Wavelet Signs: A New Tool for Signal Analysis

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Abstract—**We propose a new analysis tool for signals, called signature, that is based on complex wavelet signs. The complex-valued signature of a signal at some spatial location is defined as the fine-scale limit of the signs of its complex wavelet coefficients. We show that the signature equals zero at sufficiently regular points of a signal whereas at salient features, such as jumps or cusps, it is non-zero. We establish that signature is invariant under fractional differentiation and rotates in the complex plane under fractional Hilbert transforms. We derive an appropriate discretization, which shows that wavelet signatures can be computed explicitly. This allows an immediate application to signal analysis.**

I. INTRODUCTION

The determination and classification of salient features, such as jumps or cusps, is an important task in signal processing. Classical approaches assume the interesting features of a signal to be points of low regularity. In this context, local regularity is measured in terms of the (fractional) differentiability order, e.g., in the sense of local Hölder, Sobolev or Besov regularity. Since such measures of smoothness only rely on the *modulus* of wavelet coefficients [5], [9], they do not take into account wavelet sign (or phase) information.

We may observe the shortcomings of a purely modulus based approach by considering the two functions $f(x) = \operatorname{sgn} x$ and $g(x) = 2 \log |x|$. Since f and g are related by the Hilbert transform, their wavelet coefficients are equal with respect to the order of magnitude. Hence, the locally symmetric singularity of f and the locally antisymmetric singularity of g at the origin cannot be distinguished using a purely modulus-based signal analysis.

We present a new signal analysis tool, which exclusively uses the (complex) *sign* of the wavelet coefficients. To this end, we investigate the fine scale limits of the signs of the wavelet coefficients

$$\sigma f(b) := \lim_{a \to 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle := \lim_{a \to 0} \frac{\langle f, \kappa_{a,b} \rangle}{|\langle f, \kappa_{a,b} \rangle|}, \quad (1)$$

where κ is a complex-valued wavelet, $a > 0$ the scale, and $b \in \mathbb{R}$ the location. The *complex-valued* quantity $\sigma f(b)$ is called the *signature of f at location b*. We shall see that the signature allows the local analysis of isolated salient features. Hereby, the orientation of the signature within the complex plane may be interpreted as an indicator of local symmetry or antisymmetry. In particular, we show that the signature is purely imaginary at a jump, whereas it is purely real at a cusp. Moreover, the signature is invariant under fractional Laplacians, i.e.,

$$\sigma((-\Delta)^{\frac{r}{2}} f) = \sigma f,$$

and it serves as a multiplier when acting on the fractional Hilbert transform, in the sense that

$$\sigma(\mathcal{H}^\alpha f) = e^{i\alpha \frac{\pi}{2}} \sigma f.$$

Therefore, the signature may be interpreted as being "dual" to the local Sobolev regularity index, which is invariant under fractional Hilbert transforms but shifts under fractional Laplacians. We also establish that

$$\operatorname{sing\,supp} f \not\subset \operatorname{supp} \sigma f \quad \text{and} \quad \operatorname{supp} \sigma f \not\subset \operatorname{sing\,supp} f. \quad (2)$$

Thus, a singularity in the classical sense need not coincide with a signature-type singularity.

We further introduce a method to numerically compute the signature of digital or sampled signals and validate the theoretically developed concepts by numerical experiments. There are some connections between our discretization and *phase congruency* [6]. However, the approach undertaken in [6] tends to favor unwanted large coefficients, which our method avoids.

In this short communication, we omit the proofs which the interested reader may find in [11].

II. DEFINITIONS AND BASIC PROPERTIES

We define the Fourier transform of a Schwartz function $f \in \mathscr{S}(\mathbb{R}; \mathbb{C})$ by

$$\mathscr{F}(f)(\omega) := \widehat{f}(\omega) := \int_{\mathbb{R}} e^{-i\omega x} f(x) dx.$$

Likewise, we use the above notation for the usual extension to the space of tempered distributions $\mathscr{S}'(\mathbb{R}; \mathbb{C})$. Furthermore, $\mathscr{F}^{-1}(f)$ and $f^\vee$ denote the corresponding inverse Fourier transform of f. Let us introduce the class of complex wavelets we need for the definition of signature.

Definition 1. *We call a complex-valued non-zero function $\kappa \in \mathscr{S}(\mathbb{R}; \mathbb{C})$ a signature wavelet if κ has a one-sided compactly supported Fourier transform, i.e.,*

$$\operatorname{supp} \widehat{\kappa} \subseteq [c, d], \quad 0 < c < d < \infty, \quad (3)$$

and a non-negative frequency spectrum, i.e.,

$$\widehat{\kappa}(\omega) \geq 0, \quad \text{for all } \omega \in \mathbb{R}. \quad (4)$$

The wavelet system associated with a signature wavelet κ is defined as the family of functions

$$\kappa_{a,b}(x) := \frac{1}{\sqrt{a}} \kappa \left(\frac{x - b}{a} \right), \quad \text{where } a > 0 \text{ and } b \in \mathbb{R}.$$
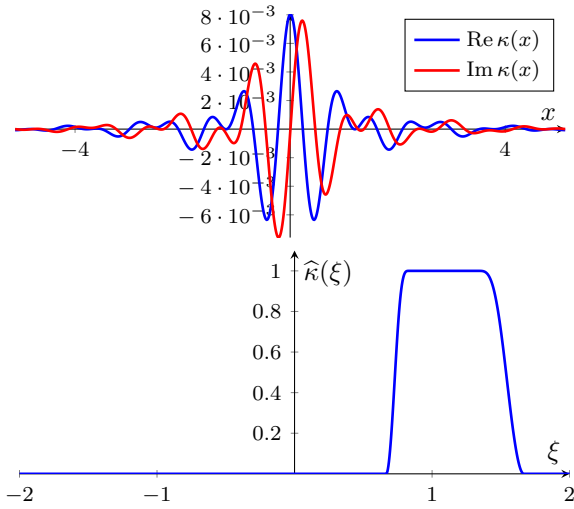
Fig. 1. The Meyer-type signature wavelet κ (left) and its Fourier transform $\widehat{\kappa}$ (right).

An example of a signature wavelet is given by the inverse Fourier transform of the (one-sided) Meyer window W, i.e.,

$$\kappa(x) = \mathscr{F}^{-1}(W)(x), \qquad (5)$$

where W is a Meyer window function (see Figure 1). We refer to [11] for the definition of W.

Recall that the sign of a complex number $z \in \mathbb{C}$ is given by

$$\operatorname{sgn} z = \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

The signature of a signal is then defined as follows.

Definition 2. *Let $f \in \mathscr{S}'(\mathbb{R}; \mathbb{R})$. If there exists a $z \in \mathbb{C}$, such that for all signature wavelets κ,*

$$\lim_{a \to 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle = z,$$

then we define the signature, σf, of f at $b \in \mathbb{R}$ by $\sigma f(b) := z$; otherwise, we set $\sigma f(b) := 0$.

Note that the signature $\sigma f(b)$ is either equal to zero or is a complex number of modulus 1. It follows directly from the definition that the signature is invariant under translations, i.e.,

$$\sigma(T_r f)(b) = (\sigma f)(b - r) \qquad (6)$$

and under dilations, i.e.,

$$\sigma(D_\nu f)(b) = (\sigma f)(\nu b). \qquad (7)$$

Here, the operator of translation by $r \in \mathbb{R}$, T_r, and dilation by $\nu \in \mathbb{R} \setminus \{0\}$, D_ν, are defined by

$$T_r f(x) := f(x - r) \quad \text{and} \quad D_\nu f(x) := \frac{1}{\sqrt{\nu}} f\left(\frac{x}{\nu}\right),$$

respectively.

Since the Fourier transform of a signature wavelet κ vanishes in a neighborhood of the origin, we have that

$$\langle p, \kappa \rangle = 0, \quad \text{for any polynomial } p. \qquad (8)$$

Therefore, the signature is well defined on the space of tempered distributions modulo polynomials $\mathscr{S}'/\mathscr{P}$, where $\mathscr{P}$ denotes the space of all polynomials.

Our first result shows that a signal of polynomial growth has signature equal to zero at a point where all derivatives are equal to zero.

Theorem 3. *Let f be a real-valued, locally integrable function of polynomial growth. Further assume that f is smooth in a neighborhood of $b \in \mathbb{R}$. If $f^{(k)}(b) = 0$, for all $k \in \mathbb{N}_0$, then $\sigma f(b) = 0$. In particular, $\operatorname{supp} \sigma f \subseteq \operatorname{supp} f$.*

An interesting consequence of Theorem 3 is the case when f is locally a polynomial.

Corollary 4. *Let f be a real-valued, locally integrable function of polynomial growth which is smooth in a neighborhood of $b \in \mathbb{R}$. If for some $k_0 \in \mathbb{N}_0$, $f^{(k)}(b) = 0$, for all $k \geq k_0$, then $\sigma f(b) = 0$. In particular, if f coincides on an open set $U \subset \mathbb{R}$ with a polynomial then $\sigma f(b) = 0$, for every $b \in U$.*

In the following example, we consider the unit step function. Here, we can compute the signature at $b = 0$ explicitly. For $b \neq 0$, we can apply Corollary 4.

Example 5. Let U be the unit step function defined by

$$U(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

For any signature wavelet κ, we have that

$$\langle U, \kappa_{a,0} \rangle = \left\langle \widehat{U}, (\kappa_{a,0})^\vee \right\rangle = \frac{i\sqrt{a}}{\pi} \int_{\mathbb{R}} \frac{\widehat{\kappa}(a\xi)}{\xi} \, d\xi. \qquad (9)$$

Hence, since $\widehat{\kappa} \geq 0$ and $\operatorname{supp} \widehat{\kappa} \subset [0, \infty)$, we obtain that $\operatorname{sgn} \langle U, \kappa_{a,0} \rangle = i$, for all $a > 0$. For $b \neq 0$, we apply Corollary 4 yielding

$$\sigma U(b) = \begin{cases} i, & \text{if } b = 0, \\ 0, & \text{else.} \end{cases}$$

In our next example, we turn our attention to the signature of a pure cusp-type singularity.

Example 6. For a fixed x_0, consider the function

$$f(x) = |x - x_0|^\gamma, \qquad \text{where } \gamma > 0.$$

In [11] we proved that the wavelet signs are given by

$$\sigma f(x_0) = \begin{cases} 0, & \text{if } \gamma \in 2\mathbb{N}_0, \\ -1, & \text{if } \gamma \in \,]0,2[\, \cup \,]4,6[\, \cup \ldots, \\ +1, & \text{if } \gamma \in \,]2,4[\, \cup \,]6,8[\, \cup \ldots. \end{cases}$$

and $\sigma f(b) = 0$, for $b \neq x_0$.

Next we show that, in general, a jump discontinuity induces a purely imaginary signature at the jump location. A function f has a *jump (or step) discontinuity* at b if the left-hand and the right-hand limits $f(b+)$ and $f(b-)$ exist but are not equal.

Theorem 7. *Let f be a real-valued, locally integrable function of polynomial growth and let $b \in \mathbb{R}$. If there exists a neighborhood U of b such that f is continuous on $U \setminus \{b\}$ and has a jump discontinuity at b, then*

$$\sigma f(b) = \begin{cases} +i, & \text{if} \quad f(b-) < f(b+), \\ -i, & \text{if} \quad f(b-) > f(b+). \end{cases} \tag{10}$$

III. FRACTIONAL LAPLACIANS AND FRACTIONAL HILBERT TRANSFORMS

We now investigate the behavior of signature under the action of fractional powers of the Laplacian and the fractional Hilbert transform. We shall see that the former leaves the signature invariant whereas the latter acts on the signature by a rotation in the complex plane.

We recall that fractional powers of the Laplacian $(-\Delta)^{\frac{r}{2}}$, acting on $f \in \mathscr{S}'(\mathbb{R})/\mathscr{P}$, are defined by

$$\widehat{(-\Delta)^{\frac{r}{2}} f} := |\bullet|^r \cdot \widehat{f}, \quad \text{for } r \in \mathbb{R}. \tag{11}$$

We show that the signature is invariant under $(-\Delta)^{\frac{r}{2}}$. Again, note that the signature is well defined for $f \in \mathscr{S}'(\mathbb{R})/\mathscr{P}$.

Theorem 8. *Let $f \in \mathscr{S}'(\mathbb{R})/\mathscr{P}$ and $r \in \mathbb{R}$. Then,*

$$\sigma\left((-\Delta)^{\frac{r}{2}} f\right)(b) = \sigma f(b), \qquad \text{for all } b \in \mathbb{R}.$$

Now we turn to the fractional Hilbert transform, which was first introduced in [7]. We follow the definition given in [8]. For $\alpha \in \mathbb{R}$, the fractional Hilbert transform $\mathcal{H}^\alpha$ is defined on $\mathscr{S}'(\mathbb{R})/\mathscr{P}$ by

$$\widehat{\mathcal{H}^\alpha f} := e^{-i\alpha \frac{\pi}{2} \cdot \mathrm{sgn}(\bullet)} \cdot \widehat{f}. \tag{12}$$

The following theorem shows that the fractional Hilbert transform $\mathcal{H}^\alpha$ acts on the signature as multiplication by $e^{i\alpha \frac{\pi}{2}}$, i.e., as a rotation in the complex plane.

Theorem 9. *Let $f \in \mathscr{S}'(\mathbb{R})/\mathscr{P}$ and $b \in \mathbb{R}$. Then*

$$\sigma\left(\mathcal{H}^\alpha f\right)(b) = e^{i\alpha \frac{\pi}{2}} \cdot \sigma f(b). \tag{13}$$

See Table I for a comparison between local Sobolev regularity index of f, denoted by s_f (cf. e.g. [4]), and the signature under action of fractional Laplacians and fractional Hilbert transforms.

The next two examples show that the points of non-zero signature in general do not coincide with the singular support, cf. (2).

Example 10. Consider the Weierstraß function (see e.g. [2])

$$f(x) = \sum_{n=0}^\infty r^n \cos(t^n x), \qquad \text{where } 0 < r < 1 \text{ and } rt \geq 1;$$

	Sobolev regularity index	Signature
Fractional differentiation	$s_{(-\Delta)^{\frac{r}{2}} f} = s_f - r$	$\sigma((-\Delta)^{\frac{r}{2}} f) = \sigma(f)$
Fractional Hilbert transform	$s_{\mathcal{H}^\alpha f} = s_f$	$\sigma(\mathcal{H}^\alpha f) = e^{i\alpha \frac{\pi}{2}} \sigma(f)$

TABLE I
THE ACTION OF FRACTIONAL LAPLACIANS AND FRACTIONAL HILBERT TRANSFORMS TO THE SOBOLEV REGULARITY INDEX s_f AND THE SIGNATURE σf.

As f is nowhere differentiable, it follows that $\mathrm{sing\,supp}\, f = \mathbb{R}$. In [11], we have proved that $\sigma f(b) = 0$, for all $b \in \mathbb{R}$. Therefore, we see that in general $\mathrm{sing\,supp}\, f \not\subseteq \mathrm{supp}\, \sigma f$.

Example 11. Let $f = e^{-\gamma x^2}$ be a Gaussian function with $\gamma > 0$, and let κ be any signature wavelet. The singular support of f is empty because f is smooth. On the other hand, as the support of $\widehat{\kappa}$ is not empty,

$$\langle f, \kappa_{a,0}\rangle = \left\langle \widehat{f}, (\kappa_{a,0})^\vee \right\rangle = \sqrt{\pi} \int_{\mathbb{R}} e^{-\frac{\omega^2}{4\gamma}} (\kappa_{a,0})^\vee(\omega) \, d\omega > 0,$$

for all $a > 0$, implying that the signature equals 1 at $b = 0$. Thus, in general, $\mathrm{supp}\, \sigma f \not\subseteq \mathrm{sing\,supp}\, f$. This shows that the converse inclusion does not hold either.

IV. DISCRETIZATION AND NUMERICAL EXPERIMENTS

Now we turn our attention to the practical computation of wavelet signs for sampled signals. In practice, only a finite number of wavelet scales $\{a_j\}_{j=1}^N$ is available. Furthermore, since we cannot test for convergence in (1) using every signature wavelet, we have to choose a suitable signature wavelet κ. Thus, we have to estimate the signature from the finite set of samples $\{\mathrm{sgn}\, \langle f, \kappa_{a_j,b}\rangle\}_{j=1}^N$.

To motivate our numerical approach, we begin by considering the following elementary convergence result for discrete samples.

Proposition 12. *Let f be a tempered distribution, $\{a_j\}_{j\in\mathbb{N}}$ a sequence such that $a_j \to 0$, and $b \in \mathbb{R}$. If $\sigma f(b) \neq 0$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \mathrm{sgn}\, \langle f, \kappa_{a_j,b}\rangle = \sigma f(b) \tag{14}$$

for all signature wavelets κ.

Proposition 12 suggests the Cesàro limit (14) as an alternative to computing a non-zero signature $\sigma f(b)$. Note that $\sigma f(b)$ is of modulus 1 and so is the Cesàro limit (14). Furthermore, the elements of the Cesàro sequence

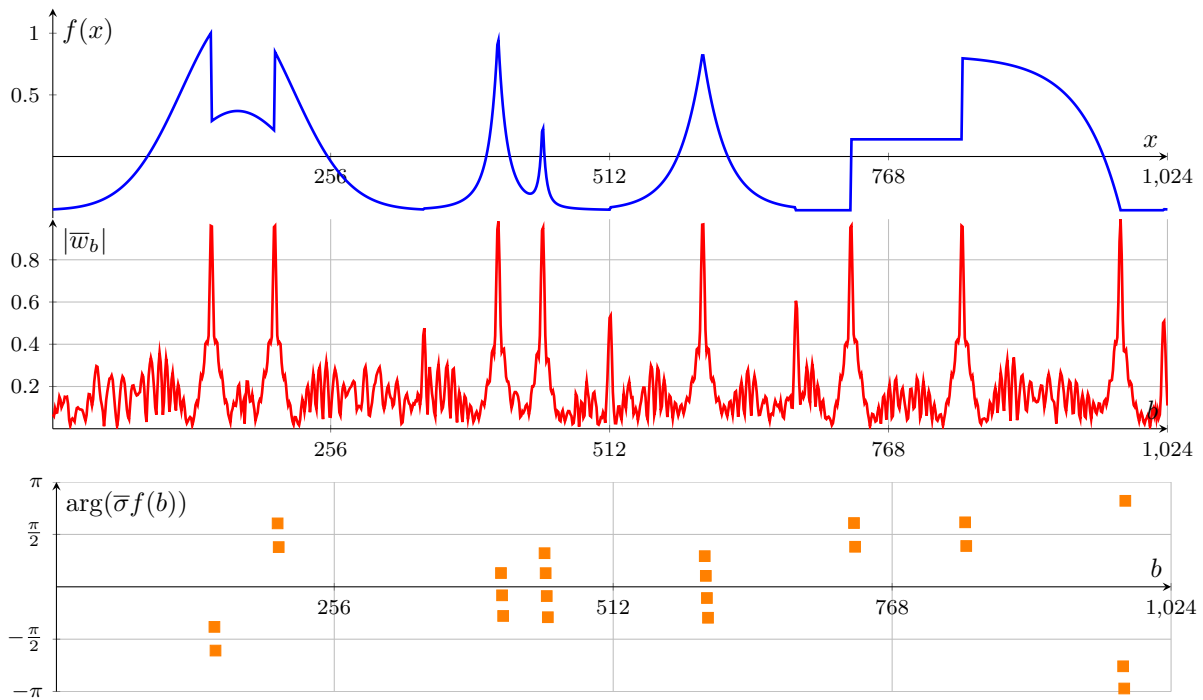$$\frac{1}{N} \sum_{j=1}^N \mathrm{sgn}\, \langle f, \kappa_{a_j,b}\rangle, \quad N \in \mathbb{N},$$

Fig. 2. The discrete signature of a sample signal (top) taken from Wavelab [1]. We observe that the absolute value of the mean $|\overline{w}_b|$ is large at the feature points and much lower at the other points (center). The bottom plot depicts the discrete signature in phase angle representation. We see that the signature clusters around the angles $\pm\frac{\pi}{2}$ at the step-like points, and around π and 0 at cusp-like points. The threshold τ is set equal to 0.7 in this experiment. If we choose a lower threshold parameter, say $\tau = 0.4$, then the discrete signature would also catch the subtle feature points, like the small jump at $x = 512$. However, in that case, we would require a non-maximum suppression to maintain the sharp localization of the pronounced feature points.

are not necessarily of modulus one, but their moduli converge to 1 as N goes to infinity. This observation motivates the following procedure for the numerical estimation of the signature.

Given a finite number of scale samples $\{a_j\}_{j=1}^{N}$, we interpret the mean of the sequence of discrete signs, given by

$$\overline{w}_b := \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \left\langle f, \kappa_{a_j,b} \right\rangle, \qquad (15)$$

as the N-th element of a Cesàro sequence. If the absolute value $|\overline{w}_b|$ is close to 1, we consider the Cesàro sequence as being convergent, with $\operatorname{sgn} \overline{w}_b$ giving an estimate of $\sigma f(b)$. On the other hand, a small value of $|\overline{w}_b|$ suggests a vanishing signature. More precisely, we consider $|\overline{w}_b|$ to be non zero if it exceeds some empirical threshold parameter τ between 0 and 1. Hence, we propose a discrete estimate $\overline{\sigma}f(b)$ of the signature of the form

$$\overline{\sigma}f(b) := \begin{cases} \operatorname{sgn} \overline{w}_b, & \text{if } |\overline{w}_b| \geq \tau, \\ 0, & \text{elsewhere.} \end{cases} \qquad (16)$$

In Figure 2, we see a numerical experiment based on the above procedure. We observe that the modulus of the mean $|\overline{w}_b|$ is large at the salient points. Furthermore, we see that the discrete signature is oriented towards the imaginary axis for jump discontinuities and oriented to the real axis for cusp singularities. This experiment illustrates that the procedure proposed above yields a reasonable way to compute the signature numerically. We used the Meyer-type signature

wavelet (5) and scale samples of the form $a_j = 2^{-\frac{j}{3}}$, with $j = 0, 1, \ldots, 15$. The threshold parameter was set to $\tau = \frac{1}{2}\sqrt{2} \approx 0.7$.

In [10], a generalization of the discrete signature to higher dimensions is proposed, which can be applied directly for sign based edge detection and edge analysis. That generalization bases on monogenic wavelets similar to those of [3].

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