Sparse Recovery with Fusion Frames via RIP

Ulaş Ayaz
Hausdorff Center for Mathematics
Institute for Numerical Simulation
University of Bonn
Endenicher Allee 60, 53115 Bonn, Germany
Email: ulas.ayaz@hcm.uni-bonn.de

Holger Rauhut
Hausdorff Center for Mathematics
Institute for Numerical Simulation
University of Bonn
Endenicher Allee 60, 53115 Bonn, Germany
Email: rauhut@hcm.uni-bonn.de

***Abstract*—We extend ideas from compressed sensing to a structured sparsity model related to fusion frames. We present theoretical results concerning the recovery of sparse signals in a fusion frame from undersampled measurements. We provide both nonuniform and uniform recovery guarantees. The novelty of our work is to exploit an incoherence property of the fusion frame which allows us to reduce the number of measurements needed for sparse recovery.**

I. Introduction

Compressed sensing (CS) predicts that one can efficiently recover a sparse vector from few measurements by solving a convex optimization problem [1]–[3]. Often signals possess more structure than mere sparsity, and exploiting such structure often allows to further reduce the amount of required measurements, see, e.g., [4]. In this paper, we investigate a structured sparsity model related to fusion frames. These were introduced as generalizations of classical frames, in order to better capture the richness of multidimensional signals with an inherent structure [5]. Here, subspaces take the role of the frame vectors.

We investigate sufficient conditions in order to recover a sparse signal in a fusion frame via mixed ℓ_1/ℓ_2 minimization. We both give nonuniform and uniform recovery guarantees. The uniform recovery result is based on the fusion RIP introduced in [6]. Hereby, we improve the recovery conditions given in [6] by exploiting the additional information inherent in the fusion frame structure.

II. Fusion Frames

A *fusion frame* for $\mathbb{R}^d$ is a collection of N subspaces $W_j \subset \mathbb{R}^d$ and associated weights v_j that satisfies

$$A\|x\|_2^2 \leq \sum_{j=1}^{N} v_j^2 \|P_j x\|_2^2 \leq B\|x\|_2^2$$

for all $x \in \mathbb{R}^d$ and for some universal fusion frame bounds $0 < A \leq B < \infty$, where $P_j \in \mathbb{R}^{d\times d}$ denotes the orthogonal projection onto the subspace W_j. For simplicity we assume that the dimensions of the W_j are equal, $\dim(W_j) = k$.

For a fusion frame $(W_j)_{j=1}^N$, let us define the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = \{(x_j)_{j=1}^N : x_j \in W_j,\ \forall j \in [N]\} \subset \mathbb{R}^{d\times N},$$

where we denote $[N] = \{1,\ldots,N\}$. The *mixed $\ell_{2,1}$-norm* of a vector $\mathbf{x} \equiv (x_j)_{j=1}^N \in \mathcal{H}$ is defined as

$$\|(x_j)_{j=1}^N\|_{2,1} \equiv \sum_{j=1}^{N} \|x_j\|_2.$$

Furthermore, the 'ℓ_0-norm' (which is actually not even a quasi-norm) is defined as

$$\|\mathbf{x}\|_0 = \sharp\{j \in [N] : x_j \neq 0\}.$$

We call a vector $\mathbf{x} \in \mathcal{H}$ s-sparse, if $\|\mathbf{x}\|_0 \leq s$. Our sparsity model requires that the 'blocks' x_j are either zero or nonzero as a whole.

A. Sparse Recovery Problem

We take m linear combinations of an s-sparse vector $\mathbf{x}^0 = (x_j^0)_{j=1}^N \in \mathcal{H}$, i.e.,

$$\mathbf{y} = (y_i)_{i=1}^m = \left(\sum_{j=1}^{N} a_{ij} x_j^0\right)_{i=1}^{m}, \quad y_i \in \mathbb{R}^d.$$

Let us denote the block matrices $\mathbf{A_I} = (a_{ij} I_d)_{i\in[m],j\in[N]}$ and $\mathbf{A_P} = (a_{ij} P_j)_{i\in[m],j\in[N]}$ that consist of the blocks $a_{ij} I_d$ and $a_{ij} P_j$ respectively. Here I_d is the identity matrix of size $d \times d$. Then we can formulate this measurement scheme as

$$\mathbf{y} = \mathbf{A_I}\mathbf{x}^0 = \mathbf{A_P}\mathbf{x}^0.$$

We can replace $\mathbf{A_I}$ by $\mathbf{A_P}$ since the relation $P_j x_j = x_j$ holds for all $\mathbf{x} \in \mathcal{H}$ and $j \in [N]$. We wish to recover $\mathbf{x}^0$ from $\mathbf{y}$. This task can be stated as

$$(L0) \quad \hat{\mathbf{x}} = \text{argmin}_{x\in\mathcal{H}} \|\mathbf{x}\|_0 \quad s.t. \quad \mathbf{A_P}\mathbf{x} = \mathbf{y}.$$

This optimization problem is NP-hard. Therefore, we instead propose the following program

$$(L1) \quad \hat{\mathbf{x}} = \text{argmin}_{x\in\mathcal{H}} \|\mathbf{x}\|_{2,1} \quad s.t. \quad \mathbf{A_P}\mathbf{x} = \mathbf{y}.$$

B. Relation with Previous Work

A special case of the sparse recovery problem above appears when all subspaces coincide with the ambient space $W_j = \mathbb{R}^d$ for all j. Then the problem reduces to the well studied *joint sparsity setup* [7] in which all the vectors have the same sparsity structure.

Furthermore, our problem is itself a special case of the *block sparsity setup* [8], with significant additional structure that allows us to enhance existing results. In fact, the fusion frame model assumes the additional prior knowledge that the x_j's are contained in the fusion frame subspaces W_j.

Finally in the case $d = 1$, the projections equal 1, and hence the problem reduces to the *classical recovery problem* $Ax = y$ with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^m$.

C. Incoherence Parameter

We define the parameter λ as a measure of the coherence of the fusion frame subspaces as

$$\lambda = \max_{i \neq j} \|P_i P_j\|_{2\to 2}, \quad i, j \in [N].$$

Note that $\|P_i P_j\|_{2\to 2}$ equals the largest absolute value of the cosines of the principle angles between W_i and W_j. Observe that if the subspaces are all orthogonal to each other, i.e., $\lambda = 0$, then only one measurement suffices to recover $\mathbf{x}^0$ as $y_1 = \sum_j a_{1j} x_j^0$ is an orthogonal decomposition. This observation suggests that fewer measurements are necessary when λ gets smaller. In this work our goal is to provide a solid theoretical understanding of this observation.

D. A Nonuniform Result

We first consider the recovery of a fixed sparse signal from random measurements. To this end, we introduce the Gaussian matrix whose entries consist of independent standard normal distributed random variables and the Bernoulli matrix where the entries are independent random variables taking the values ± 1 with equal probability.

Theorem II.1. *Let $(W_j)_{j=1}^N$ be given with parameter $\lambda \in [0, 1]$ and $\mathbf{x} \in \mathcal{H}$ be s-sparse. Let $A \in \mathbb{R}^{m\times N}$ be a Bernoulli or Gaussian matrix. Assume that*

$$m \geq C(1 + \lambda s) \ln^\alpha(\max\{N, sd\}) \ln(\varepsilon^{-1}), \tag{1}$$

where $C > 0$ is a universal constant. Then with probability at least $1 - \varepsilon$, $(L1)$ recovers $\mathbf{x}$ from $\mathbf{y} = \mathbf{A_P x}$. Here $\alpha = 1$ in the Bernoulli case and $\alpha = 2$ in the Gaussian case.

We provide an outline of the proof in [9]. We remark that Theorems II.1 is also shown to be stable with respect to noise on the measurements and under passing to approximately sparse signals.

III. Sparse recovery using "fusion" RIP

In this section we study uniform recovery of sparse fusion frame signals from their random measurements. One common way to study such recovery conditions is via the restricted isometry property (RIP). A version adapted to fusion frames has been introduced in [6].

Definition III.1 (Fusion RIP)**.** Let $A \in \mathbb{R}^{m\times N}$ and $(W_j)_{j=1}^N$ be a fusion frame for $\mathbb{R}^d$. The fusion restricted isometry constant δ_s is the smallest constant such that

$$(1 - \delta_s)\|\mathbf{x}\|_2^2 \leq \|\mathbf{A_P x}\|_2^2 \leq (1 + \delta_s)\|\mathbf{x}\|_2^2 \tag{2}$$

for all $\mathbf{x} \in \mathcal{H}$ of sparsity $\|\mathbf{x}\|_0 \leq s$.

The following result was also shown in [6].

Proposition III.2 (Fusion RIP implies recovery)**.** *Let $(A, (W_j)_{j=1}^N)$ with fusion RIP constant $\delta_{2s} < 1/3$. Then $(L1)$ recovers all s-sparse $\mathbf{x}$ from $\mathbf{y} = \mathbf{A_P x}$.*

This result shows that given a fusion frame $(W_j)_{j=1}^N$ and matrix A, for uniform recovery it is enough to check whether the block matrix $\mathbf{A_P}$ satisfies the fusion RIP. Recovery is also stable under noise and passing to compressible signals. Another result from [6] tells us that if the underlying random measurement matrix A satisfies the classical RIP, $\mathbf{A_P}$ satisfies fusion RIP with same constants. This suggests that $m \gtrsim s \ln(N/s)$ is sufficient for many random measurement ensembles (up to some log factors). However, the following main result of our work shows that the inherent structure of fusion frames provides additional information that can be exploited to derive stronger recovery conditions.

Theorem III.3. *Let $(W_j)_{j=1}^N$ be given with $\dim(W_j) = k$ and parameter $\lambda \in [0, 1]$. Let $A \in \mathbb{R}^{m\times N}$ be a Bernoulli matrix and $\delta \in (0, 1)$. Assume that*

$$m \geq C\delta^{-2} k\sqrt{\lambda s^2 + s} \ln^4(\max\{N, d\}). \tag{3}$$

Then with probability at least $1 - 2e^{-c\delta^2 m}$, the fusion RIP constant δ_s of $\tilde{\mathbf{A}}_\mathbf{P} = \frac{1}{\sqrt{m}}\mathbf{A_P}$ satisfies $\delta_s \leq \delta$. Above $C, c > 0$ are universal constants.

Theorem III.3 can be extended for the random matrices with independent subgaussian entries. Presently the uniform result (3) behaves slightly worse than the nonuniform one (1) for small λ and suffers from additional log-terms. On the other hand, we gain uniformity and stronger stability.

IV. Proof outline

Due to lack of space, we only present the outline of the proof of Theorem III.3. The detailed proof will appear in a forthcoming journal publication. Let us first give a characterization of the fusion RIP constant. The definition (2) implies that

$$\delta_s = \sup_{\mathbf{x}\in D_{s,N}} \left| \|\tilde{\mathbf{A}}_\mathbf{P}\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2 \right|,$$

where $D_{s,N} := \{\mathbf{x} \in \mathcal{H} : x_i \in W_i, \|\mathbf{x}\|_2 \leq 1, \|\mathbf{x}\|_0 \leq s\}$. Next we derive an estimate for the expectation of δ_s. To this end, we denote $\mathbf{E}_{ij}(Y) \in \mathbb{R}^{md\times Nd}$, $i \in [m], j \in [N]$, as the block matrix (consisting of $m \times N$ blocks) with a single block entry $Y \in \mathbb{R}^{d\times d}$ at position (i, j) and the entry $0 \in \mathbb{R}^{d\times d}$ elsewhere. Let ϵ_{ij} be the entries of A and observe that

$$\tilde{\mathbf{A}}_\mathbf{P}\mathbf{x} = \frac{1}{\sqrt{m}} \sum_{i\in[m], j\in[N]} \epsilon_{ij}(\mathbf{Q}_{ij}\mathbf{x}),$$

where $\mathbf{Q}_{ij} := \mathbf{E}_{ij}(P_j)$. We define the matrix $V_\mathbf{x}$ whose columns are $\frac{1}{\sqrt{m}}\mathbf{Q}_{ij}\mathbf{x}$ for all i, j, i.e.,

$$V_\mathbf{x} = \frac{1}{\sqrt{m}}(\mathbf{Q}_{11}\mathbf{x}|\mathbf{Q}_{12}\mathbf{x}| \ldots |\mathbf{Q}_{mN}\mathbf{x}).$$

Then we can write $\tilde{\mathbf{A}}_{\mathbf{P}}\mathbf{x} = V_{\mathbf{x}}\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is a Bernoulli vector of length mN. Denoting the set $\mathcal{A} = \{V_{\mathbf{x}} : \mathbf{x} \in D_{s,N}\}$, we have

$$\delta_s = \sup_{\mathbf{x}\in D_{s,N}} \left| \|V_{\mathbf{x}}\boldsymbol{\epsilon}\|_2^2 - \|\mathbf{x}\|_2^2 \right| = \sup_{A\in\mathcal{A}} \left| \|A\boldsymbol{\epsilon}\|_2^2 - \mathbb{E}\|A\boldsymbol{\epsilon}\|_2^2 \right|.$$

Following Krahmer et al. [10] where they use chaining methods in order to get bounds for this type of random variables, we obtain

$$\mathbb{E}\sup_{A\in\mathcal{A}} \left| \|A\boldsymbol{\epsilon}\|_2^2 - \mathbb{E}\|A\boldsymbol{\epsilon}\|_2^2 \right| \lesssim d_F(\mathcal{A})d_{2\to 2}(\mathcal{A}) + \left(d_F(\mathcal{A})\gamma_2(\mathcal{A}, \|\cdot\|_{2\to 2}) + \gamma_2(\mathcal{A}, \|\cdot\|_{2\to 2})^2\right). \quad (4)$$

Here, $d_F(\mathcal{A})$ and $d_{2\to 2}(\mathcal{A})$ denote the radius of $\mathcal{A}$ in the Frobenius and the operator norms, respectively. For the definition of Talagrand's γ_2-functional we refer to [11]. It is easy to check that

$$d_{2\to 2}(\mathcal{A}) = \sup_{\mathbf{x}\in D_{s,N}} \|V_{\mathbf{x}}\|_{2\to 2} \le 1/\sqrt{m} \quad \text{and} \quad d_F(\mathcal{A}) = 1.$$

The γ_2-functional can be estimated by the well-known Dudley integral [11]

$$\gamma_2(\mathcal{A}, \|\cdot\|_{2\to 2}) \lesssim \int_0^{d_{2\to 2}(\mathcal{A})} \sqrt{\ln \mathcal{N}(\mathcal{A}, \|\cdot\|_{2\to 2}, u)}\, du, \quad (5)$$

where the covering number $\mathcal{N}(T, d, u)$ is defined as the smallest number of open balls of radius u in (T, d) needed to cover T. Therefore estimating the expectation in (4) amounts to estimating covering numbers which will perform in two different ways similar to [12].

a) Small values of u: For $S \subset [N]$ we introduce the set $B_S^2 := \{\mathbf{x} : \mathrm{supp}(\mathbf{x}) \subset S, \|\mathbf{x}\|_2 \le 1\}$. Furthermore define the norm $|||\mathbf{x}||| := \|V_{\mathbf{x}}\|_{2\to 2}$. Observe that $|||\mathbf{x}||| \le \frac{1}{\sqrt{m}}\|\mathbf{x}\|_2$. Then using subadditivity of covering numbers and a standard volumetric argument (see, e.g., [13, Chapter 8.4]) we obtain

$$\begin{aligned}
&\mathcal{N}(\mathcal{A}, \|\cdot\|_{2\to 2}, u) = \mathcal{N}(D_{s,N}, |||\cdot|||, u) \\
&\le \sum_{\substack{S\subset[N]\\|S|=s}} \mathcal{N}\left(B_S^2, |||\cdot|||, u\right) \le \sum_{\substack{S\subset[N]\\|S|=s}} \mathcal{N}\left(B_S^2, \frac{\|\cdot\|_2}{\sqrt{m}}, u\right) \\
&= \sum_{\substack{S\subset[N]\\|S|=s}} \mathcal{N}\left(B_S^2, \|\cdot\|_2, u\sqrt{m}\right) \le \left(\frac{eN}{s}\right)^s \left(1 + \frac{2}{u\sqrt{m}}\right)^{sk}.
\end{aligned}$$

For $u > 0$, it thus holds

$$\ln \mathcal{N}(\mathcal{A}, \|\cdot\|, u) \le s\ln(eN/s) + sk\ln\left(1 + \frac{2}{u\sqrt{m}}\right). \quad (6)$$

b) Large values of u: We define the set

$$B_{2,1} := \left\{\mathbf{x} \in \mathcal{H} : \|\mathbf{x}\|_{2,1} \le 1, \|\mathbf{x}\|_2 \le \frac{1}{\sqrt{s}}\right\}.$$

Then it is evident that $D_{s,N} \subset \sqrt{s}B_{2,1}$. Therefore,

$$\begin{aligned}
&\mathcal{N}(\mathcal{A}, \|\cdot\|_{2\to 2}, u) = \mathcal{N}(D_{s,N}, |||\cdot|||, u) \\
&\le \mathcal{N}(\sqrt{s}B_{2,1}, |||\cdot|||, u) = \mathcal{N}\left(B_{2,1}, |||\cdot|||, \frac{u}{\sqrt{s}}\right). \quad (7)
\end{aligned}$$

For the task of estimating $\mathcal{N}(B_{2,1}, |||\cdot|||, u)$, we invoke the so-called *empirical method of Maurey*. We fix $u > 0$ and $\mathbf{x} \in B_{2,1}$. The idea is to approximate $\mathbf{x}$ by a finite set of very sparse vectors of ℓ_2-norm 1. In order to construct this set, we discretize the unit sphere of each frame subspace W_j. Denote $S_j = \{\mathbf{y} \in \mathcal{H} : \|y_j\|_2 = 1; y_i = 0,\ i \ne j\}$. A volumetric argument yields that

$$\mathcal{N}(S_j, \|\cdot\|_2, \tilde{\varepsilon}) \le \left(1 + \frac{2}{\tilde{\varepsilon}}\right)^k.$$

For each j, let $T_j \subset S_j$ be the covering set of S_j with this cardinality. We will use 1-sparse elements from the set $\mathcal{T} = \bigcup_{j\in[N]} T_j$ in order to find a vector $\mathbf{z}$ that is close to $\mathbf{x}$. To this end, we define a random vector $\tilde{\mathbf{Z}}$ as follows

$$\mathbb{P}\left(\tilde{\mathbf{Z}} = \overrightarrow{\mathbf{E}}_j\left(\frac{x_j}{\|x_j\|_2}\right)\right) = \|x_j\|_2 \quad \text{for } j \in [N],$$

and $\tilde{\mathbf{Z}} = 0$ with probability $1 - \|\mathbf{x}\|_{2,1}$. Here the notation $\overrightarrow{\mathbf{E}}_j(x)$ corresponds to the block column vector of size N with the vector x in j-th position and 0 elsewhere. Observe that $\mathbb{E}\tilde{\mathbf{Z}} = \mathbf{x}$. Let M be a number to be determined later. Let $\tilde{\mathbf{Z}}_1, \dots, \tilde{\mathbf{Z}}_M$ be independent copies of $\tilde{\mathbf{Z}}$, and put

$$\tilde{\mathbf{z}} = \frac{1}{M}\sum_{\ell=1}^M \tilde{\mathbf{Z}}_\ell.$$

We now denote $\mathbf{Z}_\ell \in \mathcal{T}$ as the closest vector to $\tilde{\mathbf{Z}}_\ell$ in the set $\mathcal{T}$ for all ℓ. Then we have $\|\tilde{\mathbf{Z}}_\ell - \mathbf{Z}_\ell\|_2 \le \tilde{\varepsilon}$. The M-sparse vector $\mathbf{z} = \frac{1}{M}\sum_{\ell=1}^M \mathbf{Z}_\ell$ will be our candidate to approximate $\mathbf{x}$. By the triangle inequality

$$|||\mathbf{z} - \mathbf{x}||| \le |||\mathbf{z} - \tilde{\mathbf{z}}||| + |||\tilde{\mathbf{z}} - \mathbf{x}|||. \quad (8)$$

With the choice $\tilde{\varepsilon} = \frac{u\sqrt{m}}{2}$, it is not hard to deduce $|||\mathbf{z} - \tilde{\mathbf{z}}||| \le u/2$. It remains to show that $|||\tilde{\mathbf{z}} - \mathbf{x}||| \le u/2$ with nonzero probability for large enough M. Since $|||\tilde{\mathbf{z}} - \mathbf{x}||| = \left\|\frac{1}{M}\sum_{\ell=1}^M (V_{\tilde{\mathbf{Z}}_\ell} - V_{\mathbf{x}})\right\|_{2\to 2}$ is a sum of centered random matrices, we may invoke the noncommutative Bernstein inequality due to Tropp [14, Theorem 1.6] in order the bound the tail probability of this norm. This leads to the condition

$$M \ge \ln(md + mN)\left(\frac{16\sqrt{\lambda + 1/s}}{mu^2} + \frac{3}{\sqrt{m}u}\right), \quad (9)$$

which implies the existence of a realization of the vector $\tilde{\mathbf{z}}$ for which $|||\tilde{\mathbf{z}} - \mathbf{x}||| \le u/2$. Together with (8) this yields $|||\mathbf{z} - \mathbf{x}||| \le u$. Since each $\mathbf{Z}_\ell \in \mathcal{T}$ takes at most

$$|\mathcal{T}| = \bigcup_{j\in[N]} |T_j| \le N\left(1 + \frac{4}{u\sqrt{m}}\right)^k$$

many values, $\mathbf{z}$ can take at most $N^M\left(1 + \frac{4}{u\sqrt{m}}\right)^{kM}$ values. Setting M to the least integer that satisfies (9), we deduce that the covering numbers can be estimated by

$$\sqrt{\ln \mathcal{N}(B_{2,1}, |||\cdot|||, u)} \le \sqrt{\ln\left[N^M\left(1 + \frac{4}{u\sqrt{m}}\right)^{kM}\right]}$$

$$\leq \sqrt{\frac{16\sqrt{\lambda + 1/s}}{mu^2} + \frac{3}{\sqrt{mu}}} \sqrt{k \ln(D) \ln\left[N\left(1 + \frac{4}{u\sqrt{m}}\right)\right]}, \quad (10)$$

where $D := md + mN$. Finally we estimate the Dudley integral (5) by integrating (6) from 0 to a suitable $\kappa \in (0, 1/\sqrt{m})$ and (10) from κ to $1/\sqrt{m}$ with replacing u by $u/\sqrt{s}$ due to (7). Plugging all estimates derived for $d_{2\to2}(\mathcal{A})$, $d_F(\mathcal{A})$ and $\gamma_2(\mathcal{A}, \|\cdot\|_{2\to2})$ into (4), we obtain $\mathbb{E}\delta_s \leq \delta$, provided Condition (3) of Theorem III.3 holds with an appropriate constant.

The probability estimate for δ_s is derived by applying a concentration inequality provided also in [10] having all complexity parameters at hand. This completes the proof.

V. Numerical Experiments

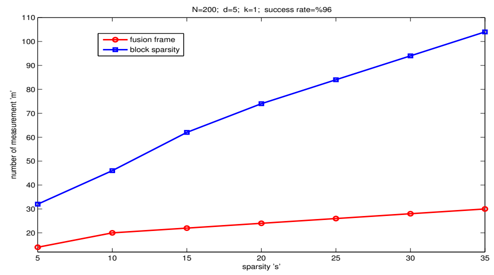
In this section, we compare two sparsity models: Fusion frame and block sparsity. We present numerical experiments that illustrate that the additional knowledge about the fusion frame subspaces, that is $\mathbf{x} \in \mathcal{H}$, significantly improves the recovery compared to the block sparsity case where we do not assume such a knowledge. (See Section II-B.) In all of our experiments, we use SPGL1 [15], [16] to solve the minimization problems.

a) In Fig.1a, we fix a fusion frame with $N = 200$ subspaces in $\mathbb{R}^d$, $d = 5$ with $k = 1$. Then we vary the sparsity level s from 5 to 35, and generate an s-sparse vector $\mathbf{x}$ in the fusion frame. We form $\mathbf{y} = \mathbf{A_P x}$ with a randomly generated Gaussian matrix $A \in \mathbb{R}^{m\times N}$ for different values of m and solve the minimization problem $(L1)$ with and without the constraint that $\mathbf{x} \in \mathcal{H}$. Repeating this test 50 times for each s for both cases, we record the values of m which yield a recovery success rate of at least $\%96$.

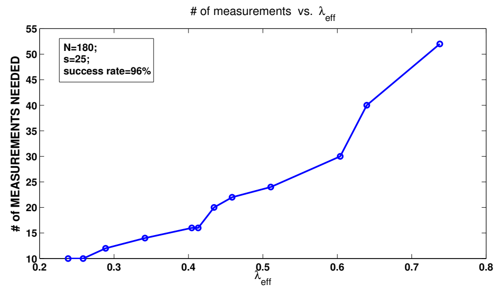
b) Fig.1b depicts a relation between the number of measurements needed m and the incoherence parameter λ_{eff} where

$$\lambda_{\text{eff}} = \frac{1}{s} \max_{i\in[N]} \sum_{j\in S} \|P_i P_j\|_{2\to2}.$$

In the Bernoulli case, the parameter λ in (1) can be replaced by λ_{eff} which is smaller. To that end, we fix the sparsity level to $s = 25$ and generate various fusion frames with $N = 180$ and different values of λ_{eff}. Then we generate an s-sparse vector in each fusion frame and find the number of measurements m which yields an empirical recovery rate of 96%.



(a)



(b)

Fig. 1. (a) $N = 200, d = 5$ fixed, we vary s and plot the number of measurements needed m for the cases where we assume the knowledge the subspaces (fusion frame) and the general block sparsity case. (b) $N = 180, s = 25$ fixed, we plot the number of measurements m vs. λ_{eff}.

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