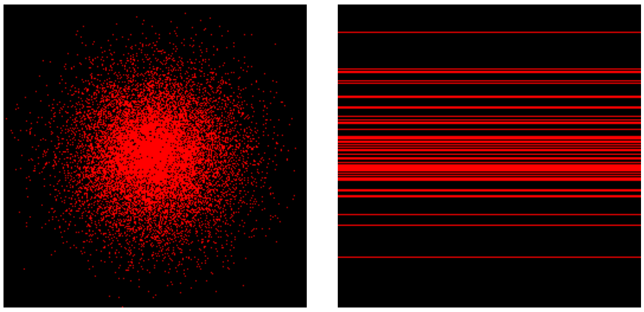

(a) (b)

Fig. 1. **An example of two sampling schemes in the 2D Fourier domain with an undersampling factor** $R = 4$ (a): Isolated points and radial distribution. (b): Corresponding acquisition in the case of block measurements that consist of lines in the 2D Fourier domain.

In this paper, we deal with the case where the blocks are predefined. We give some conditions on the choice of the drawing probability of the blocks and on the number of measurements that are sufficient to obtain an exact recovery by ℓ^1 minimization. We finish by illustrating these results on various examples, and we discuss their connection to the literature on CS.

II. Problem setting

A. Notation

We consider an orthogonal matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ which denotes the full sensing matrix. Matrix $\boldsymbol{A}$ is given a block structure, as follows: $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B}_1 \\ \vdots \\ \boldsymbol{B}_M \end{bmatrix}$, where the blocks $(\boldsymbol{B}_j)_{1\le j\le M}$ are non-overlapping and such that $\boldsymbol{B}_j \in \mathbb{C}^{n_j\times n}$ with $\sum_{j=1}^M n_j = n$. We set $\|\boldsymbol{A}\|_\infty = \max_{1\le i,j\le n} |\boldsymbol{A}_{ij}|$.

Let $(\pi_j)_{1\le j\le M}$ be positive weights with $\sum_{j=1}^M \pi_j = 1$, and let Π be a discrete probability distribution on the set of integers $\{1,\dots,M\}$, associated to these weights. Throughout $(J_k)_{1\le k\le m}$ denotes a sequence of i.i.d. discrete random variables taking their value in $\{1,\dots,M\}$ with distribution Π.

Let $S \subset \{1,\dots,n\}$ be a set of cardinality s. For a matrix $\boldsymbol{M} \in \mathbb{C}^{m\times n}$, we define

$$\boldsymbol{M}^S = (\boldsymbol{M}_{ij})_{1\le i\le m, j\in S}\,.$$

B. The sampling strategy

In this paper, we consider the following sampling strategy. We randomly select m blocks among $(\boldsymbol{B}_j)_{1\leq j\leq M}$, according to the discrete probability distribution Π, which leads to consider the sequence of i.i.d. random blocks $(\boldsymbol{X}_k)_{1\leq k\leq m}$ defined by

$$\boldsymbol{X}_k = \frac{1}{\sqrt{\pi_{J_k}}}\boldsymbol{B}_{J_k},\ k = 1\ldots m \tag{1}$$

We consider the following random sampling matrix

$$\widetilde{\boldsymbol{A}_m} = \frac{1}{\sqrt{m}}\begin{bmatrix}\boldsymbol{X}_1\\ \vdots\\ \boldsymbol{X}_m\end{bmatrix}. \tag{2}$$

It satisfies $\mathbb{E}\left[\widetilde{\boldsymbol{A}_m}^*\widetilde{\boldsymbol{A}_m}\right] = \mathrm{Id}_n$ by construction.

C. Minimization problem

Let $\boldsymbol{y} = \widetilde{\boldsymbol{A}_m}\boldsymbol{x}$ denote a set of $q = \sum_{k=1}^m n_{J_k}$ linear measurements of a signal $\boldsymbol{x}$. To reconstruct $\boldsymbol{x}$, the following standard ℓ_1-minimization problem is solved:

$$\min_{\boldsymbol{z}\in\mathbb{C}^n}\|\boldsymbol{z}\|_1 \quad \text{subject to} \quad \widetilde{\boldsymbol{A}_m}\boldsymbol{z} = \boldsymbol{y}. \tag{3}$$

III. A NON-UNIFORM RECOVERY RESULT

Let us first introduce a new quantity of interest that will be shown to be of primary importance to obtain exact recovery.

Definition III.1

For $S \subset \{1,..,n\}$ we denote by ρ_k^S for $1\leq k\leq M$ any set of positive reals that satisfies

$$\rho_k^S \geq \left\|\left(\boldsymbol{B}_k^S\right)^*\boldsymbol{B}_k^S\right\|_2$$

where $\|\boldsymbol{C}\|_2$ is the spectral norm of a matrix $\boldsymbol{C}$.

The following theorem is the main result of the paper. It gives a set of sufficient conditions for exact recovery of $\boldsymbol{x}$ with large probability.

Theorem III.2

Let $S \subset \{1\ldots n\}$, be a set of cardinality $\sharp\{S\} = s$ and let $\boldsymbol{\epsilon} = (\epsilon_\ell)_{\ell\in S}\in\mathbb{C}^s$ be a sequence of independent random variables that are uniformly distributed on $\{-1;1\}$ (or on the torus $\{z\in\mathbb{C}\,,\,|z|=1\}$).
Let $\boldsymbol{x}$ be a sparse vector with support S and $\mathrm{sgn}(\boldsymbol{x}^S) = \boldsymbol{\epsilon}$.
Let $\widetilde{\boldsymbol{A}_m}$ be the sampling matrix built as above (see (2)).
Assume that

$$\begin{cases} m \geq Cs\ln^2\left(\frac{2^{3/4}n}{\varepsilon}\right)\max\limits_{1\leq k\leq M}\frac{\|\boldsymbol{B}_k^*\boldsymbol{B}_k\|_\infty}{\pi_k} & (4)\\ m\geq C\ln\left(\frac{2^{3/4}s}{\varepsilon}\right)\max\limits_{1\leq k\leq M}\frac{\rho_k^S}{\pi_k} & (5)\end{cases}$$

with $C = 256\kappa^2$, $C' = 32\kappa^2$ and $\kappa^2 = \left(\frac{\sqrt{17}+1}{4}\right)^2$.
Then with probability at least $1-\varepsilon$ the vector $\boldsymbol{x}$ is the unique solution to the ℓ_1-minimization problem (3).

The proof of Theorem III.2 is too long to be written here. It will appear in a forthcoming preprint. The approach is inspired by the results in [4]. To derive Theorem III.2, we had to extend probabilistic tools such as symmetrization and Rudelson's lemma [4] from the vectorial case to the matricial one.

Remark : We can notice that the bounding above of $\left\|\left(\boldsymbol{B}_k^S\right)^*\boldsymbol{B}_k^S\right\|_2$ by ρ_k^S should not be too coarse, at the risk of making the required number of measurements too large.

IV. DISCUSSION AND EXAMPLES

Conditions (4) and (5) may lead to a different optimal drawing probabilty Π^*, in the sense that they can be used to minimize a lower bound on the number m of block measurements. Indeed

- if the right-hand side (rhs) of Inequality (4) is greater than the rhs of Inequality (5), an optimal drawing probability Π^* is defined as follows: $\forall k\in\{1,\ldots,M\}$
$$\pi_k^* = \frac{\|\boldsymbol{B}_k^*\boldsymbol{B}_k\|_\infty}{\sum_{\ell=1}^M\|\boldsymbol{B}_\ell^*\boldsymbol{B}_\ell\|_\infty}.$$
- On the contrary, if the rhs of Inequality (5) prevails, then an optimal drawing probability Π^* turns to be: $\forall k\in\{1,\ldots,M\}$
$$\pi_k^* = \frac{\rho_k^S}{\sum_{\ell=1}^M\rho_\ell^S}.$$

Let us illustrate Theorem III.2 on practical examples.

A. One row blocks - the case of isolated measurements

First, let us show that our result matches the standard setting where blocks are made of only one row. This is the case considered e.g. by [2], [4]. Thus $M = n$,

$$\boldsymbol{A} = \begin{pmatrix}\boldsymbol{B}_1\\ \vdots\\ \boldsymbol{B}_M\end{pmatrix} = \begin{pmatrix}\boldsymbol{a}_1^*\\ \vdots\\ \boldsymbol{a}_n^*\end{pmatrix}$$

where $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n$ are vectors of $\mathbb{C}^n$, and $\forall k\in\{1,\cdots,M\}$, $\boldsymbol{B}_k = \boldsymbol{a}_k^*$. We can set

$$\rho_k^S = s\|\boldsymbol{a}_k\|_\infty^2$$

with $\sharp S = s$. Then, the required number of measurements will be minimized for the following drawing probability: $\forall k\in\{1,\ldots,M\}$

$$\pi_k^* = \frac{\|\boldsymbol{a}_k\|_\infty^2}{\sum_{\ell=1}^n\|\boldsymbol{a}_\ell\|_\infty^2}.$$

According to Theorem III.2 the number of isolated measurements sufficient to obtain perfect reconstruction with high probability is

$$m\geq Cs\ln^2\left(\frac{2^{3/4}3n}{\varepsilon}\right)\sum_{\ell=1}^n\|\boldsymbol{a}_\ell\|_\infty^2. \tag{6}$$

This condition is consistent with [4] for the non-uniform recovery, up to a constant. This additional factor is not too serious, since Theorem III.2 should be mainly considered as a guide to construct sampling patterns and not as a requirement for perfect recovery. Surprisingly, a better drawing probability distribution reducing the required number of measurements is not the uniform one, as commonly used in [8], [4], but the one depending on the ℓ_∞-norm of the considered row.

B. Block diagonal case

Let us assume that $\boldsymbol{A}$ is orthogonal, and $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{B}_1 \\ \vdots \\ \boldsymbol{B}_M \end{pmatrix} = \begin{pmatrix} \boldsymbol{D}_1 & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{D}_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{D}_M \end{pmatrix}.$$

Then $\|\boldsymbol{B}_k^* \boldsymbol{B}_k\|_\infty = 1$ and ρ_k^S can be taken equal to 1 for all $k \in \{1, \dots, M\}$, since $\boldsymbol{D}_k$ is orthogonal. Thus, the block diagonal case corresponds to a uniform bound for ρ_k^S. Therefore, both Inequalities (4) and (5) entail a uniform drawing probability as an optimal choice. Here, we see that no matter how large the block is, an optimal drawing probability Π^* is the uniform one: $\forall k \in \{1, \dots, M\}$,

$$\pi_k^* = \frac{1}{M}.$$

Moreover, with such a choice for Π^*, and by Theorem III.2 the number of block measurements sufficient to obtain perfect reconstruction with high probability is

$$m \geq C s \ln^2 \left(\frac{2^{3/4} 3n}{\varepsilon} \right) M. \tag{7}$$

C. 2D Fourier matrix

We now turn to a more realistic setting where signals are sparse in the Dirac basis and blocks of frequencies are probed in the 2D Fourier domain. We consider blocks that consist of discrete lines in the 2D Fourier space as in Fig 1(b). This scenario is close to what can be encountered in MRI, echography or some tomographic devices.

We assume that $\sqrt{n} \in \mathbb{N}$ and that $\boldsymbol{A}$ is the 2D Fourier matrix applicable on $\sqrt{n} \times \sqrt{n}$ images. For all $p_1 \in \{1, \dots, \sqrt{n}\}$,

$$\boldsymbol{B}_{p_1} = \left[\frac{1}{\sqrt{n}} \exp\left(2i\pi \left(\frac{p_1 \ell_1 + p_2 \ell_2}{\sqrt{n}} \right) \right) \right]_{(p_1, p_2)(\ell_1, \ell_2)} \tag{8}$$

with $1 \leq p_2 \leq \sqrt{n}, 1 \leq \ell_1, \ell_2 \leq \sqrt{n}$. Let $S \subset \{1, \dots, \sqrt{n}\} \times \{1, \dots, \sqrt{n}\}$ denote the support of $\boldsymbol{x}$, with $\sharp S = s$. We can write $S = \{(S_{1,1}, S_{1,2}), (S_{2,1}, S_{2,2}), \dots, (S_{s,1}, S_{s,2})\}$, and we call $S_1 = \{S_{1,1}, S_{2,1}, \dots, S_{s,1}\}$ and $S_2 = \{S_{1,2}, S_{2,2}, \dots, S_{s,2}\}$. We can rewrite $\boldsymbol{B}_{p_1}^S$ as

$$\underbrace{\left(\frac{1}{n^{1/4}} e^{-2i\pi p_2 \frac{\ell_2}{\sqrt{n}}} \right)_{\substack{1 \leq p_2 \leq \sqrt{n} \\ \ell_2 \in S_2}}}_{\substack{\boldsymbol{M}^S \\ \sqrt{n} \times s \text{ matrix}}} \underbrace{\begin{pmatrix} \ddots & 0 & 0 \\ 0 & \frac{1}{n^{1/4}} e^{-2i\pi p_1 \frac{\ell_1}{\sqrt{n}}} & 0 \\ 0 & 0 & \ddots \end{pmatrix}_{\ell_1 \in S_1}}_{\substack{\boldsymbol{D}_{p_1} \\ s \times s \text{ diagonal matrix}}}$$

so
$$\begin{aligned} \left\| \left(\boldsymbol{B}_{p_1}^S\right)^* \boldsymbol{B}_{p_1}^S \right\|_2 &= \left\| \boldsymbol{D}_{p_1}^* \boldsymbol{M}^{S*} \boldsymbol{M}^S \boldsymbol{D}_{p_1} \right\|_2 \\ &\leq \left\| \boldsymbol{D}_{p_1}^* \right\|_2 \left\| \boldsymbol{M}^{S*} \boldsymbol{M}^S \right\|_2 \left\| \boldsymbol{D}_{p_1} \right\|_2 \\ &\leq \frac{1}{n^{1/2}} \left\| \boldsymbol{M}^{S*} \boldsymbol{M}^S \right\|_2 . \end{aligned}$$

In fact, we can see $\boldsymbol{M}^S$ as 1D Fourier matrix $\boldsymbol{M}$, from which we select columns, eventually repeated, the indexes of which are in S_2. Now we have to evaluate the quantity $\left\| \boldsymbol{M}^{S*} \boldsymbol{M}^S \right\|_2$. To do so, let us denote by $(s_j)_{j=1..\sqrt{n}}$ the number of repetitions of the j-th element of $\{1, \dots, \sqrt{n}\}$ in S_2. We have that $\sum_{j=1}^{\sqrt{n}} s_j = s$, and $0 \leq s_j \leq \sqrt{n}$, $\forall j \in \{1, \dots, \sqrt{n}\}$.

Simple calculation leads to the following upper bound:

$$\left\| \boldsymbol{M}^{S*} \boldsymbol{M}^S \right\|_2 \leq \frac{\max_{j=1,\dots,\sqrt{n}} s_j}{\sqrt{n}} \leq \frac{\min(s, \sqrt{n})}{\sqrt{n}},$$

which leads to the choice

$$\rho_k^S = \frac{\min(s, \sqrt{n})}{n}, \ k = 1, \dots, M.$$

By definition of the 2D Fourier matrix of size $n \times n$, $\|\boldsymbol{B}_k^* \boldsymbol{B}_k\|_\infty = 1/\sqrt{n}$, for all $k \in \{1, \dots, \sqrt{n}\}$. Then, the choice of the optimal drawing probability is given by $\forall k \in \{1, \dots, \sqrt{n}\}$

$$\pi_k^* = \frac{1}{\sqrt{n}}.$$

We deduce that the number of block measurements sufficient to ensure exact recovery with high probability is

$$m \geq C s \ln^2 \left(\frac{2^{3/4} n}{\varepsilon} \right).$$

D. Wavelet Transform

Here, we consider that $\boldsymbol{A}$ is a dyadic wavelet transform matrix, with $n = 2^\alpha$, $\alpha \in \mathbb{N}$. To each resolution level $k \in \{0, \dots, \alpha\}$ ($k = 0$, corresponding to the scaling function), we associate the block $\boldsymbol{B}_k$

$$\boldsymbol{B}_k = \left(\Psi_{k,j}(\ell) \right)_{\substack{j = 1 \dots n_k \\ 1 \leq \ell \leq n}}, \tag{9}$$

where $\Psi_{k,j}$ is the discrete wavelet at scale k and location parameter j, ℓ is the time variable and n_k is the number of wavelets (or scaling function) at scale k defined as follows

$$n_k = \begin{cases} 1 & \text{if} \quad k = 0 \\ 2^{k-1} & \text{if} \quad k \geq 1. \end{cases}$$

Although this example is not realistic in practice, it provides an interesting illustration of Theorem III.2. Let S be a set of indexes of cardinality s. Then $\boldsymbol{B}_k^S$ can be defined by restricting ℓ to belong to S, i.e.

$$\boldsymbol{B}_k^S = \left(\Psi_{k,j}(\ell) \right)_{\substack{j = 1 \dots n_k \\ \ell \in S}}.$$

As a consequence, $\left(\boldsymbol{B}_k^S\right)^* \boldsymbol{B}_k^S$ is an $s \times s$ matrix, and

$$\left[\left(\boldsymbol{B}_k^S\right)^* \boldsymbol{B}_k^S\right]_{(\ell,\ell')\in S^2} = \left(\sum_{j=1}^{n_k} \boldsymbol{\psi}_{k,j}(\ell)\, \boldsymbol{\psi}_{k,j}(\ell')\right)_{(\ell,\ell')\in S^2}. \tag{10}$$

By the results in [9], for wavelets with compact support, such as Haar's wavelets, we obtain that $\left\|\left(\boldsymbol{B}_k^S\right)^* \boldsymbol{B}_k^S\right\|_2 \leq \left\|\left(\boldsymbol{B}_k^S\right)^* \boldsymbol{B}_k^S\right\|_\infty s \leq \frac{n_k}{n} s$. Hence, one can take $\rho_k^S = \frac{n_k}{n} s$, and the required number of measurements satisfies the bounds

$$\begin{cases} m \geq C s \ln^2\left(\frac{2^{3/4} n}{\varepsilon}\right) \frac{1}{n} \max\limits_{1\leq k\leq K} \frac{n_k}{\pi_k} & (11) \\ m \geq C' s \ln\left(\frac{2^{3/4} s}{\varepsilon}\right) \frac{1}{n} \max\limits_{1\leq k\leq K} \frac{n_k}{\pi_k} & (12) \end{cases}$$

that m is still proportional to s. If (12) is the strongest condition on m, then an optimal choice for the drawing probability Π^* is

$$\pi_k^* = \frac{n_k}{\sum_{q=0}^{\alpha} n_q}\ k \in \{1, \ldots, K\}.$$

In this setting, the drawing probability is growing with the resolution level k and it is proportional to the block size.

V. Conclusion

In this paper, we have introduced some theoretical tools for the study of the exact recovery of sparse signals from blocks of measurements selected randomly from an orthogonal sensing matrix. We introduced the new quantities ρ_k^S and $\|\boldsymbol{B}_k^* \boldsymbol{B}_k\|_\infty$. They play a central role to derive optimal sampling strategies and to assess the number of block measurements that is necessary to exactly reconstruct sparse signals by ℓ_1-minimization. We plan to calibrate their for orthogonal matrices that appear in applications such as the product of a discrete Fourier transform with a wavelet transform. The extension of this work to overlapping blocks, as presented in Figure 2, offers much more versatility in the sampling patterns.

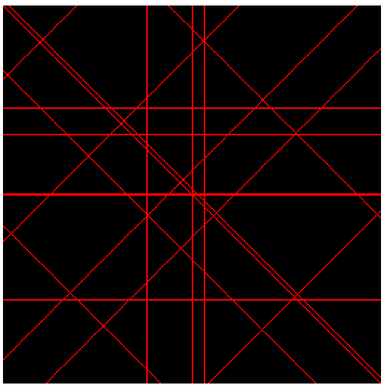

Fig. 2. An example of overlapping blocks of measurements in the 2D Fourier domain.

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