Multivariate sampling Kantorovich operators: approximation and applications to civil engineering

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***Abstract*—In this paper, we present the theory and some new applications of linear, multivariate, sampling Kantorovich operators. By means of the above operators, we are able to reconstruct pointwise, continuous and bounded signals (functions), and to approximate uniformly, uniformly continuous and bounded functions. Moreover, the reconstruction of signals belonging to Orlicz spaces are also considered. In the latter case, we show how our operators can be used to approximate not necessarily continuous signals/images, and an algorithm for image reconstruction is developed. Several applications of the theory in civil engineering are obtained. Thermographic images, such as masonries images, are processed to study the texture of the buildings, thus to separate the stones from the mortar and finally a real-world case-study is analyzed in terms of structural analysis.**

I. Introduction

In [1] the authors introduced the linear sampling Kantorovich operators and studied, in particular, their convergence in the general setting of Orlicz spaces, in one-dimesional case. Later these results have been extended in [8] to the multivariate setting, in [12], [9] to the nonlinear case and in a more general context in [13], [2].

In this paper, we obtain applications to civil engineering by using the linear multivariate sampling Kantorovich operators $(S_w)_{w>0}$, defined by

$$(S_w f)(\underline{x}) \ := \ \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) \ d\underline{u} \right], \quad \textbf{(I)}$$

for every $\underline{x} \in \mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$. Here $\chi : \mathbb{R}^n \to \mathbb{R}$ is a kernel function satisfying suitable properties, $t_{\underline{k}} = (t_{k_1}, ..., t_{k_n})$ is a vector where $(t_{k_i})_{k_i \in \mathbb{Z}}$, $i = 1, ..., n$ is a sequence of real numbers with some properties and where

$$R^w_{\underline{k}} \ := \ \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w} \right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w} \right] \times ... \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w} \right],$$

$w > 0$ and $A_{\underline{k}} = \Delta_{k_1} \cdot \Delta_{k_2} \cdot ... \cdot \Delta_{k_n}$ with $\Delta_{k_i} := t_{k_i+1} - t_{k_i}$, $i = 1, ..., n$. For the study of the above family **(I)**, see [8].
The sampling series **(I)** represents a Kantorovich version of the generalized sampling operators introduced by P.L. Butzer and his school at Aachen (see e.g. [4]). Here, in place of the sample values $f(\underline{k}/w)$ one has an average of f in a small pluri-rectangle containing $\underline{k}/w$ (here instead of $\underline{k}$, we have a general sequence $t_{\underline{k}}$, obtaining a non uniform sampling). This situation very often occurs in Signal Processing, when one cannot match exactly the "node" $t_{\underline{k}}/w$: this represents the so called "time-jitter error". Therefore our theory reduces time-jitter errors calculating the information in a neighborhood of a point rather than exactly at that point.
For the sampling Kantorovich operators **(I)**, we study the pointwise convergence for continuous and bounded functions, the uniform convergence, for uniformly continuous and bounded functions, and the modular convergence, for functions belonging to Orlicz spaces (see e.g. [3]). The latter case, allows to treat the case of L^p-signals, i.e., not necessarily continuous signals; note that in multivariate setting, when one deals with images, discontinuities are concentrated in the contours or edges of the image itself, in terms of jumps of grey levels (see [8], [9]). To show the versatility of our theory, we study various applications to civil engineering images. In this subject the images, in particular thermographic images, are used to make non-invasive investigations of structures, to analyze the story of the buildings or of the building walls, to make diagnosis and monitoring buildings, and to make structural measurements. The thermography is a remote sensing technique, performed by the image acquisition in the infrared. Moreover, these images are also used in civil engineering for image texture, i.e., for the separation between the bricks and the mortar in masonries images. Unfortunately, the direct application of the image texture algorithm to the thermographic images, can produce errors, as an incorrect separation between the bricks and the mortar. Then, we use the sampling Kantorovich operators to process the thermographic images before to apply the texture. In this way, the result produced by the texture becomes more refined and therefore we can apply structural analysis to a real-world case-study after the calculation of the various parameters involved.

A. Approximation results

In this section, we treat the main approximation results for the multivariate sampling Kantorovich operators. In what follows, we denote by $t_{\underline{k}} = (t_{k_1}, ..., t_{k_n})$ a vector where each $(t_{k_i})_{k_i \in \mathbb{Z}}$, $i = 1, ..., n$ is a sequence of real numbers with $-\infty < t_{k_i} < t_{k_{i+1}} < +\infty$, $\lim_{k_i \to \pm\infty} t_{k_i} = \pm\infty$, for every

$i = 1, ..., n$, and such that there exist Δ, $\delta > 0$ for which $\delta \le \Delta_{k_i} := t_{k_{i+1}} - t_{k_i} \le \Delta$, for every $i = 1, ..., n$.

A function $\chi : \mathbb{R}^n \to \mathbb{R}$ will be called a kernel if it satisfies the following properties:

- $(\chi 1)$ $\chi \in L^1(\mathbb{R}^n)$ and is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$;
- $(\chi 2)$ For every $\underline{u} \in \mathbb{R}^n$, $\displaystyle\sum_{\underline{k} \in \mathbb{Z}^n} \chi(\underline{u} - t_{\underline{k}}) = 1$;
- $(\chi 3)$ For some $\beta > 0$,
$$m_{\beta,\Pi^n}(\chi) = \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} \left|\chi(\underline{u} - t_{\underline{k}})\right| \cdot \left\|\underline{u} - t_{\underline{k}}\right\|_2^\beta < +\infty,$$
where $\|\cdot\|_2$ denotes the usual Euclidean norm.

We may now state the following theorem for the linear multivariate sampling Kantorovich operators **(I)** based upon the kernel function χ.

Theorem 1: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and bounded function. Then, for every $\underline{x} \in \mathbb{R}^n$,

$$\lim_{w \to +\infty} (S_w f)(\underline{x}) = f(\underline{x}).$$

In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous and bounded, then

$$\lim_{w \to +\infty} \|S_w f - f\|_\infty = 0,$$

where $\|\cdot\|_\infty$ denotes the usual sup-norm.

We now recall some basic fact concerning Orlicz spaces, see e.g. [11], [3].
Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a φ-function, i.e. φ satisfies the following assumptions:

1) $\varphi(0) = 0$, $\varphi(u) > 0$ for every $u > 0$;
2) φ is continuous and non decreasing on $\mathbb{R}_0^+$;
3) $\lim_{u \to \infty} \varphi(u) = +\infty$.

For a fixed φ-function φ, one can consider the functional $I^\varphi : M(\mathbb{R}^n) \to [0, +\infty]$, where $M(\mathbb{R}^n)$ denotes the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. We define

$$I^\varphi[f] := \int_{\mathbb{R}^n} \varphi(|f(\underline{x})|)\, d\underline{x}, \qquad (f \in M(\mathbb{R}^n)).$$

The Orlicz space generated by φ is defined by

$$L^\varphi(\mathbb{R}^n) := \{f \in M(\mathbb{R}^n) : I^\varphi[\lambda f] < \infty, \text{ for some } \lambda > 0\}.$$

We can introduce in $L^\varphi(\mathbb{R}^n)$, a notion of convergence, called "modular convergence", which induces a topology (modular topology) on the space ([11], [3]). Namely, we will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is modularly convergent to a function $f \in L^\varphi(\mathbb{R}^n)$ if

$$\lim_{w \to +\infty} I^\varphi[\lambda(f_w - f)] = 0$$

for some $\lambda > 0$.

Now, by means of a modular estimate for the operators **(I)** and using a density result, we may state the following modular convergence theorem for the sampling Kantorovich operators (based upon the kernel function χ) in Orlicz spaces.

Theorem 2: Let φ be a convex φ-function. For every $f \in L^\varphi(\mathbb{R}^n)$, there exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} I^\varphi[\lambda(S_w f - f)] = 0.$$

Now, choosing $\varphi(u) = u^p$, $1 \le p < \infty$, we have $L^\varphi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $I^\varphi[f] = \|f\|_p^p$, where $\|\cdot\|_p$ is the usual L^p-norm. Then, from Theorem 2 we obtain the following corollary.

Corollary 1: For every $f \in L^p(\mathbb{R}^n)$, $1 \le p < +\infty$,

$$\lim_{w \to +\infty} \|S_w f - f\|_p = 0.$$

The corollary above, allows us to reconstruct L^p-signals (in L^p-sense), therefore not necessarily continuous. Other examples of Orlicz spaces for which the theory can be applied, are given by the Zygmund spaces (or interpolation spaces) and by the exponential spaces, see e.g. [11], [3], [1], [8].

B. The choice of the kernels

In the theory of sampling Kantorovich operators an important role is played by the kernels χ. A procedure to construct examples of multivariate kernel is to use product kernels by means of one-dimensional kernels. For a sake of simplicity, we consider our operators in case of uniform sampling ($t_{\underline{k}} = \underline{k}$), and denote by $\chi_1, ..., \chi_n$, the one-dimensional kernels $\chi_i : \mathbb{R} \to \mathbb{R}$ satisfying conditions $(\chi 1)$, $(\chi 2)$ and $(\chi 3)$ for $n = 1$. In [4], [8] is proved that the multivariate function

$$\chi(\underline{x}) := \prod_{i=1}^n \chi_i(x_i),$$

for every $\underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, is a multivariate kernel for our operators $(S_w)_{w>0}$ satisfying the assumption of the theory. Then, it is now sufficient to give examples of one-dimensional kernels satisfying $(\chi 1)$, $(\chi 2)$ and $(\chi 3)$. Remarkable examples of kernels with compact support, are given by the well-known central B-spline of order $k \in \mathbb{N}$, defined by

$$M_k(x) := \frac{1}{(k-1)!} \sum_{i=0}^k (-1)^i \binom{k}{i} \left(\frac{k}{2} + x - i\right)_+^{k-1}.$$

where the function $(x)_+ := \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$ (see [1], [12], [8]). Other well-known examples of one-dimensional kernels are given by the Jackson-type kernels, defined by

$$J_k(x) = c_k \mathrm{sinc}^{2k}\left(\frac{x}{2k\pi\alpha}\right), \qquad x \in \mathbb{R},$$

with $k \in \mathbb{N}$, $\alpha \ge 1$, for a suitable constant c_k and where the sinc-function is defined by

$$\mathrm{sinc}(x) := \begin{cases} 1, & x = 0, \\ \dfrac{\sin(\pi x)}{\pi x}, & \text{otherwise}, \end{cases}$$

(see [5], [1]).

It is also possible to consider kernels which are not of product type. For instance, one can take into consideration *radial kernels*, i.e., functions for which the value depends on the Euclidean norm of the argument only. Example of such a kernel can be given, for example, by the Bochner-Riesz kernel, defined as follows

$$b^{\alpha}(\underline{x}) := 2^{\alpha}\Gamma(\alpha+1)\|\underline{x}\|_2^{-(n/2)+\alpha}\mathcal{B}_{(n/2)+\alpha}(\|\underline{x}\|_2),$$

for $\underline{x} \in I\!R^n$, where $\alpha > (n-1)/2$, $\mathcal{B}_\lambda$ is the Bessel function of order λ and Γ is the Euler function. For more details about this matter, see e.g. [4].

C. Applications to Image Processing

In this section, we show how the multivariate sampling Kantorovich operators can be applied to process digital images, see [8], [9]. Every bi-dimensional grey scale image A (matrix) can be modeled as a step function I, with compact support, belonging to $L^p(I\!R^2)$, $1 \leq p < +\infty$. The most natural way to define I is:

$$I(x,y) \ := \ \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \cdot \mathbf{1}_{ij}(x,y) \quad ((x,y) \in I\!R^2),$$

where $\mathbf{1_{ij}(x,y)}$, $i,j = 1,2,...,m$, are the characteristics functions of the sets $(i-1,\ i]\times(j-1,\ j]$ (i.e. $\mathbf{1_{ij}(x,y)} = \mathbf{1}$, for $(x,y) \in (i-1,\ i]\times(j-1,\ j]$ and $\mathbf{1_{ij}(x,y)} = \mathbf{0}$ otherwise). Note that the above function $I(x,y)$ is defined in such a way that, to every pixel (i,j) it is associated the corresponding grey level a_{ij}. Then, we can now consider the family of bivariate sampling Kantorovich operators applied to the function I, $(S_w I)_{w>0}$ (for some kernel χ) that approximate I in L^p-sense. Now, in order to obtain a new image (matrix) that approximates the original one, it is sufficient to sample $S_w I$ (for some $w > 0$) with a fixed sampling rate. In particular, we can reconstruct the approximating images (matrices) taking into consideration different sampling rates and this is possible since we know the analytic expression of $S_w I$.

Obviously, if the sampling rate is chosen higher than the original sampling rate, one can get a new image that has a better resolution than the original one's. The above procedure has been implemented by using MATLAB, in order to obtain an algorithm based on the multivariate sampling Kantorovich theory.

D. Applications to civil engineering images

In this section, we propose some new applications of the algorithm, based on the multivariate sampling Kantorovich operators, to civil engineering images.

The most widely used images in this areas are the thermographic images, largely used to make diagnosis and monitoring buildings, and to make structural measurements. The thermography is a remote sensing technique, performed by the image acquisition, in the infrared. The thermographic images are obtained by the thermograph, that in practice consists in a thermal camera for detecting radiation in the infrared range of the electromagnetic spectrum, and perform measurements related with the emission of this radiation. This tool is able to detect the temperatures of the bodies analyzed by measuring the intensity of infrared radiation emitted by the body under examination. All the objects at a temperature above absolute zero emit radiation in the infrared range. The thermography allows to avoid the use of invasive techniques of investigation for buildings. Moreover, these images are also used in civil engineering for image texture, i.e., for the separation between the bricks and mortar in masonries images. The image texture algorithm performs as follows: first of all we apply a median filter to the image using a suitable mask, then the image is converted into a black and white image by means of a suitable thresholding, in order to obtain a consistent separation of the phases; the area consisting of white pixels denote the inclusions (stones or bricks) and the remaining areas of black pixels denotes the mortar joints. Finally, morphological operators are used to enhance the quality of the separation of the phase: closing of the area to eliminate salt-and-pepper noise, erosion and dilation to smooth the contours of the inclusions. The image obtained is characterized by a consistent separation of phases, where each stone is surrounded by mortar joints and unrealistic conjunction of inclusions is avoided as much as possible (see e.g. [6]).

The direct application of the image texture algorithm to the thermographic images, can produce errors (see e.g. Figure 2 (left) and (right)), as an incorrect separation between the bricks and the mortar. Then, we can use the sampling Kantorovich operators (see in Figure 1 (left) for the original termographic image of a masonry, and Figure 1 (right) for a reconstruction) to process the thermographic images before to apply the texture, in order to obtain images suitable for the application of the texture algorithm (see e.g. the comparison between Figure 2 (left) and (right)).

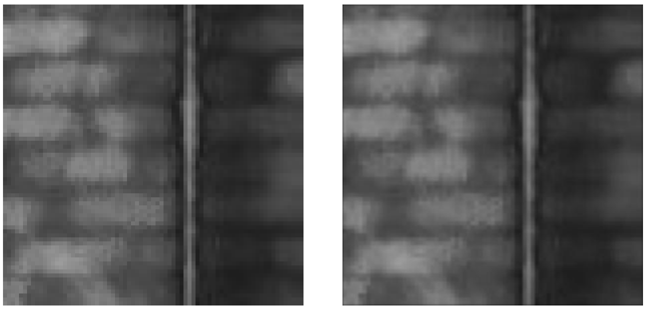

Fig. 1. Recontruction of the original image (left, 75 × 75 pixel) by the sampling Kantorovich operators with the bivariate Jackson kernel with $k = 4$ and $\alpha = 1$, for $w = 40$ (right, 450 × 450 pixel)

In order to perform structural analysis, the mechanical characteristics of an homogeneous material equivalent to the original heterogeneous material are sought (see e.g. [7]). The equivalence is in the sense that, when subjected to the same boundary conditions, the overall response in terms of mean values of stresses and deformations is the same, see e.g. [10]. In particular, the equivalent elastic properties taking into account the effective characteristics of the micro-structure can be estimated by a suitable choice of two kinds of boundary conditions: i) in terms of displacements (essential boundary conditions); ii) in terms of forces (natural boundary conditions). In order to solve the boundary condition problem, the Finite Element Method (F.E.M.) is used.

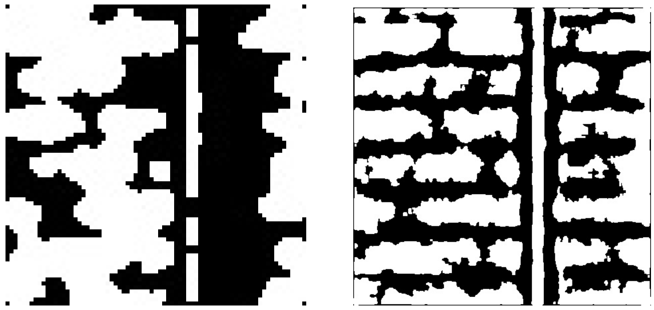

Fig. 2. On the left, we have the image texture of the original image of Figure 1 (left). On the right, we have the image texture of Figure 1 (right), reconstructed by the sampling Kantorovich operators.

The estimated mechanical properties can be used to analyze a real-world case-study. In particular the proposed approach allows to overcome some difficulties, that arise when dealing with the vulnerability analysis of existing structures, which are: i) the knowledge of the actual geometry of the walls (in particular the identification of hidden doors and windows); ii) the identification of the actual texture of the masonry and the distribution of inclusions and mortar joints, and from this iii) the estimation of the elastic characteristics of the masonry. It is noteworthy that, for item i) the engineer has limited knowledge, due to the lack of documentation, while for items ii) and iii) he usually use tables proposed in technical manuals and standards which however give large bounds in order to encompass the generality of the real masonries.

E. Future developments

A future development of the present paper is to study applications of the algorithm based on the sampling Kantorovich operators to biomedical images. In biomedicine, images cover a fundamental role for the clinical diagnosis, surgery (Endovascular aneurysm repair - EVAR), and for the patient follow up. For this purpose, it reveals of a certain importance that the contours of the biomedical images are clearly visible. Then becomes important having at disposal an algorithm for image reconstruction and enhancement. Our aim is to treat images in the field of Vascular Surgery, in collaboration with a group of radiologists and vascular surgeons of the sections of Vascular and Endovascular Surgery and Diagnostic Radiology and Interventional of the University of Perugia. In particular our aim is to apply our algorithm to images related to the aneurysmal aortic and steno-obstructive pathology of epiaortic and peripheral vessels in order to improve the medical diagnosis.

II. Conclusion

In this paper, we present the theory of the multivariate sampling Kantorovich operators. Approximation results are given in various settings. Applications of the theory to Image Processing are also shown. In particular, new applications of the algorithm based on the sampling Kantorovich operators to civil engineering images are obtained. The applications related to image texture algorithm is significant, and of practical utility in seismic engineering.

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