On the Noise-Resilience of OMP with BASC-Based Low Coherence Sensing Matrices

Henning Zörlein, Dejan E. Lazich and Martin Bossert
Institute of Communications Engineering
Ulm University
89081 Ulm, Germany
{henning.zoerlein, dejan.lazich, martin.bossert}@uni-ulm.de

***Abstract*—In Compressed Sensing (CS), measurements of a sparse vector are obtained by applying a sensing matrix. With the means of CS, it is possible to reconstruct the sparse vector from a small number of such measurements. In order to provide reliable reconstruction also for less sparse vectors, sensing matrices are desired to be of low coherence. Motivated by this requirement, it was recently shown that low coherence sensing matrices can be obtained by Best Antipodal Spherical Codes (BASC) [1]. In this paper, the noise-resilience of the Orthogonal Matching Pursuit (OMP) used in combination with low coherence BASC-based sensing matrices is investigated.**

I. Introduction

In Compressed Sensing (CS), one is particularly interested in the sparsest solution to an underdetermined system of $\mathcal{M}$ linear equations:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

This is commonly interpreted as acquiring a sufficiently k-sparse vector $\boldsymbol{x} \in \mathbb{R}^{\mathcal{N}}$ from a small number of measurements. A so called sensing matrix $\boldsymbol{A} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ describes these measurements enlisted in $\boldsymbol{b} \in \mathbb{R}^{\mathcal{M}}$, where $\mathcal{M}$ is significantly smaller than $\mathcal{N}$.

However, for practical applications, measurement noise will always be present. Therefore, an additional noise term $\boldsymbol{n} \in \mathbb{R}^{\mathcal{M}}$ consisting of Gaussian distributed elements with zero mean is usually considered in the system model (e.g. [2]):

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{n} = \boldsymbol{b} + \boldsymbol{n}.$$

There are multiple approaches to reconstruct the sparse vector $\hat{\boldsymbol{x}}$ out of its measurements $\boldsymbol{b}$, e.g. the Basis Pursuit (BP) and Basis Pursuit De-Noising (BPDN) algorithms [3] based on convex relaxation, or greedy algorithms like the Orthogonal Matching Pursuit (OMP) [4].

The selection of suitable sensing matrices $\boldsymbol{A}$ is crucial for a successful reconstruction. There are multiple properties providing conditions on sensing matrices, e.g. the worst-case coherence μ between columns of the sensing matrix [5]–[9]. The worst-case coherence is defined by the maximal absolute value of the inner product between two distinct columns of $\boldsymbol{A}$:

$$\mu = \max_{i \neq j} |\boldsymbol{a}_i \cdot \boldsymbol{a}_j|, \tag{1}$$

where $\boldsymbol{a}_i$ is the ith column. Motivated by these coherence properties, the construction of Best Antipodal Spherical Codes (BASC)-based sensing matrices with low worst-case coherence has been proposed in [1].

Other approaches for guarantees on successful reconstruction utilize the Restricted Isometry Property (RIP) [10]. Normalized Gaussian random matrices are often used as sensing matrices, because they fulfill the RIP with high probability [10]. Due to its combinatorial nature, the direct evaluation of a matrix for its RIP is not tractable. However, Monte Carlo experiments can be performed, which indicate the suitability of BASC-based sensing matrices with respect to the RIP [1].

The reason for using BASC-based sensing matrices and their construction is briefly summarized in Section II as given in [1]. An analysis on the noise-resilience based on numerical simulations is given in Section III. In Section IV, conclusions are provided as well.

II. BASC-based Sensing Matrices

A. Spherical Codes

Any finite set of M points placed on the surface of the N-dimensional unit sphere centered at the origin of the N-dimensional Euclidean space $\mathbb{R}^N$ is called a spherical code and denoted by $C_s(N,M)$ [11]–[13]. A point of $C_s(N,M) = \{\boldsymbol{s}_m\}_{m=1}^{M}$ is determined by its corresponding code word $\boldsymbol{s}_m = (s_{m1}, \ldots, s_{mn}, \ldots, s_{mN})$ representing a unit position vector ($|\boldsymbol{s}_m| = 1, m = 1, \ldots, M$) whose components $s_{mn} \in \mathbb{R}$ are the coordinates of the point in some reference Cartesian coordinate system centered at the origin. Best Spherical Codes (BSC), $C_{\mathrm{bs}}(N,M)$, are spherical codes which maximize the minimal Euclidean (or angular) distance $d_{ml} = |\boldsymbol{s}_m - \boldsymbol{s}_l|$ between any two points (or equivalently, minimize the maximal inner product of the corresponding code words). All rotations of a BSC are usually regarded as the same, therefore, a BSC is characterized only by its distance distribution $\mathfrak{D} = \{d_{ml}\}_{m<l}$. For some specific (N,M) pairs $(N > 2, M > N)$, the corresponding BSC can be unique, however, there also exist (N,M) pairs with more than one corresponding BSC (these BSCs have different distance distributions but the same minimal distance). For BASC, $C_{\mathrm{bas}}(N,M)$, the antipodal of each code word is also a code word:

$$\boldsymbol{s}_m \in C_{\mathrm{bas}}(N,M) \iff -\boldsymbol{s}_m \in C_{\mathrm{bas}}(N,M).$$

This property can be used in order to construct low coherence sensing matrices [1].

B. Sensing Matrices Based on Spherical Codes

The $M = \mathcal{N}$ code words of a spherical code $C_s(N, M)$ can be regarded as $\mathcal{N}$ columns of a sensing matrix $\boldsymbol{A} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$. It can be easily shown that the squared Euclidean distance between code words is proportional to their inner product. However, the worst-case coherence is defined over the absolute value of the inner product, see Equation (1). Consequently, the search for sensing matrices with smallest worst-case coherence μ is transformed into the search for BASC with $N = \mathcal{M}$ and $M = 2\mathcal{N}$. The $\frac{M}{2}$ non-antipodal code words of $C_{bas}(N, M)$ are the columns of the desired sensing matrix.

C. Obtaining BSC

The points of spherical codes can be considered as M charged particles on the unit sphere acting in some field of repelling forces [14]. Starting from any initial position, such particles will move until the total potential energy of the system approaches some local minimum. In any one of these local minima the particles will settle causing a stable or unstable equilibrium of mutual repelling forces. In [15], such a generalized potential function, $g(\mathfrak{D})$, was introduced. For a specific form of $g(\mathfrak{D})$ given in [16] by

$$g(\mathfrak{D}) = \sum_{m<l} |\boldsymbol{s}_m - \boldsymbol{s}_l|^{-(\nu-2)}, \tag{2}$$

where $\nu \in \mathbb{N}$ $(\nu > 2)$, it was shown that the global minimum of $g(\mathfrak{D})$ is attained by a BSC if $\nu \to \infty$.

As further summarized in [1], the set of fixed points of two mappings can be regarded as the desired minima.

The first mapping $\boldsymbol{F}$ can be interpreted as collection of effective forces $\boldsymbol{f}_m$ acting on the code words $\boldsymbol{s}_m$ of a spherical code and is given by

$$\boldsymbol{F}[C_s(N,M)] = \left\{\underline{\boldsymbol{f}}_m(C_s(N,M))\right\}_{m=1}^{M}$$
$$= \left\{ \boldsymbol{f}_m = \frac{\sum\limits_{l\neq m} \left[(\boldsymbol{s}_m - \boldsymbol{s}_l)/|\boldsymbol{s}_m - \boldsymbol{s}_l|^\nu\right]}{\left|\sum\limits_{l\neq m} \left[(\boldsymbol{s}_m - \boldsymbol{s}_l)/|\boldsymbol{s}_m - \boldsymbol{s}_l|^\nu\right]\right|} \right\}_{m=1}^{M}$$

or, with the underlined denotation of unit vectors $\underline{\boldsymbol{u}} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|}$, by

$$\left\{ \underline{\boldsymbol{f}}_m = \underline{\sum_{l\neq m} \frac{\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l}{|\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l|^\nu}} = \underline{\sum_{l\neq m} \boldsymbol{\delta}_{ml}} \right\}_{m=1}^{M}. \tag{3}$$

With the help of $\boldsymbol{F}$ a second mapping can be defined by

$$\boldsymbol{\Phi}[C_s(N,M)] = \left\{\underline{\underline{\boldsymbol{s}}_m + \alpha \underline{\boldsymbol{f}}_m}\right\}_{m=1}^{M}, \tag{4}$$

where $\underline{\boldsymbol{f}}_m$ is given by (3) and $\alpha \in \mathbb{R}$. It is evident that the mappings $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ have the same set of fixed points. For a small enough "damping factor" α, the iterative process

$$C_s(N,M)^{(k+1)} = \boldsymbol{\Phi}(C_s(N,M)^{(k)});\ k = 0, 1, \ldots \tag{5}$$

converges to one of the fixed points of the function $\boldsymbol{\Phi}$, and consequently of $\boldsymbol{F}$.

It was also shown [17] that, generally, for ν large enough, all fixed points correspond to spherical codes whose minimal distances are close enough to the minimal distance of corresponding BSCs. Consequently by finding any fixed point using (5) with ν large enough, the corresponding spherical code will be very close to the best one.

D. Obtaining BASC

The construction of BSC can be easily adapted for BASC, by considering additional antipodal points [1], leading to a new mapping and new forces acting on the particles respectively:

$$\left\{ \underline{\boldsymbol{f}}_m = \underline{\sum_{l\neq m} \left[\frac{\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l}{|\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l|^\nu} + \frac{\underline{\boldsymbol{s}}_m + \underline{\boldsymbol{s}}_l}{|\underline{\boldsymbol{s}}_m + \underline{\boldsymbol{s}}_l|^\nu} \right]} \right\}_{m=1}^{M}. \tag{6}$$

After the mapping (4) is applied, the antipodal points need to be updated. The resulting algorithm is given in Fig. 1.

III. Noise-Resilience Determined by Numerical Evaluations

The frequency of successful reconstruction[1] is evaluated over the sparsity of $\boldsymbol{x}$, where the non-zero values are drawn from a Gaussian distribution with zero mean and unit variance, and over the Signal-to-Noise-(power)-Ratio SNR, with

$$\mathsf{SNR}\ [\mathsf{dB}] = 10 \cdot \log_{10} \left(\frac{\sum\limits_{i=1}^{\mathcal{M}} |b_i|^2}{\sum\limits_{i=1}^{\mathcal{M}} |n_i|^2} \right),$$

where b_i and n_i are the components of the corresponding vectors $\boldsymbol{b}$ and $\boldsymbol{n}$. For the construction of BASC-based matrices, we used the initial values as given in the algorithm description presented in Fig. 1. For the stopping criterion of the OMP algorithm, we assume knowledge of the noise power: If the ℓ_2-norm of the residual is less than the ℓ_2-norm of the noise plus some small threshold (10^{-6}), the OMP algorithm will stop. Our simulations indicate that OMP also performs well for an overestimation of the noise power, therefore, the assumption of a known noise power is not too restrictive. All simulations have been performed in MATLAB® [18].

Column-normalized random matrices with entries drawn from a Gaussian distribution have also been considered for comparisons. For the numerical evaluation, a version of each matrix type has been computed. The frequency of successful reconstruction has been determined over 7500 simulations. The corresponding result is shown in Fig. 2 and Fig. 3 for matrices of size 64×128. We repeated such simulations for multiple different realizations of the discussed matrices, however, the results did not show significant differences.

As it can be seen in Fig. 2 and Fig. 3, the signal must be strong enough in order to allow sparse recovery by the OMP. For high SNR levels, the sparsity is the dominating factor,

[1]The reconstruction is considered to be successful, if the condition $|\hat{\boldsymbol{x}} - \boldsymbol{x}| < 10^{-3}$ is fulfilled.

procedure BASC-BASED SENSING MATRIX($\mathcal{M}, \mathcal{N}$)
  $N \leftarrow \mathcal{M}$
  $M \leftarrow 2\mathcal{N}$
  $\alpha_{init} \leftarrow 0.9$
  $\nu \leftarrow 2$
  $\nu_{max} \leftarrow 2^{10}$
  $i_{max} \leftarrow 10^5$
  $\epsilon \leftarrow 10^{-10}$
  $C_s \leftarrow$ arbitrary ▷ Random spherical code
  $C_{as} \leftarrow [C_s \quad -C_s]$ ▷ Antipodal spherical code
  $\alpha \leftarrow \alpha_{init}$
  while $\nu < \nu_{max}$ **do**
    FixedPoint $\leftarrow$ **false**
    $i \leftarrow 0$
    while $i < i_{max}$ **AND** FixedPoint $=$ **false do**
      for $m = 1$ to $\frac{M}{2}$ **do**
        $\underline{\boldsymbol{f}}_m \leftarrow \boldsymbol{0}$
        for $l = 1$ to M **do**
          if $l \neq m$ **AND** $l \neq m + \mathcal{N}$ **then**
            $\underline{\boldsymbol{f}}_m \leftarrow \underline{\boldsymbol{f}}_m + \frac{\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l}{\left|\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l\right|^{\nu}}$
          end if
        end for
      end for
      $\{s_m\}_{m=1}^{\frac{M}{2}} \leftarrow \left\{\underline{\boldsymbol{s}}_m + \alpha \underline{\boldsymbol{f}}_m\right\}_{m=1}^{\frac{M}{2}}$
      $\{s_m\}_{m=1+\frac{M}{2}}^{M} \leftarrow \{-\underline{\boldsymbol{s}}_m\}_{m=1}^{\frac{M}{2}}$
      if all $\left|\underline{\boldsymbol{f}}_m - \underline{\boldsymbol{s}}_m\right| < \epsilon$ **then**
        FixedPoint $\leftarrow$ **true**
      end if
      $i \leftarrow i + 1$
    end while
    $\nu \leftarrow 2\nu$
    $\alpha \leftarrow \frac{\alpha_{init}}{\nu - 1}$
  end while
  return $A \leftarrow \{\underline{\boldsymbol{s}}_m\}_{m=1}^{\frac{M}{2}}$
end procedure

Fig. 1. The construction algorithm for BASC-based sensing matrices.

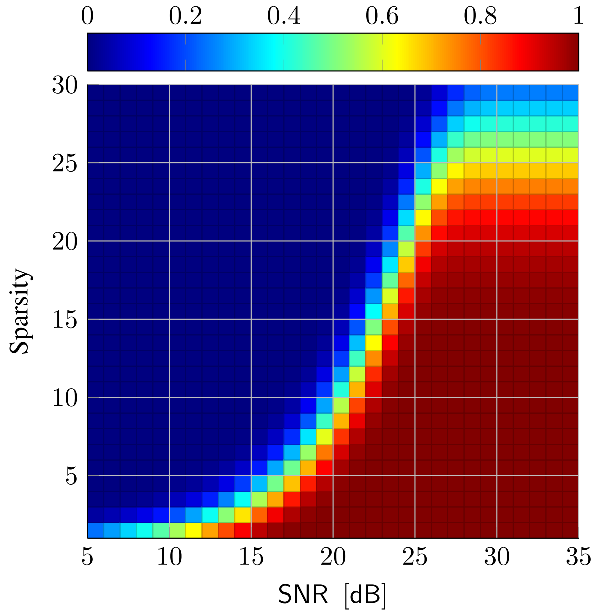

Fig. 2. Frequency of exact reconstruction for normalized Gaussian matrices with $\mathcal{M} = 64$ and $\mathcal{N} = 128$.

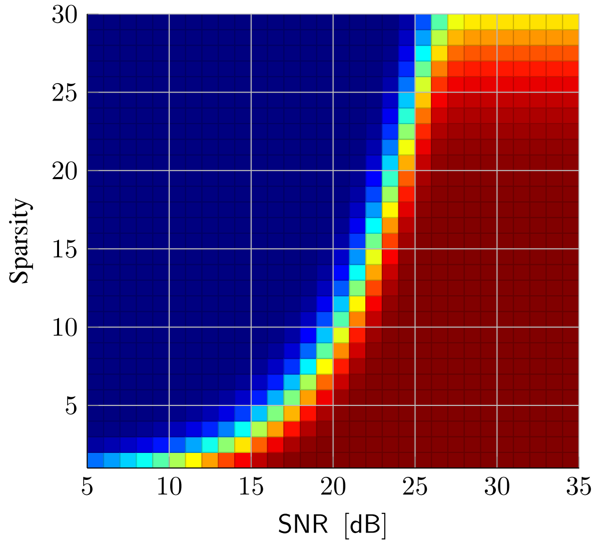

Fig. 3. Frequency of exact reconstruction for BASC-based matrices with $\mathcal{M} = 64$ and $\mathcal{N} = 128$.

and it can be seen that the OMP performs better for BASC-based matrices. Comparing the presented results with those of [1], where a noise-free setting was investigated with a BP algorithm, it is obvious that OMP gains more from the low coherence BASC-based matrices than the BP algorithm.

In Fig. 4, the difference of the reconstruction frequencies is shown in order to give a clearer comparison. Green areas indicate that the OMP algorithm was more often successful with the BASC-based matrix then with the normalized Gaussian matrix. Red areas would indicate better results in favor of the Gaussian matrices. Obviously, BASC-based matrices work on average better with the OMP algorithm. However, it should also be noted that this superiority is not always observable. For certain individual realizations of the noise $\boldsymbol{n}$ and the sparse vector $\boldsymbol{x}$, the Gaussian matrices performed slightly better for low SNR levels ($10 - 20$db). Taking more simulations into account, these differences average out cf. Fig. 4.

IV. Conclusions

The OMP algorithm clearly benefits more from the low coherence of BASC-based sensing matrices than the BP algorithm (cf. [1]).

For higher SNR levels, the lower coherence between the columns of the BASC-based sensing matrices can be exploited by the OMP, and therefore, a better performance can be achieved in such situations.

However, no significant gain in performance can be expected for lower SNR regions.

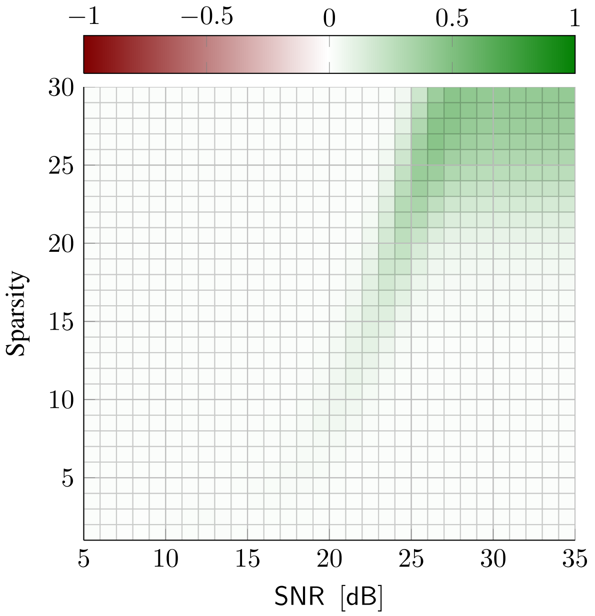

Fig. 4. Difference of the frequency of exact reconstruction between BASC-based matrices and normalized Gaussian matrices. Both are of size $\mathcal{M} = 64$ and $\mathcal{N} = 128$. Green areas indicate better performance of BASC-based matrices, whilst red areas show the same for Gaussian matrices.

Acknowledgments

The authors would like to thank the anonymous reviewers for their insightful and constructive comments, and their students Faisal Akram and Kotchapol Yoosooksawat for the assisting simulations.

This work was supported by the German research council Deutsche Forschungsgemeinschaft (DFG) under Grant Bo 867/27-1.

References

[1] D. E. Lazich, H. Zörlein, and M. Bossert, "Low coherence sensing matrices based on best spherical codes," in *9th International ITG Conference on Systems, Communications and Coding 2013 (SCC'2013)*, Munich, Germany, Jan. 2013.

[2] T. Cai and L. Wang, "Orthogonal matching pursuit for sparse signal recovery with noise," *IEEE Transactions on Information Theory*, vol. 57, no. 7, pp. 4680 – 4688, Jul. 2011.

[3] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Journal on Scientific Computing*, vol. 20, pp. 33 – 61, 1998.

[4] Y. Pati, R. Rezaiifar, and P. S. Krishnaprasad, "Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition," in *Proceedings of the 27 th Annual Asilomar Conference on Signals, Systems, and Computers*, 1993, pp. 40 – 44.

[5] D. L. Donoho and M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ_1 minimization," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 100, no. 5, pp. 2197 – 2202, 2003.

[6] M. Elad and A. M. Bruckstein, "A generalized uncertainty principle and sparse representation in pairs of bases," *IEEE Transactions on Information Theory*, vol. 48, no. 9, pp. 2558 – 2567, Sep. 2002.

[7] R. Gribonval and M. Nielsen, "Sparse representations in unions of bases," *IEEE Transactions on Information Theory*, vol. 49, no. 12, pp. 3320 – 3325, Dec. 2003.

[8] W. Bajwa, R. Calderbank, and S. Jafarpour, "Model selection: Two fundamental measures of coherence and their algorithmic significance," in *Information Theory Proceedings (ISIT), 2010 IEEE International Symposium on*, Jun. 2010, pp. 1568 – 1572.

[9] W. U. Bajwa, R. Calderbank, and D. G. Mixon, "Two are better than one: Fundamental parameters of frame coherence," *Applied and Computational Harmonic Analysis*, vol. 33, no. 1, pp. 58 – 78, 2012.

[10] E. J. Candès and T. Tao, "Decoding by Linear Programming," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4203 – 4215, Dec. 2005.

[11] J. Conway and N. Sloane, *Sphere Packings, Lattices, and Groups*, 3rd ed., ser. Grundlehren der Mathematischen Wissenschaften. Springer, 1999.

[12] T. Ericson and V. Zinoviev, *Codes On Euclidean Spheres*, ser. North-Holland Mathematical Library. Elsevier, 2001.

[13] N. J. A. Sloane, "Spherical codes: Nice arrangements of points on a sphere in various dimensions." [Online]. Available: http://www2.research.att.com/ njas/packings/

[14] J. Leech, "Equilibrium of sets of particles on a sphere," *The Mathematical Gazette*, vol. 41, no. 336, pp. 81 – 90, 1957.

[15] D. E. Lazic, "Class of block codes for the gaussian channel," *Electronics Letters*, vol. 16, no. 5, pp. 185 – 186, Feb. 1980.

[16] D. E. Lazic, T. Bece, and P. J. Krstajic, "On the construction of the best spherical code by computing the fixed point," in *IEEE International Symposium on Information Theory, Abstract of papers*, Ann Arbor, Michigan, USA, 1986, p. 74.

[17] D. E. Lazic, V. Senk, and R. Zamurovic, "An efficient numerical procedure for generating best spherical arrangements of points," in *Proceedings of the International AMSE'88, Istanbul Conference "Modelling and Simulation"*, vol. 1C, Istanbul, Turkey, 1988, pp. 267 – 278.

[18] MathWorks®, "MATLAB® Version R2012b," Natick, Massachusetts. [Online]. Available: http://www.mathworks.com/