Shift-Variance and Cyclostationarity of Linear Periodically Shift-Variant Systems

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***Abstract*—We study shift-variance and cyclostationarity of linear periodically shift-variant (LPSV) systems. Both input and output spaces are assumed to be of continuous-time. We first determine how far an LPSV system is away from the space of linear shift-invariant systems. We then consider cyclostationarity of a random process based on its autocorrelation operator. The results allow us to investigate properties of output of an LPSV system when its input is a random process. Finally, we analyze shift-variance and cyclostationarity of generalized sampling-reconstruction processes.**

Keywords: cyclostationarity, generalized sampling processes, linear periodically shift-variant systems, shift-variance

I. Introduction

Shift-variance and cyclostationarity are two important issues in the study of linear shift-variant systems and random processes. They have found applications in many fields, including communication and signal processing. See [2], [4] and the reference therein. Recently, Aach and Führ studied shift-variance properties of multirate filterbanks with either deterministic or random inputs [2]. They analyzed shift-variance of the filterbank and calculated the cyclostationarity of its output. For generalized sampling processes, we also performed shift-variance analysis in the deterministic setting [11]. It is the purpose of this paper to report our extension of the results to linear periodically shift-variant LPSV systems whose inputs and outputs are both of continuous-time.

As in [2], we also consider the effect of LPSV systems on the deterministic and random signals. We apply a norm in a Hilbert space of linear systems. The distance between the LPSV system and the space of linear shift-invariant (LSI) systems is then used to measure the shift-variance of the LPSV system. To study cyclostationarity of random processes, we also follow the idea of [2] to link the cyclostationarity to the shift-variance of the associated autocorrelation operator (or function). This is because a random process is wide sense stationary (WSS) if and only if (iff) the operator is shift-invariant; and it is wide sense cyclostationary (WSCS) iff the operator is LPSV. We then obtain a kind of cyclostationarity based on the shift-variance level of the autocorrelation operator. This cyclostationarity also characterizes the distance from the autocorrelation of a random process to the autocorrelation of a nearest WSS process.

Finally we treat generalized sampling-reconstruction processes as a particular application. For minimum error reconstruction, we assume that the sampling and reconstruction kernels form Riesz dual basis [9]. The expected shift-variance and cyclostationarity of the output signal are then determined. Two illustrative examples are provided.

For brevity most derivations and proofs are omitted.

II. Shift-variance of LPSV Systems

We start this section with some basic definitions. The main aim is to determine the nearest shift-invariant system for any LPSV system.

Let L^2 be the Hilbert space of square integrable continuous-time functions. Let $H(L^2 \to L^2)\colon x(t) \mapsto y(t)$ be a bounded linear system. Denote by $\mathcal{B}$ the linear space of all bounded systems. For each $T > 0$, $\mathcal{B}_T$ denotes the subspace of bounded LPSV systems with period T (T-LPSV); and $\mathcal{B}_0$ the subspace of all bounded shift-invariant systems. Note that $\mathcal{B}_0 \subset \mathcal{B}_T$.

For every $H \in \mathcal{B}_T$, we can specify it with its response to shifted impulse function $\delta_s(\cdot) = \delta(\cdot - s)$. Let the response be $H\delta_s(t) = h(t, t-s)$. Then the output of H is given as

$$y(t) = Hx = \int_{-\infty}^{\infty} h(t,s)x(t-s)\,\mathrm{ds} \tag{1}$$

Throughout the paper we assume that $H \in \mathcal{B}_T$, or equivalently $h(t+T, s) = h(t,s)$.

Since $h(t,s)$ is periodic in t with period T, we can express the impulse response as Fourier series

$$h(t,s) = \sum_{k\in Z} h_k(s)\, e^{jk\omega_0 t} \tag{2}$$

where $\omega_0 = 2\pi/T$ and the coefficients are

$$h_k(s) = \frac{1}{T}\int_0^T h(t,s)\, e^{-jk\omega_0 t}\,\mathrm{dt} \tag{3}$$

Let $\hat{h}(t,\xi)$ be Fourier transform of $h(t,s)$ with respect to s. As a function of t, $\hat{h}(t,\xi)$ is also periodic with period T. Thus we can express it as Fourier series

$$\hat{h}(t,\xi) = \sum_{k\in\mathbb{Z}} \hat{h}_k(\xi)\, e^{jk\omega_0 t} \tag{4}$$

where

$$\hat{h}_k(\xi) = \frac{1}{T}\int_0^T \hat{h}(t,\xi)\, e^{-jk\omega_0 t}\,\mathrm{dt} \tag{5}$$

Note that $\hat{h}_k(\xi)$ is actually the Fourier transform of $h_k(s)$.

We define a norm of H by

$$\|H\|^2 = \frac{1}{T}\int_0^T \|H\delta_s(\cdot)\|_2^2 \,\mathrm{ds} \tag{6}$$

By change of variable, we get

$$\|H\|^2 = \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} |h(t,s)|^2 \,\mathrm{ds}\,\mathrm{dt} \tag{7}$$

And using Parseval's relation, we can express the norm in the Fourier domain:

$$\begin{aligned}\|H\|^2 &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |h_k(s)|^2 \,\mathrm{ds} \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |\hat{h}_k(\xi)|^2 \,\mathrm{d}\xi \end{aligned}\tag{8}$$

Let $G \in \mathcal{B}_0$ and g be its impulse response i.e., $g(t) = G\delta(t)$. The distance (squared) between H and G can be calculated as

$$\begin{aligned} d^2(H,G) &= \|H-G\|^2 \\ &= \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} |h(t,s)-g(s)|^2 \,\mathrm{ds}\,\mathrm{dt} \\ &= \int_{-\infty}^{\infty} (|h_0(s)-g(s)|^2 + \sum_{k\neq 0} |h_k(s)|^2)\,\mathrm{ds} \end{aligned}\tag{9}$$

The above expression allows us to determine the nearest system $G_0 \in \mathcal{B}_0$. It is specified by the impulse response

$$g_0(s) = h_0(s) = \frac{1}{T}\int_0^T h(t,s)\mathrm{dt} \tag{10}$$

Note that G_0 is the orthogonal projection of H onto the subspace $\mathcal{B}_0$ and that the impulse response h_0 is the DC component of $h(t,s)$.

Then we have the distance between H and $\mathcal{B}_0$:

$$d^2(H,\mathcal{B}_0) = \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} |h(t,s)-g_0(s)|^2 \,\mathrm{ds} \tag{11}$$

That is,

$$d^2(H,\mathcal{B}_0) = \sum_{k\neq 0} \int_{-\infty}^{\infty} |h_k(s)|^2 \mathrm{ds} \tag{12}$$

or

$$d^2(H,\mathcal{B}_0) = \frac{1}{2\pi}\sum_{k\neq 0} \int_{-\infty}^{\infty} |\hat{h}_k(\xi)|^2 \,\mathrm{d}\xi \tag{13}$$

Note that $h(t,s) - g_0(s)$ is in the orthogonal complement space of the shift-invariant subspace $\mathcal{B}_0$. Thus the LPSV system $H - G_0$ can be considered the shift-variant part of H. Following [2], we can also define $d(H,\mathcal{B}_0)$ as the shift-variance level (denoted by $\mathrm{SV}_2(H)$) of H.

III. Cyclostationarity of Random Processes

In this section we shall study cyclostationarity of a random process by linking it to the shift-variance of a linear system that is determined by autocorrelation function of the process.

Let $z : \mathbb{R} \to \mathbb{C}$ be a zero-mean continuous-time random process with $\mathcal{E}\{|z(t)|^2\} < \infty$, $t \in \mathbb{R}$, where $\mathcal{E}$ denotes the expectation operator. The autocorrelation function of z is defined as $r_z(t,s) = \mathcal{E}\{z(t+s)\, z^*(t)\}$. The random process z is called WSS if $r_z(t,s)$ is independent of time, t; and it is WSCS with period T (T-WSCS) if $r_z(t{+}T,\tau) = r_z(t,s)$. The notions for discrete-time random process are similarly defined.

We consider the autocorrelation operator R_z as a deterministic linear system whose impulse responses are specified as $R_z\,\delta_s = r_z(t,t-s)$. It is assumed that $R_z \in \mathcal{B}$. Note that z is WSS iff R_z is shift-invariant system; and z is T-WSCS iff R_z is T-LPSV system. This suggests that we can characterize cyclostationarity of random process z by shift-variance of linear system R_z. The amount of cyclostationarity of z can be assessed in terms of the shift-variance measure of R_z:

$$\mathrm{Cyc}(z) = \mathrm{SV}_2(R_z) \tag{14}$$

This measure quantifies the distance between the autocorrelation function $r_z(t,\tau)$ and the nearest autocorrelation function of a WSS random process.

We point out that the degree of cyclostationarity (DCS) defined in [9] is a normalized version of $\mathrm{Cyc}^2(z)$, specifically

$$\mathrm{DCS}(z) = \frac{\mathrm{Cyc}^2(z)}{\int_{-\infty}^{\infty} |r_{z_0}(s)|^2 \,\mathrm{ds}} \tag{15}$$

where $r_{z_0}(t)$ is the impulse response of the nearest system in $\mathcal{B}_0$.

IV. Expected Shift-variance of LPSV Systems With Random Input

Now assume that the input is random (for example, a WSS process), how can we quantify the shift-variance of an LPSV system? This problem was considered by Aach and Führ for multirate discrete-time systems. They introduced the notation of expected shift-variance, which is related not just to the system itself, but also to the random input.

Similar to [1], introduce the commutator

$$[H,\tau_s] = H\tau_s - \tau_s H \tag{16}$$

where $\tau_s\colon x(t) \mapsto x(t-s)$ is the shift operator. The expected shift-variance of H with input x can then be defined as

$$\mathrm{ESV}^2(H,x) = \frac{1}{T^2}\int_0^T \int_0^T \mathcal{E}(|[H,\tau_s]x(t)|^2)\,\mathrm{ds}\,\mathrm{dt} \tag{17}$$

After some tedious calculations, we obtain in the time-domain that

$$\mathrm{ESV}^2(H,x) = 2\sum_{k\neq 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_k^*(t)\, h_k(t-s) r_x(s)\,\mathrm{ds}\,\mathrm{dt} \tag{18}$$

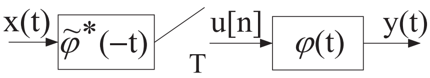

Fig. 1. A generalized sampling and reconstruction process

and in the Fourier domain that

$$\mathrm{ESV}^2(H,x) = \frac{1}{\pi}\sum_{k\neq 0}\int_{-\infty}^{\infty} |\hat{h}_k(\xi)|^2\, S_x(\xi)\,\mathrm{d}\xi \qquad (19)$$

where $S_x(\xi)$ is the power spectral density of x, i.e., the Fourier transform of r_x [7]. Note that the ESV tells how different the expected value of the output to a shifted input from that of shifted output.

Note that the ESV is zero iff the system is LSI. And the Fourier domain expression (19) provides some insight as when an LPSV system becomes LSI (see the examples at the end of Section V).

V. Generalized Sampling-reconstruction processes

Sampling-reconstruction process plays an important role in signal processing and communication. In particular, the generalized sampling-reconstruction theory of Unser and Aldroubi [9] offers a versatile framework in studying many problems of sampling beyond Shannon.

In this section, we investigate cyclostationarity and shift-variance of generalized sampling-reconstruction processes show in Fig. 1, where x is a zero-mean WSS random process; and for minimum error between input signal and the output signal (which is in the space of spanned by $\{\varphi(\cdot - nT)\}_n$), $\tilde{\varphi}(t)$ and $\varphi(t)$ are assumed to be dual Riesz basis [9]. It is well-known that sampling generally results in shift-variance whereas reconstruction introduces cyclostationarity.

Consider the sampling first. The output of sampling $u[n]$ is given by [1]

$$\begin{aligned} u[n] &= \langle x, \tilde{\varphi}(\cdot - nT)\rangle \\ &= \int_{-\infty}^{\infty} \tilde{\varphi}^*(t-nT)\,x(t)\,\mathrm{dt} \end{aligned} \qquad (20)$$

Note that u is of discrete-time and has autocorrelation function

$$r_u[n,k] = \mathcal{E}\{\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi^*(t_1-(n+k)T)x(t_1)\varphi(t_2-nT)x^*(t_2)\,\mathrm{d}t_1\mathrm{d}t_2\} \qquad (21)$$

By change of variable $t_1 - nT \to t_1$ and $t_2 - nT \to t_2$ we get

$$r_u[n,k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\varphi}^*(t_1+kT)\,\tilde{\varphi}(t_2)\,r_x(t_1-t_2)\mathrm{d}t_1\mathrm{d}t_2 \qquad (22)$$

Since r_u above is independent of n, thus it is a WSS discrete random process and the power spectral density of u is

$$S_u(e^{j\xi T}) = \frac{1}{T}\sum_{n\in Z} |\hat{\tilde{\varphi}}(\xi + 2n\pi/T)|^2\, S_x(\xi+2n\pi/T) \qquad (23)$$

[1] Note that the integration for random signals is in the mean square sense [5].

In the reconstruction part, the output is

$$y(t) = \sum_{n\in Z} u[n]\,\varphi(t-nT) \qquad (24)$$

and its autocorrelation function becomes

$$r_y(t,s) = \sum_{n_1,n_2\in Z} \varphi(t+s-n_1T)\varphi^*(t-n_2T)r_u[n_1-n_2] \qquad (25)$$

Note that $r_y(t+T,s) = r_y(t,s)$, thus y is T-WSCS.

In order to analyze the shift-variance of system H in Fig. 1, we need to determine its input-output relation. By direct substitution and change of variable, we obtain that

$$y(t) = Hx = \int_{-\infty}^{\infty} h(t,s)\,x(t-s)\,\mathrm{ds} \qquad (26)$$

where

$$h(t,s) = \sum_{n\in Z} \tilde{\varphi}^*(t-s-nT)\,\varphi(t-nT) \qquad (27)$$

is the impulse response. It can be shown that $h(t+T,s) = h(t,s)$ and

$$\hat{h}_k(\xi) = \frac{1}{T}\hat{\tilde{\varphi}}^*(\xi)\hat{\varphi}(\xi + k\omega_0) \qquad (28)$$

Since $\hat{\tilde{\varphi}}(\xi)\sum_{k\in\mathbb{Z}}|\hat{\varphi}(\xi+k\omega_0)|^2 = T\hat{\varphi}(\xi)$ [6], hence

$$\|H\|^2 = \int_{-\infty}^{\infty} |\varphi(t)|^2\mathrm{dt} \qquad (29)$$

Apply the results in previous sections, we can obtain the following results:

$$\mathrm{Cyc}^2(y) = \frac{1}{2\pi\,T^2}\sum_{k\neq 0}\int_{-\infty}^{\infty} |\hat{\varphi}(\xi)\,\hat{\varphi}(\xi+2\pi k/T)\,S_u(e^{j\xi T})|^2\,\mathrm{d}\xi \qquad (30)$$

and

$$\mathrm{ESV}^2(H,x) = \frac{1}{\pi T^2}\sum_{k\neq 0}\int_{-\infty}^{\infty} |\hat{\tilde{\varphi}}(\xi)\hat{\varphi}(\xi+2\pi k/T)|^2 S_x(\xi)\mathrm{d}\xi \qquad (31)$$

Finally, let us consider two examples. The first one is about the traditional Shannon's sampling. In this case the kernels $\tilde{\varphi}(t) = \varphi(t) = \mathrm{sinc}(t/T)/\sqrt{T}$. From equation (25) and (27) it is not immediate that the output y is WSS for WSS input and that the sampling-reconstruction system is LSI. On the other hand if we examine (30) and (31), we can easily see that $\mathrm{Cyc}(y) = \mathrm{ESV}(H,x) = 0$ for each x, since the Fourier transform of φ is zero for $|\xi| > \pi/T$. Consequently the output is WSS and the sampling-reconstruction process is LSI.

In the other example, φ is taken to be B-spline of various order n [8] which is normalized such that $\int_{-\infty}^{\infty}|\varphi(t)|^2\mathrm{dt} = 1$. And for the input we take the unit variance white Gaussian noise, hence $S_x(\xi) = 1$. Now the expected shift-variance turns out to be equivalent to cyclostationarity: $\mathrm{ESV}(H,x) = \sqrt{2}\,\mathrm{Cyc}(y)$. Furthermore from (30) it follows that

$$\mathrm{Cyc}^2(y) = 1 - \frac{1}{2\pi\,T^2}\int_{-\infty}^{\infty} |\hat{\varphi}(\xi)\,\hat{\tilde{\varphi}}(\xi)|^2\,\mathrm{d}\xi \qquad (32)$$

Consequently $0 \le \mathrm{Cyc}(y) \le 1$.

For the zero order B-spline (a box), we obtain $\mathrm{Cyc}(y) = 0.5773 > 0.5$. This result indicates that output is quite non-stationary (in the wide sense). We also obtain numerical values of $\mathrm{Cyc}(y)$ for other orders: they are 0.3546 ($n = 1$), 0.2864 ($n = 2$), 0.2485 ($n = 3$), and 0.2227 ($n = 4$). Again the output is not WSS for all cases, but now the output y seems to be more stationary than non-stationary as the order n increases. We expect that $\mathrm{Cyc}(y)$ can become arbitrary small for n large enough.

VI. Conclusion

We reported our latest study on shift-variance and cyclostationarity analysis of LPSV systems. We extended recent similar results to systems with continuous-time input and output, rendering our treatment of generalized sampling-reconstruction processes. The extension enables us to define and compute the following:

- a distance of an LPSV system to the nearest linear shift-invariant system.
- a cyclostationarity of a WSCS random process
- the expect shift-variance of a generalized sampling process and cyclostationarity of its output when the input is WSS.

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