Orlicz Modulation Spaces

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***Abstract*—In this work we extend the definition of modulation spaces associated to Lebesgue spaces to Orlicz spaces and mixed-norm Orlicz spaces. We give the definition of the Orlicz spaces L^Φ, a generalisation of the L^p spaces of Lebesgue. Therefore we characterise the Young function Φ and give some basic properties of this spaces. We collect some facts about this spaces that we need for the time frequency analysis, then we introduce the Orlicz modulation spaces. Finally we present a discretisation of the Orlicz space and mixed-norm Orlicz space and a characterisation of the modulation space by discretisation.**

I. Introduction

The modulation spaces were introduced in 1983 by H. Feichtinger. The idea is to impose a norm on the short-time Fourier transform and to define Banach spaces of signals with a given time-frequency behavior. Especially, the modulation space $M^{p,q}$ consists of all tempered distributions such that the short-time Fourier transform is a function in the mixed-norm Lebesgue space $L^{p,q}$. We will extend this concept and examine modulation spaces associated to Orlicz spaces and mixed-norm Orlicz spaces. The Orlicz spaces L^Φ are a generalisation of the L^p spaces of Lebesgue. For the Young function $\Phi(x) = |x|^p$ with $p \geq 1$, $L^\Phi(\mu) = L^p(\mu)$. In general, the function Φ is a convex function, precisely a Young function. The mixed-norm Orlicz spaces $L^{\Phi_1\Phi_2}$ are vector-valued L^{Φ_2} spaces where Φ_1, Φ_2 are Young functions. Since the function $x \mapsto f(\cdot, x)$ takes values in the Banach space L^{Φ_2}, the mixed-norm Orlicz spaces $L^{\Phi_1\Phi_2}$ arise by taking a L^{Φ_2} norm with respect to the time variable x and an L^{Φ_1} norm with respect to the frequency variable w. This can be considered as a generalisation of the mixed-norm Lebesgue spaces $L^{p,q}$. As general setting let (Ω, Σ, μ) be a measure space, where Ω is a set, Σ is a σ-algebra of Ω and μ a σ-additive measure on Σ and $f : \Omega \to \overline{\mathbb{C}}$ is a measurable function. We also assume that the measure μ has the finite subset property, i.e., for $E \in \Sigma$ with $\mu(E) > 0$ there exists a subset $F \in \Sigma$ with $F \subset E$ and $0 < \mu(F) < \infty$.

II. Orlicz spaces and mixed-norm Orlicz spaces

A. Definition and properties

Firstly we give the definition of a Young function Φ and the Δ_2-condition, which is a growth condition.
After that we introduce the Orlicz spaces and characterise norms so that these spaces are Banach spaces. Then we determine their corresponding dual spaces.

This section is based on the book [8] *Theory of Orlicz spaces* of Rao and Ren.

Definition 1: (Young function) A convex function $\Phi : \mathbb{R} \to \overline{\mathbb{R}^+}$ which satisfies the conditions:

1) $\Phi(-x) = \Phi(x)$,$\Phi(0) = 0$,
2) $\lim_{x\to\infty}\Phi(x) = +\infty$,

is called *Young function*.
In the theory of Lebesgue spaces, the conjugate exponent q to p is related to the dual space. By analogy, one can define the so called complementary function, this function is the counterpart to the conjugate exponent.

Definition 2: (Complementary function) If $\Psi : \mathbb{R} \to \overline{\mathbb{R}^+}$ is defined by $\Psi(y) = \sup\{x|y| - \Phi(x); x \geq 0\}$. Then Ψ is called the *complementary function* to the Young function Φ.

In the structure theory of Orlicz spaces a classification of the Young function based on properties of their growth plays a central role. Of particular importance for us will be the Δ_2-condition.

Definition 3: (Δ_2-condition) A Young function $\Phi : \mathbb{R} \to \mathbb{R}^+$ is said to satisfy the Δ_2-*condition*, if there exists a constant $K > 0$ and $x_0 \in \mathbb{R}_0^+$, such that

$$\Phi(2x) \leq K\Phi(x) \qquad \text{for all } x \geq x_0 \geq 0.$$

Hereafter we say that a Δ_2-condition for Φ is *regular* if it holds locally (for a $x_0 > 0$) when the measure in $L^\Phi(\mu)$ is finite and globally (for $x_0 = 0$) when the measure is infinite.

Definition 4: (Orlicz space) The function space

$$L^\Phi(\mu) = \Big\{ f\colon \Omega \to \overline{\mathbb{C}} \text{ (equivalence classes of) } \Sigma\text{-measurable:} \int_\Omega \Phi(\alpha|f|)\, \mathrm{d}\mu < \infty \text{ for at least one } \alpha > 0 \Big\}$$

with $\Phi : \mathbb{R} \to \mathbb{R}^+$ a Young function, is called *Orlicz space*.

We next define norms on $L^\Phi(\mu)$.

Definition 5: (Gauge norm and Orlicz norm) The norm

$$N_\Phi(f) = \inf\left\{ k > 0 : \int_\Omega \Phi\left(\frac{|f|}{k}\right) \mathrm{d}\mu \leq 1 \right\}$$

is called *gauge norm* of the Orlicz space $L^\Phi(\mu)$ for a Young function $\Phi : \mathbb{R} \to \mathbb{R}^+$.
By using the complementary Young function we can define another norm on $L^\Phi(\mu)$.

Let (Φ, Ψ) be a complementary pair of Young functions, then we define the *Orlicz norm* as:

$$\|.\|_{L^\Phi}\colon f \mapsto \|f\|_{L^\Phi} = \sup\left\{\int_\Omega |fg|\,\mathrm{d}\mu : \int_\Omega \Psi\left(|g|\right)\,\mathrm{d}\mu \le 1\right\}.$$

The two norms defined on the Orlicz spaces are equivalent, furthermore the Orlicz spaces with the corresponding norms are Banach spaces.

Theorem 1: [8] [Proposition 4 3.3.III, Corollary 12 III.3.3] Let (Ω, Σ, μ) be a measure space, (Φ, Ψ) be a complementary Young pair, then $N_\Phi(f) \le \|f\|_{L^\Phi} \le 2N_\Phi(f)$ for $f \in L^\Phi(\mu)$. $\left(L^\Phi(\mu), N_\Phi(\cdot)\right)$ and $\left(L^\Phi(\mu), \|\cdot\|_{L^\Phi}\right)$ are Banach spaces.

Since it is often useful to work with duality arguments in proofs, we give a characterisation of the dual space to the Orlicz space in the next theorem.

Theorem 2: [8] [Theorem 7, Corollary 9 IV.4.1] Let (Φ, Ψ) be a complementary Young pair and Φ be Δ_2-regular and (Ω, Σ, μ) be σ-finite. Then $\left(L^\Phi(\mu)\right)^*$ is isometrically isomorphic to $L^\Psi(\mu)$.

We next extend the Orlicz space theory of $\mathbb{C}$-valued functions $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ to functions $f : \Omega \subset \mathbb{R}^d \to X$ whose values lie in a Banach space X. Candidates for X are Orlicz spaces L^{Φ_2} associated to a Young function Φ_2.

Definition 6: (Mixed-norm Orlicz space) Let $(\Omega_i, \Sigma_i, \mu_i)$ be measure spaces, (Φ_i, Ψ_i) be complementary Young pairs for $i = 1, 2$. Then the *mixed-norm Orlicz space* is

$$\begin{aligned} L^{\Phi_1\Phi_2} &= L^{\Phi_1}(\mu_1, L^{\Phi_2}(\mu_2)) \\ &= \Big\{ f\colon \Omega_1 \to L^{\Phi_2}(\mu_2) \text{ strongly measurable on } (\Omega_1, \Sigma_1, \mu_1)\colon \\ &\qquad \int_{\Omega_1} \Phi_1(\alpha N_{\Phi_2}(f))\,\mathrm{d}\mu_1 < \infty \text{ for some } \alpha > 0 \Big\}. \end{aligned}$$

The corresponding *gauge norm* $N_{\Phi_1\Phi_2}(\cdot) = N_{\Phi_1}(N_{\Phi_2}(\cdot))$ is given by:

$$N_{\Phi_1\Phi_2}(f) = \inf\left\{k > 0 : \int_{\Omega_1} \Phi_1\left(\frac{|N_{\Phi_2}(f(\cdot, w_1))|}{k}\right) \mathrm{d}\mu_1(w_1) \le 1\right\}.$$

The *Orlicz norm* is similarly defined by

$$\|f\|_{\Phi_1\Phi_2} = \sup\left\{\int_{\Omega_1} \left| \|f(\cdot, w_1)\|_{L^{\Phi_2}} \cdot g(w_1)\right| \mathrm{d}\mu_1(w_1) : \int_{\Omega_1} \Psi_1(|g(w_1)|)\,\mathrm{d}\mu_1(w_1) \le 1\right\},$$

As in the case of the Orlicz spaces the mixed-norm Orlicz spaces are also Banach spaces and it can be shown that the norms are equivalent.

Theorem 3: Let $(\Omega_i, \Sigma_i, \mu_i)$ be measure spaces, (Φ_i, Ψ_i) be complementary Young pairs for $i = 1, 2$, then $(L^{\Phi_1}(\mu_1, L^{\Phi_2}(\mu_2)), N_{\Phi_1\Phi_2}(\cdot))$ and $(L^{\Phi_1}(\mu_1, L^{\Phi_2}(\mu_2)), \|\cdot\|_{L^{\Phi_1\Phi_2}})$ are Banach spaces and the norms are equivalent. Furthermore it follows

$$N_{\Phi_1\Phi_2}(f) \le \|f\|_{L^{\Phi_1\Phi_2}} \le 4N_{\Phi_1\Phi_2}(f) \text{ for } f \in L^{\Phi_1\Phi_2}.$$

If we assume that the Young functions are also strictly convex the dual space to L^{Φ_1,Φ_2} is isometrically isomorphic to the space L^{Ψ_1,Ψ_2} to the complementary functions.

Theorem 4: [8] [Theorem 4 VII.7.5] Let $(\Omega_i, \Sigma_i, \mu_i)$ be measure spaces, (Φ_i, Ψ_i) be complementary Young pairs which are Δ_2-regular and strictly convex for $i = 1, 2$. Then $\left(L^{\Phi_1\Phi_2}\right)^*$ is isometrically isomorphic to $L^{\Psi_1\Psi_2}$.

B. Useful properties for time frequency analysis

In this section we list properties of the Orlicz spaces which are useful for time-frequency analysis. At first we mention that the Orlicz norm and the mixed Orlicz norm are invariant under translations, if the measure spaces are the Lebesgue space $(\Omega_i, \Sigma_i, \mu_i) = (\mathbb{R}^d, \mathcal{B}^d, \boldsymbol{\lambda}^d)$ for $i = 1, 2$.

Lemma 1: Let Φ_i be Young functions for $i = 1, 2$, then $L^{\Phi_1}(\boldsymbol{\lambda}^d)$ and $L^{\Phi_1\Phi_2}(\boldsymbol{\lambda}^{2d}) = L^{\Phi_1}(\boldsymbol{\lambda}^d, L^{\Phi_2}(\boldsymbol{\lambda}^d))$ are invariant under $T_zF := F(\cdot - z)$ and we have

$$N_{\Phi_1}(T_zF) = N_{\Phi_1}(F) \text{ for } F \in L^{\Phi_1}(\boldsymbol{\lambda}^d), z \in \mathbb{R}^d \quad \text{and}$$
$$N_{\Phi_1\Phi_2}(T_zF) = N_{\Phi_1\Phi_2}(F) \text{ for } F \in L^{\Phi_1\Phi_2}(\boldsymbol{\lambda}^{2d}), z \in \mathbb{R}^{2d}.$$

Futher one can also prove a Hölder inequality for Orlicz spaces.

Lemma 2: (Hölder inequality)[8] [Proposition 1 III.3.3] Let $(\Omega_i, \Sigma_i, \mu_i) = (\mathbb{R}^d, \mathcal{B}^d, \boldsymbol{\lambda}^d)$ and (Φ_i, Ψ_i) be complementary Young pairs for $i = 1, 2$. If $F \in L^{\Phi_1}(\boldsymbol{\lambda}^d)$ and $G \in L^{\Psi_1}(\boldsymbol{\lambda}^d)$, then one has $\displaystyle\int_{\mathbb{R}^d} |F \cdot G|\,\mathrm{d}\boldsymbol{\lambda}^d \le 2 \cdot N_{\Phi_1}(F)N_{\Psi_1}(G).$

If we assume in addition that Φ_2 is Δ_2-regular, then one has for $F \in L^{\Phi_1\Phi_2}(\boldsymbol{\lambda}^{2d})$ and $G \in L^{\Psi_1\Psi_2}(\boldsymbol{\lambda}^{2d})$ the estimate

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |F \cdot G|\,\mathrm{d}\boldsymbol{\lambda}^d\,\mathrm{d}\boldsymbol{\lambda}^d \le 4 \cdot N_{\Phi_1\Phi_2}(F)N_{\Psi_1\Psi_2}(G).$$

Now, we have a look at inclusion properties. If Φ is continuous the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is embedded into the Orlicz space $L^\Phi(\boldsymbol{\lambda}^d)$ and if also the complementary function Ψ is continuous then the functions in the Orlicz space define tempered distributions.

Lemma 3: Let (Φ_i, Ψ_i) be pairs of complementary Young functions and Φ_i be continuous for $i = 1, 2$, then

$$\mathcal{S}(\mathbb{R}^d) \subset L^{\Phi_1}(\boldsymbol{\lambda}^d),$$
$$\text{and} \quad L^{\Phi_1}(\boldsymbol{\lambda}^d) \subset \mathcal{S}'(\mathbb{R}^d), \quad \text{if } \Psi_1 \text{ is continuous.}$$

And

$$\mathcal{S}(\mathbb{R}^{2d}) \subset L^{\Phi_1\Phi_2}(\boldsymbol{\lambda}^{2d}),$$
$$\text{and} \quad L^{\Phi_1\Phi_2}(\boldsymbol{\lambda}^{2d}) \subset \mathcal{S}'(\mathbb{R}^{2d}), \quad \text{if } \Psi_1, \Psi_2 \text{ are continuous.}$$

With the fact that $(L^\Phi)^* \cong L^\Psi$, we can extend a well known convolution relation $L^1(\mathbb{R}^d) * L^p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ of the Lebesgue spaces to the Orlicz spaces. Further one can prove the following Young inequality.

Theorem 5: If $F \in L^1(\mathbb{R}^{2d})$, $G \in L^\Phi(\boldsymbol{\lambda}^{2d})$ and Φ is a Δ_2-regular Young function, then

$$\|F * G\|_{L^\Phi} \le 2\|F\|_{L^1}\|G\|_{L^\Phi}.$$

If $F \in L^{1,1}(\mathbb{R}^{2d})$, $G \in L^{\Phi_1\Phi_2}$ and Φ_i are Δ_2-regular and strictly convex Young functions for $i = 1, 2$, then

$$\|F * G\|_{L^{\Phi_1\Phi_2}} \le 4\|F\|_{L^{1,1}}\|G\|_{L^{\Phi_1\Phi_2}}.$$

III. Orlicz Modulation spaces and mixed-norm Orlicz modulation spaces

We now have all the tools in place that we need to define and analyse the modulation space associated to the Orlicz space.

Definition 7: (Orlicz modulation space)
Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and a Young function Φ. Then the *Orlicz modulation space* $M^\Phi(\mathbb{R}^d)$ is defined by

$$M^\Phi(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : V_g f \in L^\Phi(\mathbb{R}^{2d})\}.$$

The norm on M^Φ is $\quad \|f\|_{M^\Phi} = \|V_g f\|_{L^\Phi}.$

In the same way we define the mixed-norm Orlicz modulation space.
Therefore we replace only the Orlicz space L^Φ by the mixed-norm Orlicz space $L^{\Phi_1\Phi_2}$.

Definition 8: (Mixed-norm Orlicz modulation space)
Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and Young functions Φ_i for $i = 1, 2$. Then the *Orlicz modulation space* $M^{\Phi_1\Phi_2}(\mathbb{R}^d)$ is defined by

$$M^{\Phi_1\Phi_2}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : V_g f \in L^{\Phi_1\Phi_2}(\mathbb{R}^{2d})\}.$$

The norm on $M^{\Phi_1\Phi_2}$ is $\quad \|f\|_{M^{\Phi_1\Phi_2}} = \|V_g f\|_{L^{\Phi_1\Phi_2}}.$

Remark 1: Modulation spaces are a special case of the coorbit spaces defined by H. Feichtinger and K.H. Gröchenig [1], and Orlicz spaces are mentioned, without proof, as classes of Banach function spaces Y suitable to define coorbit spaces CoY. In this paper we make this remark more explicit by providing additional details such as associated discrete coefficient spaces, relationship to tempered distributions, dual spaces, etc. We would also like to point out that to our knowledge, mixed-norm Orlicz spaces have not been considered previously.

Now we analyse a few properties of the Orlicz modulation spaces. We start with the observation that the definitions of these spaces are independent of the choice of a window g. In addition, if the Young function is Δ_2-regular, these spaces are also Banach spaces.

Theorem 6: Assume that Φ is a Δ_2-regular Young function and its complementary function Ψ is continuous. Then the definition of $M^\Phi(\mathbb{R}^d)$ is independent of the window $g \in \mathcal{S}(\mathbb{R}^d)$ and $M^\Phi(\mathbb{R}^d)$ is a Banach space.

If we assume that the Young functions are also strictly convex, we can show an analogous statement for the mixed-norm Orlicz spaces.

Theorem 7: Let (Φ_i, Ψ_i) be complementary Young pairs which are Δ_2-regular, strictly convex and continuous for $i = 1, 2$. Then the definition of $M^{\Phi_1\Phi_2}(\mathbb{R}^d)$ is independent of the window $g \in \mathcal{S}(\mathbb{R}^d)$ and $M^{\Phi_1\Phi_2}(\mathbb{R}^d)$ is a Banach space.

Furthermore, the duality between the Orlicz spaces L^Φ and L^Ψ suggests a similar statement for their modulation spaces. This can be proved in the following theorem by using the Δ_2-condition for the Young function.

Theorem 8: If (Φ, Ψ) is a complementary Young pair and if Φ is Δ_2-regular and continuous, then $\left(M^\Phi(\mathbb{R}^d)\right)^* \cong M^\Psi(\mathbb{R}^d)$ under the duality

$$\langle f, h\rangle = \iint_{\mathbb{R}^{2d}} V_{g_0} f(z)\overline{V_{g_0} h(z)}\,\mathrm{d}z$$

for $f \in M^\Phi(\mathbb{R}^d)$ and $h \in M^\Psi(\mathbb{R}^d)$, $g_0 \in \mathcal{S}(\mathbb{R}^d)$.
Let (Φ_i, Ψ_i) be complementary Young pairs which are Δ_2-regular, strictly convex and continuous for $i = 1, 2$. Then $\left(M^{\Phi_1\Phi_2}(\mathbb{R}^d)\right)^* \cong M^{\Psi_1\Psi_2}(\mathbb{R}^d)$ under the duality

$$\langle f, h\rangle = \iint_{\mathbb{R}^{2d}} V_{g_0} f(z)\overline{V_{g_0} h(z)}\,\mathrm{d}z$$

for $f \in M^{\Phi_1\Phi_2}(\mathbb{R}^d)$ and $h \in M^{\Psi_1\Psi_2}(\mathbb{R}^d)$, $g_0 \in \mathcal{S}(\mathbb{R}^d)$.

IV. Discrete Orlicz space and discrete mixed-norm Orlicz space

This space consists of all sequences for which the discrete norm defined by the next definition is finite.

Definition 9: (Discrete Orlicz space) Let Φ be a Young function, then the *discrete Orlicz space* is defined by

$$l^\Phi(\mathbb{Z}^d) = \{a = (a_n)_{n\in\mathbb{Z}^d} : n_\Phi(a) < \infty\},$$

where $\quad n_\Phi(a) = \inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}^d} \Phi\left(\frac{|a_n|}{\lambda}\right) \le 1\right\}.$

Definition 10: (Discrete mixed-norm Orlicz space) Let Φ_1, Φ_2 be Young functions, then the *discrete mixed-norm Orlicz space* is defined by

$$l^{\Phi_1\Phi_2}(\mathbb{Z}^{2d}) = \{a = (a_{kn})_{k,n\in\mathbb{Z}^d} : n_{\Phi_1\Phi_2}(a) < \infty\},$$

where

$$n_{\Phi_1\Phi_2}(a) = \inf\left\{\lambda > 0 : \sum_{k,n\in\mathbb{Z}^d} \Phi_1\left(\frac{n_{\Phi_2}(|a_{kn}|)}{\lambda}\right) \le 1\right\}.$$

With these definitions we can apply the theory of Atomic Decomposition of H. G. Feichtinger and K. H. Gröchenig presented in the paper [1]. In the context of Orlicz modulation spaces we get the following result.

Theorem 9: (The Atomic Decomposition in M^Φ) [1] Let Φ be a Δ_2-regular Young function. For any $g \in \mathcal{S}(\mathbb{R}^d)$ there exist positive constants C_0 and C_1 (depending only on g) and a neighbourhood U of the identity such that for an arbitrary U-dense and relatively separated family $X = (x_i)_{i\in I} \subset \mathbb{R}^{2d}$ the following is true:

1) Analysis: There exists a bounded linear operator $A : M^\Phi \to l^\Phi(X)$, i.e., writing $\Lambda := (\lambda_i)_{i\in I} := A(f)$ one has $n_\Phi(\Lambda) \le C_0\|f\|_{M^\Phi}$, such that every $f \in M^\Phi$ can be represented as $f = \sum_{i\in I} \lambda_i \rho(x_i) g$, where ρ is the Schrödinger representation.
2) Synthesis: Conversely, assuming that $X = (x_i)_{i\in I}$ is relatively separated, every $\Lambda \in l^\Phi$ defines an element $f = \sum_{i\in I} \lambda_i \rho(x_i) g$ in M^Φ with $\|f\|_{M^\Phi} \le C_1 n_\Phi(\Lambda)$.

In both cases convergence takes place in the norm of M^Φ.

Moreover by using the results in [3] of H.G. Feichtinger, K. H. Gröchenig and D. Walnut the orthonormal Wilson bases are unconditional bases for some Orlicz modulation spaces. Consequently in these cases M^Φ and l^Φ are isomorphic Banach spaces. Simple Wilson bases of exponential type are given by the following construction.

Definition 11: [3] A real-valued function ψ constructed, such that $|\psi(x)| \leq Ce^{-a|x|}$ and $|\hat{\psi}(t)| \leq Ce^{-b|t|}$ and such that the $\psi_{ln}, l \in \mathbb{N}, n \in \mathbb{Z}$, defined by
$\psi_{0n}(x) = \psi(x-n)$
$\psi_{ln}(x) = \sqrt{2}\psi\left(x - \frac{n}{2}\right)\cos(2\pi lx) \quad l \neq 0, l+n \in 2\mathbb{Z}$
$\psi_{ln}(x) = \sqrt{2}\psi\left(x - \frac{n}{2}\right)\sin(2\pi lx) \quad l \neq 0, l+n \in 2\mathbb{Z}+1$
constitute an orthonormal basis for $L^2(\mathbb{R})$.

In their work [3], H.G. Feichtinger, K. H. Gröchenig and D. Walnut use the density of the functions with compact support in the Banach function space. The following lemma gives a characterisation of this density for Orlicz spaces.

Lemma 4: Let Φ be a Young function, $\Phi(x) = 0$ if and only if $x = 0$, and $L^\Phi(\mathbb{R}^2)$ be the associated Orlicz space on $\mathbb{R}^2$. Then the bounded functions with compact support are dense in $L^\Phi(\mathbb{R}^2)$ if the Young function satisfies the Δ_2 condition.

If we have the Δ_2-regularity of the Young function it follows from [3].

Theorem 10: Assume that the Young function Φ satisfies the Δ_2 condition. Then the orthonormal Wilson bases are unconditional bases for $M^\Phi(\mathbb{R})$.

Moreover we can characterise inclusion properties of Orlicz modulation spaces by using properties of the corresponding Orlicz sequence spaces as in [2]. Additionally we can translate this to a comparison of Young functions.

Theorem 11: Let $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ unbounded Young functions. Then $M^{\Phi_1\Phi_1'} \subset M^{\Phi_2\Phi_2'}$ if and only if $l^{\Phi_1\Phi_1'} \subset l^{\Phi_2\Phi_2'}$ if and only if there are constants $C_1, C_2 > 0$ and $t_1, t_2 \geq 0$ such that $\Phi_2(t) \leq C_1\Phi_1(t)$ for all $0 \leq t \leq t_1$ and $\Phi_2'(t) \leq C_2\Phi_1'(t)$ for all $0 \leq t \leq t_2$.

Next, we wanted to give, without proof, an example of an embedding relation between Fourier-Lebesgue spaces and a concrete Orlicz modulation spaces. This result is an extension of the embedding theorems that Y.V. Galperin and K.H. Gröchenig gave in her work [5].

Theorem 12: Suppose that $g \in \mathcal{S}(\mathbb{R}^d), f \in \mathcal{S}'(\mathbb{R}^d)$, $C > 0, N \geq 0$ and $|V_g f(x,w)| \leq C(1+|x|+|w|)^N$ for alle $x, w \in \mathbb{R}^d$ and $0 < p \leq 2, p \leq r, s \leq 2, \frac{1}{s} + \frac{1}{s'} = 1, \frac{1}{r} + \frac{1}{r'} = 1$. If

$$\left(\frac{ap-N}{pd} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{bp-N}{pd} + \frac{1}{s} - \frac{1}{p}\right) > \left(\frac{N}{pd} + \frac{1}{p} - \frac{1}{s'}\right)\left(\frac{N}{pd} + \frac{1}{p} - \frac{1}{r'}\right)$$

with all factors positive, then $L_a^r \cap \mathcal{F}L_b^s \hookrightarrow M^{p,p}_{\frac{N}{p},\frac{N}{p}} \subset M_{l^p \ln l}$.

V. Conclusion and Outlook

In this work we have presented and analysed modulation spaces associated to Orlicz spaces and mixed-norm Orlicz spaces. It is possible to extend the theory of modulation spaces associated to Lebesgue space (understood as spaces of tempered distributions) to more general Orlicz spaces and mixed-norm Orlicz spaces. For some results the adaptation was straightforward, but in other cases further conditions on the Young function, in particular the Δ_2 condition, are necessary to obtain analogs to the results known for classical modulation spaces.
The most general approach to modulation spaces follows [1] and [2]. Here, the Δ_2 condition is needed to characterise duals of Orlicz modulation spaces (in particular in the mixed-norm setting), but also to establish density of bounded functions with compact support in the Orlicz space (needed, e.g., in *Lemma 4*, and subsequently in *Theorem 10*).
A more accessible, but less general approach is developed in [6]. The adaptation of the arguments in [6] is often feasible (and instructive), however, since duality plays a stronger role here, the Δ_2 condition is needed more often than in the general case.

Furthermore we derive embedding results between Orlicz modulation spaces by using the discretisation of the Orlicz spaces, especially by using comparison of Young functions.
For a special Orlicz modulation space we can also give an embedding relation of Fourier-Lebesgue spaces into this Orlicz modulation space. But at this time it isn't clear if this result has also an interpretation as uncertainty principles as in [5].
Another topic of interest for further work are relations applications to entropy estimates.

Acknowledgment

The authors would like to thank H. G. Feichtinger for his helpful comments.

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