Finite Rate of Innovation Signals: Quantization Analysis with Resistor-Capacitor Acquisition Filter

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***Abstract*—Sampling and perfect reconstruction of Finite rate of innovation (FRI) signals, which are usually not bandlimited, was introduced by Vetterli, Marziliano, and Blu [1].**

A typical FRI reconstruction algorithm requires solving for FRI signal parameters from a power-sum series. This in turn requires annihilation filters and polynomial root-finding techniques. These steps complicate the analysis of FRI signal reconstruction in the presence of *quantization*. In this work, we introduce a *three-channel* resistor-capacitor filter bank for the acquisition and reconstruction of FRI signals consisting of stream of Diracs and nonuniform splines. The effect of quantization error is derived for our three-channel filter-bank scheme. However, the sampling-rate required for our scheme is larger than the minimum sampling-rate of FRI signals.

I. Introduction

Parametric signals with finite degrees of freedom per unit time can be nonbandlimited [1]. E.g., for an integer $K_0 > 0$

$$x(t) = \sum_{k=0}^{K_0-1} c_k \delta(t - t_k), \quad c_k, t_k \in \mathbb{R}, \tag{1}$$

for all $0 \leq k \leq K_0 - 1$ is a parametric signal specified by the $2K_0$ real-valued parameters $\{(t_0, c_0), (t_1, c_1), \ldots, (t_{K_0-1}, c_{K_0-1})\}$. However, the Fourier bandwidth of $x(t)$ is infinite. If a signal is formed by the superposition of shifted and scaled versions of a known pulse, then the shifts and amplitudes of this pulse constitute its degrees of freedom rather than its Fourier bandwidth. Parametric signal class is large and it includes piecewise polynomials and non-uniform splines [1]. Signals, which can be specified by a finite number (or finite rate) of parameters are finite rate of innovation (FRI) signals. For an FRI signal, the degrees of freedom per unit time is the fundamental quantity to be used for determining the sampling rate [1].

The stream of Dirac delta signals in (1) has been widely studied due to its applicability in biomedical signal modeling, ultra-wideband communications, and global positioning system (e.g., see [2], [3]). Typically a power sum series has to be solved to obtain the parameters of an FRI signal. The solution involves annihilation filter and polynomial root finding [1], [4], [5], [6]. This approach is not amenable to quantization error analysis [7]. To the best of our knowledge, closed-form upper-bounds for quantization error in FRI signals are not known.

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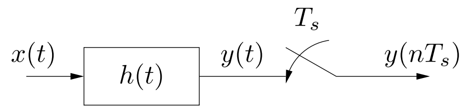

Fig. 1. *The FRI signal acquisition setup of Vetterli et al. [1] is illustrated. The filter $h(t)$ spreads the spikes thereby making $y(t)$ suitable for sampling. Typically $h(t)$ is a Gaussian or a sinc-filter.*

The acquisition filter $h(t)$ in Fig. 1 is a design choice. Consider FRI signals consisting of a stream of Dirac delta signals or nonuniform splines. For these FRI signals, a new sample acquisition setup consisting of a *three channel* resistor-capacitor (RC) filter-bank is proposed in this work. For this setup, *closed-form upper bounds* on error in FRI signal parameters due to quantization will be derived. Two channel or multi-channel resistor-capacitor filter banks for the reconstruction of stream of Diracs or other FRI signals, respectively, have been considered in the past for perfect reconstruction [8], [9]. However, these works do not study quantization and a detailed scheme for the sampling of nonuniform splines have not been suggested in them. The FRI signal reconstruction method proposed in this work *does not* involve a power sum series. This simplification, though, comes at a faster sampling rate than the minimum required by FRI signal sampling. [1]
Organization: Section II describes the signal model and our acquisition filter. Related work is reviewed in Section III. Perfect reconstruction and quantized reconstruction are discussed in Section IV. Conclusions are presented in Section V.

II. Modeling Assumptions

A finite-duration FRI signal is completely characterized by a finite number of parameters. Within the wide class of FRI signals, we consider the following signal model in this work,

$$x(t) = \sum_{k=0}^{K_0-1} c_{k,0}\delta(t - t_{k,0}) + \ldots + \sum_{k=0}^{K_p-1} c_{k,p}\delta^{(p)}(t - t_{k,p}) \tag{2}$$

[1]See C1 in Sec. IV-A for the exact condition.

where $\delta^{(r)}(t)$ denotes the r^{th} order derivative of the Dirac delta signal. The time epochs $t_{k,j} \neq t_{l,i}$ if $i \neq j$. Apart from modeling neural signals, it is also known that non-uniform splines can be reduced to the form of (2) after differentiation operations [1]. Due to space constraints, the signal model will be limited to stream of Dirac delta signal and its first derivative. The analysis extends in an analogous fashion to stream of Dirac delta signal and its higher order derivatives. Denote $x_l^m := (x_l, x_{l+1}, \ldots, x_m)$ for $m > l$. Given this signal, the parameters $\{(c_i)_0^{K_i-1}, (t_i)_0^{K_i-1}\}_{i=0}^{p-1}$ are to be (approximately) obtained from a set of sampled and quantized values obtained after filtering $x(t)$.

An ideal first-order RC filter will be used to facilitate the sampling of FRI signal in (2). Its impulse response is

$$h_{\text{rc}}(t) = e^{-\lambda t} u(t),$$

where $\lambda > 0$ is the *decay-rate* and $u(t)$ is the unit-step function. This filter is *causal* and can be implemented by a circuit consisting of single resistance of value R and single capacitor of value C. The decay-rate is $\lambda = 1/(RC)$. This passive filter is one of the simplest to implement in practice.

III. Prior Art

Sampling and perfect reconstruction of FRI signals with Gaussian and ideal lowpass acquisition filters was recently studied by Vetterli, Marziliano, and Blu [1]. These filters transform the problem of unknown Dirac delta signal (or its derivative) locations t_0^{K-1} to that of frequency estimation of a power sum series; the frequencies are estimated using annihilation filters. This method works well for perfect reconstruction.

FRI signal in (1) has been studied in application areas such as biomedical signal processing, ultra wideband communications, and global positioning system (e.g., see [2], [3]). Quantization noise analysis of FRI sampling and reconstruction has not been addressed [1], [5], [6] since the annihilation filter and polynomial root finding technique are complicated. Some quantization and oversampling results pertaining to FRI signal sampling are known in the literature [10]. Any *closed-form error analysis* due to quantization is mostly unsolved to the best of our knowledge.

In the presence of statistical sensing noise, the estimation of FRI signal parameters has also been studied in the literature. A qualitative analysis related to the numerical stability of some of these algorithms is presented in [5]. Related work includes the derivation of Cramer-Rao lower bounds for estimated poles of the power-sum series under additive Gaussian noise in [11]. This analysis, however, is restricted to a maximum of two delta functions.

IV. FRI signal reconstruction and quantization

Our sampling scheme for FRI signals in (2) is presented in two parts. Perfect reconstruction is presented first and quantization analysis is discussed in the later section.

A. *Perfect reconstruction of Dirac delta signals with RC filters*

Conceptually, a term of the form $c_k \delta^{(p)}(t - t_k)$, with $p = 0, 1$ has three degrees of freedom, namely, the constant c_k, the unknown order p, and the time instant t_k. Three RC-filters in parallel will be used to identify these three parameters. The general signal model is given by

$$x(t) = \sum_{k=0}^{K_0-1} c_{k,0} \delta(t - t_{k,0}) + \sum_{k=0}^{K_1-1} c_{k,1} \delta^{(1)}(t - t_{k,1}). \quad (3)$$

where the constants K_0, K_1 are positive integers. The time epochs $t_{k,j} \neq t_{l,i}$ if $i \neq j$. This signal class is obtained when piecewise linear signals are subjected to two differentiation operations. Consider the acquisition system shown in Fig. 2. There are three parallel RC filters with distinct decay-rate λ_1, λ_2, and λ_3. These filter outputs will be used to reconstruct the three degrees of freedom associated with every Dirac delta signal or its derivative present in $x(t)$.

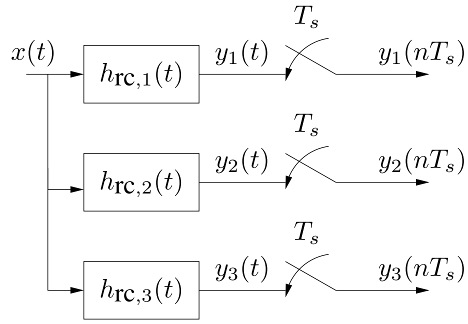

Fig. 2. *The three RC filters in parallel can be used to sample the signal in (3), provided T_s satisfies Condition C1 and $\lambda_1, \lambda_2, \lambda_3$ are distinct.*

Consider $\delta^{(1)}(t - t_0)$ as the input to an RC filter. Since delay and differentiation are linear time-invariant operations, the output of an RC filter with decay-rate λ_1 is given by

$$\begin{aligned} \frac{\mathrm{d}h(t - t_0)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} [\exp(-\lambda_1(t - t_0)) u(t - t_0)] \\ &= \delta(t - t_0) - \lambda_1 \exp[-\lambda_1(t - t_0)] u(t - t_0). \quad (4) \end{aligned}$$

Observe that, except at $t = t_0$ and a proportionality constant dependent on λ_1, these outputs are the same as the response of an RC filter to a Dirac delta signal at $t = t_0$. Define $\mathcal{T} := \{t_{0,0}, \ldots, t_{0,K_0-1}, t_{1,0}, \ldots, t_{1,K_1-1}\}$. The set $\mathcal{T}$ consists of all the points where Dirac delta signal or its derivative is present in the signal $x(t)$. Using linearity and the derivation in (4), it is straightforward to show that if $x(t)$ in (3) is the input to the system in Fig. 2, the output of the first filter is given by

$$y_1(t) = \sum_{k=0}^{K_0-1} c_{k,0} h_1(t - t_{k,0}) + \sum_{k=0}^{K_1-1} c_{k,1}(-\lambda_1) h_1(t - t_{k,1}).$$

The elements of $\mathcal{T}$ will be reordered for clarity in the analysis. Reorder the elements of set $\mathcal{T}$ in an ordered set $\{t_0, t_1, \ldots, t_{K-1}\}$ where $K = K_0 K_1$. Due to re-ordering,

$t_i = t_{l_i,j_i}$ for some unique (l_i, j_i) pair for each i. Thus,

$$y_1(t) = \sum_{k=0}^{K-1} c_k h_1(t - t_k) = e^{-\lambda_1 t} \sum_{k=0}^{K-1} c_k u(t - t_k)$$

for $t \notin \mathcal{T}$, where $c_k = c_{l_k,j_k}(-\lambda_1)^{j_k}$.[2] The parameter c_k depends on λ_1. The decay-rate λ_1 is known but c_{l_k,j_k} and j_k are parameters to be obtained or approximated. After sampling at nT_s and multiplication by $e^{\lambda_1 nT_s}$, the following readings are obtained provided $nT_s \notin \mathcal{T}$:

$$e^{\lambda_1 nT_s} y_1(nT_s) = \sum_{k=0}^{K-1} c_k e^{\lambda_1 t_k} u(nT_s - t_k). \tag{5}$$

Now the following condition is assumed:

C1: $T_s < \min_i \{t_i - t_{i-1}\}$ and $nT_s \neq t_i$ for any i and n.

Under Condition C1, the different levels of the piecewise constant discrete-time signal in (5) reveal the product $c_k e^{\lambda_t t_k} = c_{l_k,j_k}(-\lambda_1)^{j_k} \exp(\lambda_1 t_{l_k,j_k})$ one by one for different values of $k = 0, 1, \ldots, K-1$. Under Condition C1, there is at least one sample between consecutive shifted Dirac or its derivative in $x(t)$; thus, for each Dirac or its derivative at t_i an integer $N_i \in \mathbb{Z}$ exists such that

$$e^{\lambda_1 N_i T_s} y_1(N_i T_s) - e^{\lambda_1 N_{i-1} T_s} y_1(N_{i-1} T_s) = c_{l_i,j_i}(-\lambda_1)^{j_i} e^{\lambda_1 t_{l_i,j_i}}. \tag{6}$$

The value of λ_1 is known. The following result of interest is stated and proved next. All the logarithms have base e.

Proposition 4.1: Assume that three RC-filters in parallel operate with distinct $(\lambda_1, \lambda_2, \lambda_3)$ and sampling rate T_s satisfies Condition C1. Then there exist indices $N_i, i = 0, 1, \ldots, K-1$ such that $e^{\lambda_1 N_i T_s} y_1(N_i T_s) = \sum_{k=0}^{i} c_k \exp(\lambda_1 t_k)$. Define $d_m(i) := e^{\lambda_m N_i T_s} y_m(N_i T_s) - e^{\lambda_m N_{i-1} T_s} y_m(N_{i-1} T_s)$ for $m = 1, 2, 3$. Choose $(\lambda_2)^2 = \lambda_1 \lambda_3$. Then the parameters of $x(t)$ in (3) are given by the following set of equations,

$$t_{l_i,j_i} = \frac{1}{\lambda_1 + \lambda_3 - 2\lambda_2} \log \left[\frac{d_1(i) d_3(i)}{d_2^2(i)} \right], \tag{7}$$

$$j_i = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} \left[\log\left[\frac{d_1(i)}{d_2(i)}\right] + (\lambda_2 - \lambda_1) t_{l_i,j_i} \right], \tag{8}$$

$$\text{and } c_i = (-\lambda_1)^{j_i} c_{l_i,j_i} = \frac{d_1(i)}{e^{\lambda_1 t_{l_i,j_i}}}. \tag{9}$$

Proof: The existence of N_i has been argued while deriving (6); it follows from the definition of the unit-step function and (5). In the following equations, m takes the values $1, 2, 3$. The output of the three channels in Fig. 2 are given by,

$$y_m(N_i T_s) = e^{-\lambda_m N_i T_s} \sum_{k=0}^{i} c_k e^{\lambda_m t_k},$$

$$\text{or} \quad e^{\lambda_m N_i T_s} y_j(N_i T_s) = \sum_{k=0}^{i} c_k e^{\lambda_m t_k}.$$

Upon successive subtraction, we get

$$d_m(i) = e^{\lambda_m N_i T_s} y_m(N_i T_s) - e^{\lambda_m N_{i-1} T_s} y_m(N_{i-1} T_s) = c_k \exp(\lambda_m t_k). \tag{10}$$

The equations in (7), (8), and (9) follow from (10) by simple algebraic manipulations and using $\lambda_2^2 = \lambda_1 \lambda_3$. Since $\lambda_1, \lambda_2, \lambda_3$ are in geometric progression. The arithmetic mean of two unequal numbers is great than their geometric mean. Hence, $\lambda_1 + \lambda_3 > 2\lambda_2$. This ensures that the expression in (7) is well defined. ■

B. Quantization error in RC filter sampling scheme

In this section, it is assumed that $y_m(nT_s)$ values are quantized. Bounds on approximated parameters obtained through Proposition 4.1 will be derived. To work with scalar quantizers and maximum pointwise error, it is assumed that $|y_m(t)|$ is bounded. Without loss of generality, $|y_m(t)| \leq 1$ for all $t \in \mathbb{R}$. A uniform scalar quantizer will be assumed for analysis, where the quantizer precision is L-bits [12]. Let $\widehat{y}_m(nT_s)$ be the quantized value of $y_m(nT_s)$. Define $e_m(nT_s) := \widehat{y}_m(nT_s) - y_m(nT_s)$ Then, the following pointwise bound

$$|e_m(nT_s)| = |\widehat{y}_m(nT_s) - y_m(nT_s)| \leq 2^{-L}$$

holds for uniform scalar quantizer [12]. With quantized samples $\widehat{y}_m(nT_s)$, the approximate variables $d_m(i)$ have an error

$$|\widehat{d}_m(i) - d_m(i)| = |e^{\lambda_m N_i T_s} e_m(N_i T_s) - e^{\lambda_m N_{i-1} T_s} e_m(N_{i-1} T_s)|.$$

The FRI signal parameters can be approximated as follows:

$$\widehat{t}_{l_i,j_i} = \frac{1}{\lambda_1 + \lambda_3 - 2\lambda_2} \log \left[\frac{\widehat{d}_1(i) \widehat{d}_3(i)}{\widehat{d}_2^2(i)} \right], \tag{11}$$

$$\widehat{j}_i = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} \left[\log\left(\frac{\widehat{d}_1(i)}{\widehat{d}_2(i)}\right) + (\lambda_2 - \lambda_1) \widehat{t}_{l_i,j_i} \right], \tag{12}$$

$$\text{and } \widehat{c}_{l_i} = (-\lambda_1)^{\widehat{j}_i} \widehat{c}_{l_i,j_i} = \frac{\widehat{d}_1(i)}{\exp(\lambda_1 \widehat{t}_{l_i,j_i})}. \tag{13}$$

Note that $\widehat{d}_m(N_i T_s) = c_i(m) e^{\lambda_m t_i} + e_m(N_i T_s) e^{\lambda_m N_i T_s} - e_m(N_{i-1} T_s) e^{\lambda_m N_{i-1} T_s}$. The constant c_i depends on m through λ_m for $m = 1, 2, 3$. The main result is stated next.

Theorem 4.1: Let $\widehat{y}_m(nT_s)$ be available with T_s satisfying Condition C1 and $m = 1, 2, 3$. Let $\lambda_1, \lambda_2, \lambda_3$ be distinct. Define the approximations for FRI signal parameters as in (11) and (13). Denote $\Delta := \lambda_1 + \lambda_3 - 2\lambda_2$. Then,

$$|\widehat{t}_i - t_i| \leq -\frac{4}{\Delta} \min_m \log \left[1 - \frac{2^{-L}(1 + e^{\lambda_m T_s})}{|c_i(m)|} \right]$$

and

$$\left| \frac{\widehat{c}_i(m) - c_i(m)}{c_i(m)} \right| \leq \left[1 + \frac{2^{-L}(1 + e^{\lambda_m T_s})}{|c_i(m)|} \right] \left[1 - \frac{2^{-L}(1 + e^{\lambda_{m^*} T_s})}{|c_i(m^*)|} \right]^{\frac{-4\lambda_m}{\Delta}} - 1,$$

where m^* is obtained by maximization as discussed in proof.

[2] At locations mentioned in this set $\mathcal{T}$, Dirac delta signal and its derivatives are present.

Proof: The proof omits algebraic steps for brevity. Define

$$\beta_{m,i} := \frac{e_m(N_iT_s)e^{\lambda_m[N_iT_s-t_i]} - e_m(N_{i-1}T_s)e^{\lambda_m[N_{i-1}T_s-t_i]}}{c_i(m)}.$$

Then

$$|\beta_{m,i}| \leq \frac{2^{-L}|e^{\lambda_m[N_iT_s-t_i]}| + 2^{-L}|e^{\lambda_m[N_{i-1}T_s-t_i]}|}{|c_i(m)|}$$
$$\leq \frac{2^{-L}(1+e^{\lambda_m T_s})}{|c_i(m)|} \quad (14)$$

since N_i can be chosen such that $0 < N_iT_s - t_i < T_s$. Using quantized estimates $\widehat{d}_m(i)$, we get

$$\frac{\widehat{d}_1(N_iT_s)}{\widehat{d}_2(N_iT_s)} = \frac{c_i(1)}{c_i(2)}e^{(\lambda_1-\lambda_2)t_i}\left[\frac{1+\beta_{1,i}}{1+\beta_{2,i}}\right].$$

Similarly,

$$\frac{\widehat{d}_3(N_iT_s)}{\widehat{d}_2(N_iT_s)} = \frac{c_i(3)}{c_i(2)}e^{(\lambda_3-\lambda_2)t_i}\left[\frac{1+\beta_{3,i}}{1+\beta_{2,i}}\right].$$

Note that $c_i(1)c_i(3) = (c_i(2))^2$. Therefore,

$$\frac{\widehat{d}_1(N_iT_s)\widehat{d}_3(N_iT_s)}{\widehat{d}_2^2(N_iT_s)} = e^{(\lambda_1+\lambda_3-2\lambda_2)t_i}\left[\frac{(1+\beta_{1,i})(1+\beta_{3,i})}{(1+\beta_{2,i})^2}\right].$$

Taking logarithms on both sides, we get

$$|\widehat{t}_i - t_i| = \frac{1}{\Delta}\left|\log\left[\frac{(1+\beta_{1,i})(1+\beta_{3,i})}{(1+\beta_{2,i})^2}\right]\right|.$$
$$\leq \frac{-4}{\Delta}\min_m \log\left[1 - \frac{2^{-L}(1+e^{\lambda_m T_s})}{|c_i(m)|}\right].$$
$$= \frac{-4}{\Delta}\log\left[1 - \frac{2^{-L}(1+e^{\lambda_{m^*} T_s})}{|c_i(m^*)|}\right]$$

The last inequality utilizes the inequality $|\log(1+x)| \leq -\log(1-x_0)$ for all $|x| \leq x_0$. For very large values of L, note that the error is *decaying exponentially* in L as $\log(1-x) \approx -x$ for very small values of x.

The error in $\widehat{c}_i(m)$ will be derived now. From the definition of $\beta_{m,i}$

$$\widehat{c}_i(m) = c_i(m)e^{\lambda_m(t_i-\widehat{t}_i)}[1+\beta_{m,i}].$$

or
$$\frac{\widehat{c}_i(m) - c_i(m)}{c_i(m)} = e^{\lambda_m(t_i-\widehat{t}_i)}[1+\beta_{m,i}] - 1.$$

Thus,

$$\left|\frac{\widehat{c}_{m,i} - c_{m,i}}{c_{m,i}}\right| = |e^{\lambda_m(t_i-\widehat{t}_i)}[1+\beta_{m,i}] - 1|$$
$$\leq |e^{\lambda_m(t_i-\widehat{t}_i)} - 1| + |\beta_{m,i}e^{\lambda_m(t_i-\widehat{t}_i)}|$$

Now we note that $|e^{\theta} - 1| \leq e^{|\theta|} - 1$ for any θ. Therefore,

$$\left|\frac{\widehat{c}_i(m) - c_i(m)}{c_i(m)}\right|$$
$$\leq e^{\lambda_m|t_i-\widehat{t}_i|} - 1 + |\beta_{m,i}|e^{\lambda_m|t_i-\widehat{t}_i|}$$
$$\leq (1+|\beta_{m,i}|)\left[1 - \frac{2^{-L}(1+e^{\lambda_{m^*}T_s})}{|c_i(m^*)|}\right]^{-4\lambda_m/\Delta} - 1.$$

Substituting the upper-bound on $\beta_{m,i}$ from (14),

$$\left|\frac{\widehat{c}_i(m) - c_i(m)}{c_i(m)}\right|$$
$$\leq \left[1 + \frac{2^{-L}(1+e^{\lambda_m T_s})}{|c_i(m)|}\right]\left[1 - \frac{2^{-L}(1+e^{\lambda_{m^*}T_s})}{|c_i(m^*)|}\right]^{\frac{-4\lambda_m}{\Delta}} - 1.$$

As for $\widehat{t}_i$, if L is very large, then the error in $\widehat{c}_i(m)$ is proportional to 2^{-L}. ■

It must be noted that $\widehat{j}_i$ is either 0 or 1. For large-enough L, this parameter can be recovered exactly since j_i is discrete. Due to space constraints the derivations for $\widehat{j}_i$ and condition on L under which it can be recovered exactly is omitted.

V. Conclusions

In this work, a new sample acquisition method for sampling and reconstruction of an important class of FRI signals was explored. The new method, consisting of RC filters in parallel, studied in this work does not require solving a power-sum series, and ensuing annihilation filters or polynomial root finding, to obtain the FRI signal parameters. The effect of quantization error, in terms of upper bound on parameter reconstruction error, was addressed for our setup. Quantization error bounds are not available with the power-sum series approach. If L bits are used for quantizing each sample, then the reconstruction error was shown to be eventually decreasing as 2^{-L}. However, the sampling-rate required for our scheme is larger than the minimum sampling-rate of FRI signals.

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