Spline-based frames for image restoration

Amir Averbuch
School of Computer Science
Tel Aviv University
Tel Aviv 69978, Israel
Email: amir@math.tau.ac.il

Pekka Neittaanmäki
Dept. of Mathematical Information Technology
University of Jyväskylä
P.O. Box 35 (Agora), Jyväskylä, Finland
Email: pekka.neittaanmaki@jyu.fi

Valery Zheludev
School of Computer Science
Tel Aviv University
Tel Aviv 69978, Israel
Email: zhel@post.tau.ac.il

***Abstract*—We present a design scheme to generate tight and semi-tight frames in the space of discrete-time periodic signals, which are originated from four-channel perfect reconstruction periodic filter banks. The filters are derived from interpolating and quasi-interpolating polynomial splines. Each filter bank comprises one linear phase low-pass filter (in most cases interpolating) and one high-pass filter, whose magnitude response mirrors that of a low-pass filter. In addition, these filter banks comprise two band-pass filters. In the semi-tight frames case, all the filters have linear phase and (anti)symmetric impulse response, while in the tight frame case, some of band-pass filters are slightly asymmetric. We introduce the notion of local discrete vanishing moments (LDVM). In the tight frame case, analysis framelets coincide with their synthesis counterparts. However, in the semi-tight frames, we have the option to swap LDVM between synthesis and analysis framelets. The design scheme is generic, and it enables to design framelets with any number of LDVM. The computational complexity of the framelet transforms, which consists of calculation of the forward and the inverse fast Fourier transforms and simple arithmetic operations, practically does not depend on the number of LDVM and on the size of the impulse response of filters. The designed frames are used for restoration of images, which are degraded by blurring, random noise and missing pixels. The images were restored by the application of the Split Bregman Iterations (SBI) method.**

I. Introduction

Restoration of corrupted and/or damaged and/or noised multidimensional signals is a major challenge that the signal/image processing community faces nowadays when rich multimedia content is the most popular data that is being transmitted over diverse networks types such as mobile. Quality degradation in multidimensional signals can come from sampling, acquisition, transmission through noisy channels, to name some. Restoration of multidimensional signals includes denoising, deblurring, recovering missing or damaged samples or fragments (inpainting in images), resolution enhancement and super resolution. Recent developments in wavelet frames (framelets) analysis provide innovative and powerful tools to meet faithfully and robustly the above challenges. Framelets produce redundant expansions whose valuable advantage is their ability to restore missing and incomplete information and to represent efficiently and compactly the data. In principle, only part of the samples/pixels is needed for (near) perfect object restoration. This approach, which is a variation of the *Compressive Sensing* methodology, proved to be extremely efficient for image restoration.

Practically, this approach is implemented via minimization of a parameterized functional where the sparse representation is reflected in the l_1 norm of the transform coefficients. The $\|\cdot\|_1$ minimization does not have an explicit solution and can be resolved only by iterative methods. The so-called *split Bregman iteration* (SBI) scheme, which was recently introduced in [1], provided a fast and stable algorithm for that. Variations of this scheme and its application to image restoration using wavelet frames are described in [2], [3], to mention a few. A variety of impressive results on image restoration were reported in the last couple of years. A survey is given in [4] while a recent development is described in [3].

Due to applications diversity, it is important to have a library of wavelet frames in order to select a frame that fits best a specific task. Forward and inverse transforms in iterative algorithms are repeated many times, therefore, members in this library must have fast and stable transforms implementation. Waveforms symmetry with the availability of vanishing moments are also important in order to avoid distortions when thresholding is used. To satisfy these requirements, most of the framelet systems that were designed so far operate with the compactly supported framelets and the transforms are implemented by finite (and short) impulse response (FIR) oversampled filter banks Thus, the number of framelet systems available for applications is very limited. This number is even smaller when the requirement is to have tight frames.

This limitation can be overcome by switching to a periodic setting, which is the subject of this presentation. A variety of four-channel PR filter banks, where the low-pass filters are derived from interpolating and quasi-interpolating polynomial splines, are designed. These filter banks generate a library of 4- framelet periodic tight and the so-called semi-tight frames with diverse properties. The transforms implementation is reduced to application of the direct and the inverse fast Fourier transforms (FFT) with simple arithmetic operations. While implementation of SBI in non-periodic setting requires multiple approximate solution of a system of equations by the conjugate gradient method, the periodic implementation makes it possible to avoid those procedures. This fact contributes significantly to reduction of the implementation cost.

The designed framelets libraries were tested for image restoration and demonstrated a high quality. Their diversity enabled us to select a frame, which best fits each specific application. In particular, in most of the experiments the semi-tight frames outperformed tight frames.

II. Periodic filter banks and frame transform

We call the N-periodic real-valued sequences $\mathbf{x} \stackrel{\text{def}}{=} \{x[k]\},\ k \in \mathbb{Z},\ x[k+N] = x[k],\ N = 2^j$, the discrete-time periodic signals, which constitute an N-dimensional vector space $\Pi[N]$. We use the notation $\omega \stackrel{\text{def}}{=} e^{2\pi i/N}$. The circular convolution $y[k] = \sum_{k=0}^{N-1} h[k-l]\, x[l]$ of the signal $\mathbf{x}$ with a signal $\mathbf{h} \in \Pi[N]$ is called p-filtering and the signal $\mathbf{h}$ is called the p-filter. P-filtering results in multiplication of the DFT: $\hat{y}[n] = \hat{h}[n]\, \hat{x}[n]$.

It is well known that the perfect reconstruction (PR) oversampled filter banks generate frames in the signal space [5]. We deal with four channel analysis $\tilde{\mathbf{H}} \stackrel{\text{def}}{=} \left\{\tilde{\mathbf{h}}^s\right\}$, and synthesis $\mathbf{H} \stackrel{\text{def}}{=} \{\mathbf{h}^s\}$, $s = 0, ..., 3$ with downsampling factor of 2, which operate in the periodic signal space $\Pi[N]$. Either of $\tilde{\mathbf{H}}$, and $\mathbf{H}$ filter banks comprises one low-pass p-filter $\tilde{\mathbf{h}}^0$ and $\mathbf{h}^0$, one high-pass $\tilde{\mathbf{h}}^1$ and $\mathbf{h}^1$ and two band-pass $\tilde{\mathbf{h}}^s$ and $\mathbf{h}^s$, $s = 2, 3$, p-filters, respectively. The subsequent application of the time-reversed analysis and synthesis filter banks to an input signal $\mathbf{x} \in \Pi[N]$ restores the signal:

$$\begin{array}{ll} y^s[l] = & \sum_{k=0}^{N-1} \tilde{h}^s[k-2l]\, x[k], \quad s = 0, \ldots, 3, \\ x[l] = & \sum_{s=0}^{S-1} \sum_{k=0}^{N/2-1} h^s[l-2k]\, y^s[k]. \end{array} \tag{1}$$

Denote by $\left\{\tilde{\psi}^s[k] = \tilde{h}^s[k]\right\}$ and $\{\psi^s[k] = h^s[k]\}$ the impulse responses of the analysis and synthesis p-filters, respectively. Equations (1) provide the frame expansion of a signal $\mathbf{x} \in \Pi[N]$:

$$x[l] = \sum_{s=0}^{3} \sum_{k=0}^{N/2-1} \psi^s[l-2k] \left\langle \mathbf{x},\, \tilde{\psi}^s[\cdot - 2k] \right\rangle. \tag{2}$$

The 2-sample shifts of the signals $\tilde{\psi}^s[k]$ and $\psi^s[k]$ form analysis and synthesis frames of the space $\Pi[N]$, respectively. Together they constitute a bi-frame $\left\{\tilde{\mathbf{F}},\, \mathbf{F}\right\}$. If the synthesis framelets can be chosen to be equal to the analysis framelets then the frame is tight.

III. Design of 4-channel PR filter banks

Denote by $\mathbf{x}_0 \stackrel{\text{def}}{=} \{x[2k]\} \in \Pi[N/2]$ and $\mathbf{x}_1 \stackrel{\text{def}}{=} \{x[2k+1]\}$ the even and odd polyphase components of a signal $\mathbf{x} \in \Pi[N]$. Then, the DFT of $\mathbf{x}$ is $\hat{x}[n] = \hat{x}_0[n] + \omega^n\, \hat{x}_1[n]$. Application of the 4-channel PR filter bank to a signal $\mathbf{x} \in \Pi[N]$ can be expressed in a matrix form. Denote $\vec{Y}[n] \stackrel{\text{def}}{=} (\hat{y}^0[n], ..., \hat{y}^3[n])^T$ and $\vec{X}[n] \stackrel{\text{def}}{=} (\hat{x}_0[n], \hat{x}_1[n])^T$. Then, we have

$$\vec{Y}[n] = \tilde{\mathbf{P}}[-n] \cdot \vec{X}[n], \quad \vec{X}[n] = \mathbf{P}[n] \cdot \vec{Y}[n],$$

where the 4×2 analysis and the 2×4 synthesis polyphase matrices are, respectively,

$$\begin{array}{ll} \tilde{\mathbf{P}}[n] \stackrel{\text{def}}{=} & \begin{pmatrix} \hat{\tilde{h}}_0^0[n] & \ldots & \hat{\tilde{h}}_0^3[n] \\ \hat{\tilde{h}}_1^0[n] & \ldots & \hat{\tilde{h}}_1^3[n] \end{pmatrix}^T, \\ \mathbf{P}[n] \stackrel{\text{def}}{=} & \begin{pmatrix} \hat{h}_0^0[n] & \ldots & \hat{h}_0^3[n] \\ \hat{h}_1^0[n] & \ldots & \hat{h}_1^3[n] \end{pmatrix}. \end{array}$$

The relations

$$\mathbf{P}[n] \cdot \tilde{\mathbf{P}}[-n] = \mathbf{I}_2, \tag{3}$$

is the condition for the pair $\left\{\tilde{\mathbf{H}},\, \mathbf{H}\right\}$ of filter banks to form a PR filter bank.

a) Design: The matrix product in Eq. (3) can be split into two products.

$$\mathbf{P}^{01}[n] \cdot \tilde{\mathbf{P}}^{01}[-n] + \mathbf{P}^{23}[n] \cdot \tilde{\mathbf{P}}^{23}[-n] = \mathbf{I}_2, \tag{4}$$

$$\mathbf{P}^{01}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{h}_0^0[n] & \hat{h}_0^1[n] \\ \hat{h}_1^0[n] & \hat{h}_1^1[n] \end{pmatrix},\ \tilde{\mathbf{P}}^{01}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tilde{h}}_0^0[n] & \hat{\tilde{h}}_1^0[n] \\ \hat{\tilde{h}}_0^1[n] & \hat{\tilde{h}}_1^1[n] \end{pmatrix},$$

$$\mathbf{P}^{23}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{h}_0^2[n] & \hat{h}_0^3[n] \\ \hat{h}_1^2[n] & \hat{h}_1^3[n] \end{pmatrix},\ \tilde{\mathbf{P}}^{23}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tilde{h}}_0^2[n] & \hat{\tilde{h}}_1^2[n] \\ \hat{\tilde{h}}_0^3[n] & \hat{\tilde{h}}_1^3[n] \end{pmatrix}.$$

A PR pair $\left\{\mathbf{H},\, \tilde{\mathbf{H}}\right\}$ of filter banks generate a tight frame if their polyphase matrices are linked as

$$\mathbf{P}[n] = \tilde{\mathbf{P}}[n]^T \Longleftrightarrow \mathbf{P}^{01}[n] = \tilde{\mathbf{P}^{01}}[n]^T \text{ and } \mathbf{P}^{23}[n] = \tilde{\mathbf{P}}^{23}[n]^T.$$

If the matrices $\mathbf{P}^{01}[n] = \tilde{\mathbf{P}}^{01}[n]^T$ and $\mathbf{P}^{23}[n] \neq \tilde{\mathbf{P}}^{23}[n]^T$, $n \in \mathbb{Z}$, *then the frame* $\left\{\tilde{\mathbf{F}},\, \mathbf{F}\right\}$ *is called semi-tight.*

The design of four-channel (semi-)tight filter banks begins from a linear phase low-pass filter $\mathbf{h}^0 = \tilde{\mathbf{h}}^0$, whose frequency response (FR) $\hat{h}^0[n] = \hat{h}_0^0[n] + \omega^{-n}\hat{h}_1^0[n]$ is a rational function of $\omega^n = e^{2\pi i n/N}$ with real coefficients that has no poles for $n \in \mathbb{Z}$. Assume $\hat{h}^0[n]$ is symmetric about the swap $n \to -n$, which implies that $\hat{h}_0^0[n] = \hat{h}_0^0[-n]$ and $\omega^{-n}\hat{h}_1^0[n] = \omega^n\, \hat{h}_1^0[-n]$. The impulse response $\left\{h^0[k]\right\}$ is symmetric about $k = 0$.

In addition, assume that $\mathbf{P}^{01}[n] = \tilde{\mathbf{P}}^{01}[n]^T$ and the product

$$\mathbf{P}^{01}[n] \cdot \mathbf{P}^{01}[-n] = \begin{pmatrix} \alpha[n] & 0 \\ 0 & \beta[n] \end{pmatrix} \tag{5}$$

is a diagonal matrix. The assumption in Eq. (5) implies the condition $\hat{h}_0^0[n]\hat{h}_1^0[-n] + \hat{h}_0^1[n]\hat{h}_1^1[-n] = 0$. The simplest way to satisfy this condition is to define

$$\begin{array}{c} \hat{h}^1[n] = -\hat{h}_1^0[-n] + \omega^{-n}\hat{h}_0^0[-n] \\ \Longrightarrow \alpha[n] = \beta[n] = \left|\hat{h}_0^0[n]\right|^2 + \left|\hat{h}_1^0[n]\right|^2. \end{array}$$

The sequence $\omega^n\hat{h}^1[n] = \omega^{-n}\hat{h}^1[-n]$ and, consequently, the impulse response $\left\{h^1[k]\right\}$ is symmetric about $k = 1$. The product

$$\mathbf{P}^{23}[n] \cdot \tilde{\mathbf{P}}^{23}[-n] = \mathbf{Q}[n] \stackrel{\text{def}}{=} \begin{pmatrix} t[n] & 0 \\ 0 & t[n] \end{pmatrix}, \tag{6}$$

where $t[n] \stackrel{\text{def}}{=} 1 - \left|\hat{h}_0^0[n]\right|^2 + \left|\hat{h}_1^0[n]\right|^2$. Thus, the design of the PR filter bank is reduced to factorization of the matrix $\mathbf{Q}[n]$.

There are many ways to factorize the matrix $\mathbf{Q}[n]$. One way is to define the matrices $\mathbf{P}^{23}[n]$ and $\tilde{\mathbf{P}}^{23}[n]$ to be diagonal:

$$\mathbf{P}^{23}[n] = \begin{pmatrix} \hat{h}_0^2[n] & 0 \\ 0 & \hat{h}_1^3[n] \end{pmatrix},\ \tilde{\mathbf{P}}^{23}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tilde{h}}_0^2[n] & 0 \\ 0 & \hat{\tilde{h}}_1^3[n] \end{pmatrix}.$$

Consequently, we have to derive four sequences $\hat{h}_0^2[n]$, $\hat{\tilde{h}}_0^2[n]$, $\hat{h}_1^3[n]$ and $\hat{\tilde{h}}_1^3[n]$ such that

$$\hat{h}_0^2[n]\,\hat{\tilde{h}}_0^2[n] = \hat{h}_1^3[n]\,\hat{\tilde{h}}_1^3[n] = t[n]. \tag{7}$$

b) Tight frame filter banks: If the following inequality holds

$$\alpha[n] = \left|h_0^0[n]\right|^2 + \left|h_1^0[n]\right|^2 > 1, \quad n \in \mathbb{Z}, \tag{8}$$

then, due to the symmetry of the rational functions $\hat{h}_0^0[n]$ and $\hat{h}_0^0[n]$, the sequence $t[n]$ is strictly positive rational function of $\cos 2\pi n/N$). Due to Riesz Lemma , it can be factorized $t[n] = T[n]\,T[-n]$, where T is a rational function of ω^n, which does not have roots for $n \in \mathbb{Z}$. Thus, we define

$$\hat{h}_0^2[n] = \hat{\tilde{h}}_0^2[n] = \hat{h}_1^3[-n] = \hat{\tilde{h}}_1^3[-n] = T[n]. \tag{9}$$

The PR filter bank, whose filters are

$$\begin{array}{ll} \hat{h}^0[n] = \hat{h}_0^0[n] + \omega^{-n}\hat{h}_0^0[n] & \hat{h}^2[n] = T[n], \\ \hat{h}^1[n] = -\hat{h}_1^0[-n] + \omega^{-n}\hat{h}_0^0[-n], & \hat{h}^3[n] = \omega^{-n}T[-n], \end{array}$$

generates a tight wavelet frame in the space $\Pi[N]$. Certainly, the symmetry of the FR $\hat{h}^0[n]$ does not guarantee the symmetry of the FR $\hat{h}^2[n]$ and $\hat{h}^3[n]$.

c) Semi-tight frame filter banks: If the condition Eq. (8) is not fulfilled then the sequence $t[n]$ can be factorized as $t[n] = T[n]\,\tilde{T}[-n]$, where $T[n] \neq \tilde{T}[n]$. Thus, we obtain the PR filter bank, whose filters are

$$\begin{array}{c} \hat{h}^0[n] = \hat{h}_0^0[n] + \omega^{-n}\hat{h}_0^0[n],\ \hat{h}^1[n] = -\hat{h}_1^0[-n] + \omega^{-n}\hat{h}_0^0[-n], \\ \hat{h}^2[n] = T^2[n], \qquad \hat{\tilde{h}}^2[n] = \tilde{T}^2[n], \\ \hat{h}^3[n] = \omega^{-n}T^3[n], \quad \hat{\tilde{h}}^3[n] = \omega^{-n}\tilde{T}^3[n], \end{array} \tag{10}$$

where $T^2[n]\,\tilde{T}^2[-n] = T[n]\,\tilde{T}^3[-n] = t[n]$. The PR filter bank defined by Eq. (10) generates a semi-tight frame in the space $\Pi[N]$.

Remark 1: Since the rational function $t[n]$ of ω^n is symmetric about the change $n \to -n$, then it can be factorized into product of two symmetric rational functions $T[n]$ and $\tilde{T}[-n]$. An additional advantage of the semi-tight design is the option to swap approximation properties between the analysis and the synthesis framelets.

As usual, a multiscale frame transform is implemented by subsequent application of the frame transform to the low-frequency array of the transform coefficients.

IV. Spline p-filters

It was described above how to design a tight or a semi-tight frame comprising four framelets starting from a low-pass p-filter. A variety of such p-filters can be derived from the theory of periodic splines ([6], for example). The p-filters possess useful properties such as linear phase, flat spectra and well localised impulse responses. The idea is to design an $N/2$-periodic spline $S^p(t)$ of order p on the grid $\{k\}$, which interpolates the even polyphase component $\mathbf{x}_0$ of a signal $\mathbf{x}$: $S^p(k) = x[2k]$. Then, in order to derive the spline's values at the intermediate points $s_1[k] \stackrel{\text{def}}{=} S^p(k+1/2)$, which, in a sense predict the odd polyphase component $\mathbf{x}_1$ of $\mathbf{x}$, the signal $\mathbf{x}_0$ should be filtered with some "prediction" p-filter: $\hat{s}_1[n] = f^p[n]\,\hat{x}_0[n]$. Then, the interpolating low-pass p-filter is defined as $\hat{h}^0[n] \stackrel{\text{def}}{=} \left(1 + \omega^{-n} f^p[n]\right)/\sqrt{2}$. The corresponding high-pass p-filter is $\hat{h}^1[n] \stackrel{\text{def}}{=} \omega^{-n}\left(1 - \omega^n f^p[-n]\right)/\sqrt{2}$. The filters $f^p[n]$ can be explicitly calculated for any order of a spline. For all the orders except for $p = 2$ (piece-wise linear spline) the p-filters have infinite impulse response. This fact does not complicate the implementation, which consists of application of the forward and inverse fast Fourier transforms and simple arithmetic operations. The finite impulse response (up to periodization) p-filters can be derived from quasi-interpolating splines.

Because the conventional notion of vanishing moments is not applicable to the periodic discrete-time setting, we use the notion of the local discrete vanishing moments (LDVM). Loosely speaking, a framelet has m LDVM if, being convolved with a signal containing fragments of sampled polynomials of degree $m-1$, it eliminates these fragments. If the FR of a p-filter comprises either the factor $\left(1-\omega^{2n}\right)^r$ or the factor $\sin^{2r}\pi n/N$ then the corresponding framelet has $2r$ LDVM.

We designed a diverse collection of tight and semi-tight frames originating from interpolating and quasi-interpolating splines of different orders. Below are two examples.

d) Example 1: quadratic interpolating spline: $p = 3$: The frequency response of low- and high-pass p-filters are

$$\begin{array}{c} \hat{h}^0[n] = \sqrt{2}\,\frac{\cos^4 \pi n/N}{\cos^4 \pi n/N + \sin^4 \pi n/N} \\ \hat{h}^1[n] = \omega^{-n}\sqrt{2}\,\frac{\sin^4 \pi n/N}{\cos^4 \pi n/N + \sin^4 \pi n/N}. \end{array}$$

In this case a symmetric factorization of the matrix $Q[n]$ defined in Eq. (6) is possible. Therefore the p-filters $\mathbf{h}^2$ and $\mathbf{h}^3$, which complete the p-filters $\mathbf{h}^0$ and $\mathbf{h}^1$ to the PR filter bank, have linear phase. Their frequency responses are

$$\hat{h}^2[n] = \frac{1}{2\sqrt{2}}\,\frac{\sin^2 2\pi n/N}{\cos^4 \pi n/N + \sin^4 \pi n/N} = -\omega^n\,\hat{h}^3[n].$$

The high-frequency framelet $\psi^1 = \mathbf{h}^1$ has four LDVM, while the framelets $\psi^l = \mathbf{h}^l$, $l = 2, 3$, have two.

e) Example 2: quadratic quasi-interpolating spline: In the tight frame derived from this spline the p-filters $\mathbf{h}^2$ and $\mathbf{h}^3$ are non-symmetric. However, the semi-tight frame, where those p-filters are antisymmetric proved to be highly efficient in applications. The frequency response of low- and high-pass p-filters are

$$\begin{array}{c} \hat{h}^0[n] = \frac{1}{\sqrt{2}}\cos^4\frac{\pi n}{N}\left(3 - \cos\frac{2\pi n}{N}\right) \\ \hat{h}^1[n] = \frac{\omega^{-n}}{\sqrt{2}}\sin^4\frac{\pi n}{N}\left(3 + \cos\frac{2\pi n}{N}\right). \end{array}$$

Denote

$$\begin{array}{c} \tilde{T}[n]_1 = \frac{\omega^{4n} - 3\omega^{2n} + 3 - \omega^{-2n}}{8} \\ T[n]_1 = \frac{\left(1-\omega^{2n}\right)\left(-\omega^{4n} - 12\omega^{2n} + 346 - 12\omega^{-2n} - \omega^{-4n}\right)}{1024}. \end{array}$$

Then,

$$\hat{h}^2[n] = T[n]_1 = -\frac{\hat{\tilde{h}}^3[-n]}{\omega^n},\ \hat{\tilde{h}}^2[n] = \tilde{T}[n]_1 = -\frac{\hat{h}^3[-n]}{\omega^n}.$$

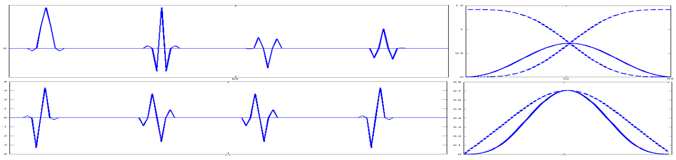

Fig. 1. Top:FB generating tight frames: impulse and magnitude responses derived from quadratic quasi-interpolating spline. Bottom: analysis and synthesis band-pass filters for semi-tight frame

The high-frequency framelet ψ^1 has four LDVM. We assign three LDVM to the analysis framelet $\tilde{\psi}^2$ leaving only one LDVM to the synthesis framelet ψ^2 and vice versa for the framelets $\tilde{\psi}^3$ and ψ^3. Figure 1 displays impulse and magnitude responses of the p-filters derived from quadratic quasi-interpolating spline, which generate the tight and the semi-tight frames. We observe that the impulse responses of the band-pass p-filters for the tight frame are non-symmetric. The semi-tight frame band-pass p-filters have anti-symmetric impulse responses.

V. Application to Image Restoration

We designed a diverse library of tight and semi-tight frames, which were extended to two dimensions via the tensor products of 1D framelets. The frames were tested in multiple image restoration experiments where images were blurred, affected by random noise and a significant number of pixels were missing. For restoration, the SBI scheme [1] with the designed frames was utilized. Performance of different frames was compared. These experiments as well as the frame design are described in details in [7]. In most cases semi-tight frames were advantageous over respective tight frames. Especially successful was the semi-tight frame presented in Example 2. However in some experiments, the frames derived from higher order splines (thus having a big number of LDVM) outperformed the frame derived from low-order splines.

f) "Boats" and "Fingerprint" images: The "Boats" image was blurred by the motion kernel and its PSNR becomes 22.88 dB. Then, 70% of pixels were randomly removed. This reduces the PSNR to 7.37 dB. The image restored using the semi-tight frame from Example 2 with PSNR =30.28 dB. The "Fingerprint" image was affected by a strong zero-mean white noise with STD $\sigma = 20$ after being blurred by the Gaussian kernel (PSNR=19.75 dB). Then, 50% of its pixels were randomly removed and this produced PSNR=9.05 dB. The frame decomposition is implemented down to the fifth level. The PSNR=23.75 dB result was achieved by the application of the tight frame derived from the interpolating spline of fifth order. In this frame, the high-frequency framelet ψ^1 has six LDVM, while either of the framelets ψ^2 and ψ^3 has three LDVM. Results of the above experiments are displayed in Fig. 2. We observe that despite a strong degradation, which made images almost undistinguishable, they are successfully restored.

g) Restoration experiments for the "Window" image: This image was taken from [8]. The image was blurred by the motion kernel. In one experiment random noise presented (PSNR= 23.56 dB), while in the other white noise with STD $\sigma = 5$ was added (PSNR= 23.19 dB). Then, 30% of pixels were randomly removed. This reduces the PSNR to 10.22 dB and 10.20 dB, respectively. We compared the restoration results with the respective results reported in [8]. In the no-noise experiment the PSNR of the image restored by the semi-tight frame described in Example 2 was 43.78 dB versus 40.25 dB in [8]. For the noise experiment the PSNR was 28.81 dB versus 27. 76 dB in [8]. The restoration results are displayed in Fig. 3.

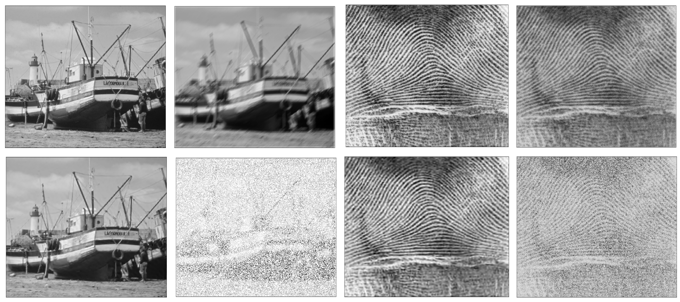

Fig. 2. "Boats" and "Fingerprint" images Top left in each quadruple – original images. Bottom left – restored images

Fig. 3. "Window" image. Left quadruple – no-noise experiment. Right quadruple – noise experiment

References

[1] T. Goldstein and S. Osher, "The split Bregman method for $L1$-regularized problems," *SIAM J. Imaging Sci.*, vol. 2, no. 2, pp. 323–343, 2009.

[2] J. Cai, S. Osher, and Z. Shen, "Split Bregman methods and frame based image restoration," *Multiscale Model. Simul.*, vol. 8, no. 2, pp. 337–369, 2009/10.

[3] J. Cai, B. Dong, S. Osher, and Z. Shen, "Image restoration: total variation, wavelet frames, and beyond," *J. Amer. Math. Soc.*, vol. 25, no. 4, pp. 1033–1089, 2012.

[4] Z. Shen, "Wavelet frames and image restorations," in *Proceedings of the International Congress of Mathematicians, Vol. IV*, R. Bhatia, Ed. New Delhi: Hindustan Book Agency, 2010, pp. 2834–2863.

[5] Z. Cvetković and M. Vetterli, "Oversampled filter banks," *IEEE Trans. Signal Process.*, vol. 46, no. 5, pp. 1245–1255, 1998.

[6] A. Averbuch and V. Zheludev, "Construction of biorthogonal discrete wavelet transforms using interpolatory splines," *Appl. Comput. Harmon. Anal.*, vol. 12, no. 1, pp. 25–56, 2002.

[7] A. Averbuch, P. Neittaanmäki, and V. Zheludev, "Spline-based frames in the space of periodic signals," *submitted*, 2012.

[8] H. Ji, Z. Shen, and Y. Xu, "Wavelet based restoration of images with missing or damaged pixels," *East Asian J. Appl. Math.*, vol. 1, no. 2, pp. 108–131, 2011.