On the Number of Degrees of Freedom of Band-Limited Functions

Tatiana Levitina
Institut Computational Mathematics
Technische Universität Braunschweig
Braunschweig, D–38106, Germany
Email: t.levitina@tu-braunschweig.de

***Abstract*—The concept of the number of degrees of freedom of band-limited signals is discussed. Classes of band-limited signals obtained as a result of successive application of the truncated direct and truncated inverse Fourier transforms are shown to posses a finite number of degrees of freedom.**

I. Introduction

Let a signal f be Ω-band-limited, i.e. representable as $f(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\Omega}^{\Omega} F(\omega)\, e^{it\omega}\, d\omega$. In accordance with the famous Whittaker–Kotel'nikov–Shannon (WKS) sampling theorem [1] f can be fully reconstructed from its uniformly distributed samples $f\left(\dfrac{\pi k}{\Omega}\right)$, $k = 0, \pm 1, \pm 2, \ldots,$

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{\Omega}\right) \frac{\sin \Omega\left(t - \pi k/\Omega\right)}{\Omega\left(t - \pi k/\Omega\right)}. \tag{1}$$

A common notion in the field is that, upon a certain duration T, this signal has no more than $2K + 1$, $K = [\Omega T/(2\pi)]$[1] degrees of freedom, since it can be *completely* recovered from just $2K + 1$ of its samples taken at the points $k\pi/\Omega \in (-T/2, T/2)$, as if $f(t) \equiv 0$ outside $(-T/2, T/2)$. This notion is refuted by the fact that a function cannot be simultaneously time- and band-limited [2].

The function f_K obtained from the sampling formula (1) by truncating the series to a finite number of terms, $|k| \leq K$, is Ω-band-limited and coincides with f at each sampling point t_k, $k = 0, \pm 1, \ldots, \pm K$. On the other hand, at any other time moment t the difference between $f(t)$ and $f_K(t)$ may be arbitrary large, depending on the values $f\left(\dfrac{\pi k}{\Omega}\right)$ from outside the interval $(-T/2, T/2)$.

Nonetheless, for band–limited functions essentially concentrated inside a finite time interval the concept of the number of degrees of freedom (NDF) makes a certain sense. For various definitions of signal concentration in the time domain we refer the reader to the monograph [3] and the literature cited therein. The $sinc$-function translates themselves are not highly concentrated inside this interval, therefore the classical sampling formula does not enable such a definition. Instead another formulation of the sampling theorem given by G. Walter and X. Shen in [4] will be helpful. The newly formulated sampling theorem employs the eigenfunctions of the finite, i.e. truncated to a finite interval, Fourier transform (TFT)[2]. The NDF of an essentially concentrated in the time domain signal can then be defined as the number of the TFT eigenfunctions which suffices to well-approximate this signal.

With a help of the sampling formula one can easily synthesize a signal of any desired NDF. This means that without an additional knowledge about the signal, the number of signal samples contributing significantly to the sampling series is not known *a priori*. However for particular classes of band–limited functions the upper bounds on this number and therewith on the NDF can be effectively computed. Thus in [4] it was shown that, if an Ω-band-limited signal is highly concentrated in $I_T = (-T/2, T/2)$ and its Fourier transform is sufficiently smooth, it has $[\Omega T/\pi] + 1$ degrees of freedom, the same number as the above erroneous explanation would give.

Yet the smoothness of the Fourier transform seems to be a too rigorous requirement. Even the TFT eigenfunctions, though proved to be the most concentrated in the time domain among other band-limited functions, have jumps in the frequency domain. The Fourier transforms of the convolutions of the Ω-band-limited TFT eigenfunctions are also discontinuous. Still they are highly concentrated in the interval $(-2\Omega, 2\Omega)$ and require no more than $2\Omega^2/\pi + 1$ of $\sqrt{2}\Omega$-band-limited TFT eigenfunctions for reconstruction via the Walter-Shen sampling formula [5].

In the present work we shall introduce a wide variety of classes of band-limited functions with a given NDF, one of them includes the convolutions of the TFT eigenfunctions as particular examples. The relevant upper bounds for the truncation error of the sampling series will be derived. We shall also touch upon a possible generalization to higher dimensions.

We begin with a brief survey of known results related to TFT eigenfunctions.

II. Band–limited functions

As is well-known [1], the Paley–Wiener space,

$$\mathcal{PW}_a := \left\{ f(x) \middle| f(x) = \frac{1}{2\pi} \int_{-a}^{a} e^{i\,xy} g(y)\, dy, g \in \mathcal{L}_2(-a, a) \right\}.$$

[1]here square brackets denote the integer part

[2]Since the acronym FFT stands commonly for the fast Fourier transform, we use the abbreviation TFT for the finite Fourier transform.

is a reproducing kernel Hilbert space with the reproducing kernel

$$G(x,y) := \frac{\sin a\,(x-y)}{\pi(x-y)}.$$

This follows from the Fourier inversion formula $\hat{F}\,[f]\,(y) = \hat{\chi}_a(y)\,g(y)$, which holds for all functions from $\mathcal{PW}_a$. Here $\hat{F}[\cdot]$ stands for the Fourier transform and $\hat{\chi}_a$ is the operator of multiplication by $\chi_a(\cdot)$, whereas $\chi_a(\cdot)$ is the characteristic function of the interval I_a

$$\chi_a(x) = \begin{cases} 1, & x \in I_a, \\ 0, & x \in \mathbb{R} \setminus I_a \end{cases} .$$

The classical WKS sampling formula

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{a}\right) \frac{\sin a\,(x - \pi k/a)}{a\,(x-\pi k/a)}, \quad \text{for } \forall f \in \mathcal{PW}_a, \tag{2}$$

reflects the fact that the sequence of point evaluation functionals $G\left(\frac{\pi\,k}{a}, y\right)$, $k = 0, \pm 1, \ldots$, forms an orthonormal basis in $\mathcal{PW}_a$ [1].

Yet Eq. (2) is easy to obtain via the direct integration of the Fourier expansion of the associated function $g \in \mathcal{L}_2(I_a)$:

$$g(y) = \sqrt{\frac{1}{2a}} \sum_{k=-\infty}^{\infty} g_k\, e^{-i\pi k y/a} = \frac{1}{2a} \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{a}\right) e^{-i\pi k y/a}, \tag{3}$$

since the Fourier coefficients g_k, $k = 0, 1, 2, \ldots$, are

$$g_k := \sqrt{\frac{1}{2a}} \int_{-a}^{a} e^{i\,\pi k\, y/a} g(y)\, dy = \sqrt{\frac{1}{2a}}\, f\left(\frac{\pi k}{a}\right).$$

The series in the right–hand side of Eq. (3) converges in $\mathcal{L}_2(-a,a)$. As a consequence, the sampling formula converges both in the $\mathcal{L}_2$–norm and uniformly on $\mathbb{R}$.

III. Prolates—eigenfunctions of the truncated Fourier transform

Another sampling formula invented and studied in [4] is written in terms of the TFT eigenfunctions. The TFT operator $\hat{F}_a$ is first introduced as acting on $\mathcal{L}_2(-a,a)$ by

$$\hat{F}_a[g](x) = \frac{1}{2\pi} \int_{-a}^{a} e^{i\,xy} g(y)\, dy, \quad x \in (-a, a). \tag{4}$$

Its eigenfunctions $\psi_l(a,x) = \psi_l(x)$ defined at $x \in (-a,a)$ via the equation

$$\mu_l\, \psi_l(x) \;=\; \frac{1}{2\pi} \int_{-a}^{a} e^{i\,xy} \psi_l(y)\, dy, \quad l = 0, 1, \ldots, \tag{5}$$

are ordered according to the absolute magnitude of the associated eigenvalues, $|\mu_0| > |\mu_1| > \ \ldots$. One can choose ψ_l to be real valued and normalized, so that

$$\|\psi_l\|_a^2 = \int_{-a}^{a} (\psi_l(x))^2\; dx \;=\; 1.$$

Eqs. (4), (5) are then used to extend the functions in the left hand side to the entire real axis. Therewith an operator is defined that maps $\mathcal{L}_2(I_a)$ on the Paley–Winer space $\mathcal{PW}_a$. We shall keep for this operator the same notation $\hat{F}_a$, similarly the extensions of the TFT eigenfunctions are hereafter denoted by ψ_l, since it shall not cause any ambiguity. One can also define $\hat{F}_a$ as the composition $\hat{F}_a := \hat{F} \circ \hat{\chi}_a : \mathcal{L}_2(\mathbb{R}) \to \mathcal{PW}_a$. Evidently ψ_l extended to $\mathbb{R}$ are the eigenfunctions of $\hat{F} \circ \hat{\chi}_a$.

The double definition of the TFT operator provides a double set of properties of its eigenfunctions. Thus functions ψ_l are pairwise orthogonal both on the finite interval I_a and on $\mathbb{R}$,

$$\int_{-a}^{a} \psi_l(x)\, \psi_s(x)\, dx \;=\; \delta_{ls}\,, \qquad \int_{-\infty}^{\infty} \psi_l(x)\, \psi_s(x)\, dx \;=\; \frac{1}{\gamma_l}\, \delta_{ls},$$

where $\gamma_l := |\mu_l|^2/2\pi$. In addition to that ψ_l form a basis both in $\mathcal{L}_2(-a,a)$ and in $\mathcal{PW}_a$.

Besides, for all $l = 0, 1, \cdots,$

$$\frac{1}{2\pi}\int_{-a}^{a} e^{-ixy} \int_{-a}^{a} e^{i\xi y} \psi_l(\xi) d\xi = \int_{-a}^{a} \frac{\sin a\,(x-\xi)}{\pi(x-\xi)}\, \psi_l(\xi)\, d\xi = \gamma_l \psi_l.$$

This means that ψ_l are eigenfunctions of the operator $\hat{G}_a : \mathcal{L}_2(I_a) \to \mathcal{L}_2(I_a)$,

$$\hat{G}_a[g](x) := \int_{-a}^{a} \frac{\sin a\,(x-y)}{\pi(x-y)}\, g(y)\, dy, \quad g \in \mathcal{L}_2(I_a), \tag{6}$$

whereas γ_l are the corresponding eigenvalues, here γ_l are the same as defined above.

Like Eq. (5), the latter equation remains valid outside the interval I_a. Note that $\hat{G}_a = \hat{F}^{-1} \circ \hat{\chi}_a \circ \hat{F} \circ \hat{\chi}_a$. In other words, $\hat{G}_a$ is the truncated direct Fourier transforms followed by the truncated inverse Fourier transform. The operator $\hat{G}_a$ plays a key role in the further consideration.

Remarkable properties of the TFT eigenfunctions have been widely discussed, see e.g. [2], [4], [6], [7]. Below we shall cite only those properties which are important for the present analysis. Among others, of special interest is the concentration property of the TFT eigenfunctions [2], namely that in the Paley-Wiener space $\mathcal{PW}_a$ the function ψ_0 is the most concentrated inside the interval I_a, since $\gamma_0 = \frac{\|\psi_0\|_{I_a}}{\|\psi_0\|_{\mathbb{R}}}$ yields the largest possible value for the ratio $\frac{\|f\|_{I_a}}{\|f\|_{\mathbb{R}}}$. In general, denote by $\mathcal{PW}_a^l$ the orthogonal complement to Span$\{\psi_0, \psi_1, \ldots, \psi_{l-1}\}$ in $\mathcal{PW}_a$, then

$$\gamma_l \;=\; \frac{\|\psi_l\|_{I_a}}{\|\psi_l\|_{\mathbb{R}}} \;=\; \max_{f \in PW_a^l} \frac{\|f\|_{I_a}}{\|f\|_{\mathbb{R}}}\,.$$

An exceptional feature of eigenvalues γ_l is that γ_l are close either to zero or to one, and the number of γ_l close to one does not exceed $L = \left[\frac{2\,a^2}{\pi}\right]$ [6].

Thus, one can see a qualitative difference between the TFT eigenfunctions of indices $l < \frac{2\,a^2}{\pi}$ and those of $l > \frac{2\,a^2}{\pi}$:

although each ψ_l is the most concentrated among functions from $\mathcal{PW}_a^l$, only the first $\left[\frac{2\,a^2}{\pi}\right]$ are *really* concentrated on I_a. The integral equation (5) does not account for this difference. In order to understand this feature, we recall that after an appropriate scaling TFT eigenfunctions coincide with the prolate spheroidal wave functions of zero order [2], and they are therefore often referred to as prolates.

IV. Prolate spheroidal wave functions

A representative overview of the prolate spheroidal wave functions (PSWF) is given in [8] (see also the literature cited therein). At any point $\xi \neq \pm 1$, a PSWF of zero order $S(c,\xi) = S(\xi)$ obeys the prolate spheroidal wave equation

$$\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}S + \left[\lambda + c^2(1-\xi^2)\right] S = 0, \tag{7}$$

remaining bounded at the singular points $\xi = \pm 1$,

$$|S(\xi)| < \infty,\ \xi \to \pm 1. \tag{8}$$

Both singularities at the points $\xi = \pm 1$ are regular and limit-point. In the neighborhood of $\xi = 1$ Eq. (7) has two linearly independent solutions [8]

$$S^{(1)}(\xi) \sim \text{const}, \quad S^{(2)}(\xi) \sim \ln(1-\xi^2), \quad \xi \to 1,$$

of which only the first one is bounded.

Solutions of Eq. (7) which are bounded at both singular points simultaneously exist not for all λ. Eq. (7) and the boundedness conditions (8) define a self-adjoint singular Sturm–Liouville eigenvalue problem on the interval $(-1,1)$. The eigenfunctions $S_l(\xi)$ of this problem are called angular PSWF. They are ordered by the number of internal zeros and normalized by the condition $\|S_l\|_{(-1,1)} = \int_{-1}^{1} S_l^2(\xi)\,d\xi = 1$. For the associated eigenvalues one can prove that

$$-c^2 < \lambda_0 < \lambda_1 < \ldots < \lambda_l < \ldots .$$

At infinity any solution of Eq. (7) vanishes as $(1/\xi)$. In particular, solutions bounded at $\xi = 1$ enjoy the asymptotic behaviour

$$S_l(\xi) = \frac{A_l}{c\xi}\cos\left(c\xi - \frac{l+1}{2}\pi\right) + \mathrm{O}\left(\frac{1}{\xi^2}\right), \quad \xi \to \infty. \tag{9}$$

In what follows A_l is chosen to match at $\xi = \pm 1$ the angular function. A simple relation links then the functions S_l and ψ_l:

$$a = \sqrt{c}, \quad \psi_l(a,x) = \frac{1}{\sqrt{a}}\, S_l\left(c, \frac{x}{a}\right), \quad l = 0,1,\cdots . \tag{10}$$

This means that the prolates ψ_l and the *sinc*-function translates have the same rate of vanishing at infinity, which seems to contradict the concentration property of prolates. However Eq. (7) gives us a key to eliminating the apparent contradiction.

As was shown in [9], the properties of solutions of Eq. (7) depend dramatically on whether λ is positive or negative. Below we add to the detailed analysis provided in [9] a few more features explaining the concentration phenomenon. To this end, we study the behavior of a bounded solution of Eq. (7) inside the interval where the product of $(1-\xi^2)$ and the potential $Q(\xi) = \lambda + c^2(1-\xi^2)$ is negative. For $-c^2 < \lambda < 0$ this is the interval $(\xi_T, 1)$, while for $\lambda > 0$ it is $(1, \xi_T)$, where $\xi_T = \sqrt{1+\lambda/c^2}$ being the turning point of Eq. (7).

The following lemma is easy to prove.

Lemma 1. *Let $-c^2 < \lambda < 0$ and ξ_T be the turning point of Eq. (7), so that $Q(x) < 0$ on the interval $(\xi_T, 1)$. If $S(\xi)$ is a solution of Eq. (7) bounded at $\xi = 1$, then neither $S(\xi)$ nor $S'(\xi)$ vanish inside the interval $(\xi_T, 1)$.*

Proof: On integrating Eq. (7) multiplied by $S(\cdot)$ over an interval $(\xi, 1)$, one obtains:

$$(1-\xi^2)S'(\xi)S(\xi) = \int_{\xi}^{1}\left\{Q(\eta)S^2(\eta) - (1-\eta^2)\left[S'(\eta)\right]^2\right\} d\eta.$$

The right hand side above is strictly negative on $(\xi_T, 1)$. ∎

As is readily seen, the logarithmic derivative of a bounded solution, $\beta(\xi) = S'(\xi)/S(\xi)$, satisfies the equation

$$(1-\xi^2)\,\beta' = -Q(\xi) + 2\,\xi\beta - \beta^2\,(1-\xi^2), \quad \xi \in (\xi_T, 1). \tag{11}$$

Besides, the expansion $\beta(\xi) = \lambda/2 + (c^2 + \lambda/2 + \lambda^2/4)(1-\xi)/2 + \ldots$ holds near the point $\xi = 1$ [10]. Straightforward but rather tiresome analysis of the direction field in (11) shows that for $\xi \in (\xi_T, 1)$

$$\beta(\xi) < \frac{Q(\xi)}{\xi + \sqrt{\xi^2 - (1-\xi^2)\,Q(\xi)}} < \frac{\lambda + c^2(1-\xi^2)}{1+\sqrt{1-\lambda}} < 0.$$

This means that $S(\xi)$ decays exponentially fast in $(\xi_T, 1)$. Thus, the smaller the index l of the eigenvalue λ_l, the higher the ratio

$$\frac{S_l(\xi_T)}{S_l(1)} = -\exp\left\{\int_{\xi_T}^{1} \beta_l(\xi)\,d\xi\right\},$$

hence the smaller the factor A_l in (9) and therewith the smaller the contribution from outside the interval$(-1,1)$ to the total norm $\|S_l\|_{\mathbb{R}}$.

Similar analysis done for $\lambda > 0$ shows that the factor A_l grows up with l in accordance with the exponential increase of S_l on $(1, \xi_T)$. As a result, the contribution from the interval $(-1,1)$ to $\|S_l\|_{\mathbb{R}}$ becomes negligibly small as $l \to \infty$.

Note that the number of negative eigenvalues λ_l of the problem (7)–(8) was proved in [9] not to exceed $2c/\pi = 2a^2/\pi$.

V. Walter-Shen Sampling formula and the range of the operator $\hat{G}_a$

In terms of prolates the sampling formula becomes [4]

$$\begin{aligned} f(x) &= \frac{\pi}{a}\sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{a}\right)\sum_{l=0}^{\infty}\gamma_l\,\psi_l\left(\frac{\pi k}{a}\right)\psi_l(x) \\ &= \frac{\pi}{a}\sum_{l=0}^{\infty}\gamma_l\left\{\sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{a}\right)\psi_l\left(\frac{\pi k}{a}\right)\right\}\psi_l(x). \end{aligned} \tag{12}$$

Here, the order of double summation is interchangeable, both double series converging in $\mathcal{L}_2$-norm and uniformly on $\mathbb{R}$.

Because of the double summation, the sampling formulae (12) look more cumbersome than (2), however in practical calculations series (12) may be more advantageous than the classical one. Moreover, the following estimate, similar to that of Lemma 4 in [4] (see also [3]), allows one to truncate the summation on k to a few terms:

$$\Theta_l := \gamma_l \sum_{|k|>a^2/\pi} \psi_l^2\left(\frac{\pi k}{a}\right) \leq C\sqrt{\gamma_l(1-\gamma_l)}. \tag{13}$$

One sees that the contribution from samples $\psi_l(\pi k/a)$, $|k| > a^2/\pi$, is small both for $l < 2a^2/\pi$ and for $l > 2a^2/\pi$.

Consider the range of the operator $\hat{G}_a$, $\mathrm{Rg}(\hat{G}_a)$. Clearly, prolates ψ_l are in $\mathrm{Rg}(\hat{G}_a)$.

Let $f(x) \in \mathrm{Rg}(\hat{G}_a)$, i.e. $f(x) = \int_{-a}^{a} \frac{\sin a\,(x-y)}{\pi(x-y)}\, g(y)\, dy$ for some $g \in \mathcal{L}_2(I_a)$. Denote the Fourier coefficients of the functions f and g in the basis of prolates by $\tilde{f}_l$ and $\tilde{g}_l$, respectively. Then the truncation error caused by neglect of the contribution from ψ_l at $l > L$ is

$$\varepsilon_L := \Big\|f - \sum_{l\leq L} \tilde{f}_l\psi_l\Big\|_{\mathbb{R}} \leq \sqrt{\gamma_{L+1}}\Big\|g - \sum_{l\leq L}\tilde{g}_l\psi_l\Big\|_{I_a},$$

which is very small, provided that $L > 2a^2/\pi$.

In view of (13), the truncation of the inner sum in (12) at some $K > a^2/\pi$ causes the error

$$\varepsilon_{L,K}^2 := \Big\|f_L(x) - \frac{\pi}{a}\sum_{l\leq L}\sum_{|k|\leq K}\gamma_l\, f\left(\frac{\pi k}{a}\right)\,\psi_l\left(\frac{\pi k}{a}\right)\,\psi_l(x)\Big\|_{\mathbb{R}}^2$$
$$\leq \sum_{|k|>K}\left[f\left(\frac{\pi k}{a}\right)\right]^2 \sum_{l\leq L}\Theta_l.$$

Summarizing, we conclude that the NDF of functions in $\mathrm{Rg}(\hat{G}_a)$ is $[2a^2/\pi] + 1$.

VI. Sampling in $\mathrm{Rg}\,(G_{\Omega,T})$

The range of the operator $\hat{G}_a$ is not the only class of band-limited functions for which the above truncation error estimates hold. Consider the operator $G_{\Omega,T}$:

$$G_{\Omega,T}\,[g]\,(x) := \frac{1}{2\pi}\hat{F}^{-1}\circ\hat{\chi}_\Omega\circ\hat{F}\circ\hat{\chi}_T\,[g]\,(x) = \int_{-T}^{T}\frac{\sin\Omega\,(x-y)}{\pi(x-y)}\,g(y)\,dy.$$

On substituting into the above equation new variables $\eta = \sqrt{\Omega/T}\,y$ and $\xi = \sqrt{\Omega/T}\,x$, we obtain

$$\tilde{f}(\xi) = f\left(\sqrt{\frac{\Omega}{T}}\xi\right) = \int_{-a}^{a}\frac{\sin a\,(\xi-\eta)}{\pi(\xi-\eta)}\,g\left(\sqrt{\frac{T}{\Omega}}\eta\right)\,d\eta,$$

where $a^2 = \Omega T$. As a result, the function $\tilde{f}(\xi) \in \mathrm{Rg}\,(G_a)$ and has $[2\Omega T/\pi] + 1$ degrees of freedom.

The convolution of two TFT eigenfunctions $\Phi_{nm}(x) := \int_{-\infty}^{\infty}\psi_n(a, x-y)\,\psi_m(a,y)\,dy$ is a–band–limited and hence

$$\Phi_{nm}(x) \approx \frac{\pi}{a}\sum_{l\leq L,\ |k|\leq K}\gamma_l\,(a)\,\Phi_{nm}\left(\frac{\pi k}{a}\right)\psi_l\left(a, \frac{\pi k}{a}\right)\psi_l(a,x)\,.$$

On the other hand, one can prove that $\Phi_{nm}(x) \in \mathrm{Rg}\,(G_{a,2a})$. Therefore $\tilde{\Phi}_{nm}(x) = \Phi_{nm}(x/\sqrt{2}) \in \mathrm{Rg}\left(G_{\sqrt{2}a}\right)$ and

$$\Phi_{nm}(x) \approx \frac{\pi}{\sqrt{2}a}\sum_{l,|k|\leq 4a^2/\pi}\gamma_l\left(\sqrt{2}a\right)\Phi_{nm}\left(\frac{\pi k}{2a}\right) \times \psi_l\Big(\sqrt{2}a, \frac{\pi k}{\sqrt{2}a}\Big)\,\psi_l\Big(\sqrt{2}a, \sqrt{2}x\Big)\,.$$

The latter sampling formula shows better accuracy than the previous one, even if the number of samples and prolates involved in calculations is the same.

VII. Generalization to higher dimensions

In [12] the eigenfunctions of the 2D Fourier transform truncated to a circle of finite radius a were represented through the eigenfunctions of the truncated Hankel transforms (THT) of different angular numbers m. Recently in [11] the number of THT eigenvalues close to one was proved not to exceed $a^2/\pi - m/2$. The same number defined the NDF of a class of Hankel-band-limited functions analogous to $\mathrm{Rg}(\hat{G}_a)$. This results can easily be generalized to the case of higher dimensions in accordance with the discussion in [13].

References

[1] J. R. Higgins, *Five short stories about the cardinal series*, Bull. Amer. Math. Soc. (N.S.) **12**(1) (1985) 45–89.

[2] D. Slepian, H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty, I*, Bell Syst. Tech. J. **40**(1) (1961) 43–64.

[3] J. A. Hogan and J. D. Lakey, *Duration and Bandwidth Limiting: Prolate Functions, Sampling, and Applications*, Birkhäuser, Boston, 2012.

[4] G. G. Walter, X. Shen, *Sampling With Prolate Spheroidal Wave Functions*, Sampl. Theory Signal Image Proc. **2**(1) (2003) 25–52.

[5] T. Levitina and E. J. Brändas, *Filter diagonalization: Filtering and postprocessing with prolates*, Comp. Phys. Comm., **180**(9) (2009) 1448–1457.

[6] H. J. Landau, *Sampling, data transmission, and the Nyquist rate*, Proc. IEEE **55** (1967) 1701–1706.

[7] H. Xiao, V. Rokhlin, and N. Yarvin, *Prolate Spheroidal Wave Functions, Quadrature and Interpolation*, Inverse Problems **17** (2001) 805–838.

[8] I. V. Komarov, L. I. Ponomarev, and S. Yu. Slavyanov, *Spheroidal and Coulomb Spheroidal Functions*, [in Russian], Nauka, Moscow, 1976.

[9] A. Osipov and V. Rokhlin, *Detailed Analysis of Prolate Quadratures and Interpolation Formulas*, arXiv:1208.4816 [math.NA]

[10] T. V. Levitina and E. J. Brändas, *Computational techniques for Prolate Spheroidal Wave Functions in Signal Processing*, J. Comp. Meth. Sci. & Engrg. **1** (2001) 287–313.

[11] T. Levitina, *On the Eigenfunctions of the Finite Hankel Transform*, Sampl. Theory Signal Image Proc. **11**, (2012), 55–79.

[12] D. Slepian, *Prolate spheroidal wave functions. Fourier analysis and uncertainty, IV: Extensions to many dimensions; generalized prolate spheroidal functions*, Bell Sys. Tech. J. **43** (1964) 3009–3058.

[13] O. Brander and B. DeFacio, *A generalisation of Slepian's solution for the singular value decomposition of filtered Fourier transforms*, Inverse Problems **2** (4) (1986) 375–393.