Spectrum Reconstruction from Sub-Nyquist Sampling of Stationary Wideband Signals

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***Abstract*—In light of the ever-increasing demand for new spectral bands and the underutilization of those already allocated, the new concept of Cognitive Radio (CR) has emerged. Opportunistic users could exploit temporarily vacant bands after detecting the absence of activity of their owners. One of the most crucial tasks in the CR cycle is therefore spectrum sensing and detection which has to be precise and efficient. Yet, CRs typically deal with wideband signals whose Nyquist rates are very high. In this paper, we propose to reconstruct the spectrum of such signals from sub-Nyquist samples in order to perform detection. We consider both sparse and non sparse signals as well as blind and non blind detection in the sparse case. For each one of those scenarii, we derive the minimal sampling rate allowing perfect reconstruction of the signal spectrum in a noise-free environment and provide recovery techniques. The simulations show spectrum recovery at the minimal rate in noise-free settings.**

I. Introduction

Spectral resources are traditionally allocated to primary users (PUs). As most are already licensed, new applications can hardly ever obtain access to free frequency bands. Paradoxally, the over-crowded spectrum is usually significantly underutilized as numerous studies have shown [1]–[3]. In order to respond to the increasing demand for spectrum usage from new users, the concept of Cognitive Radio (CR) [4], [5] has recently been considered. In this approach, secondary users opportunistically use temporarely vacant spectrum bands when their owners are inactive.

In this scheme, the CR has to constantly monitor the spectrum and detect the PUs' activity in order to select unoccupied bands, before and throughout its transmission. Obviously, the detection has to be extremely reliable and fast. On the other hand, it is worthwhile for the CR to sense a wide band of spectrum simultaneously, in order to increase the probability of finding a vacant spectral band. Nyquist rates of such wideband signals are very high and sometimes cannot even be met by today's best analog-to-digital converters (ADCs). Moreover, the tremendous amount of samples such high rates generate have to be processed by the CR, slowing down the digital detection process.

To overcome the rate bottleneck, several new sampling methods have recently been proposed [6]–[8] that reduce the sampling rate in multiband settings below the Nyquist rate. In [6]–[8], the authors derive the minimal sampling rate allowing for perfect signal reconstruction in noise-free settings and provide sampling and recovery techniques. However, when the final goal is spectrum sensing and detection, reconstructing the original signal is unnecessary. In this paper, we propose to only reconstruct the signal spectrum from sub-Nyquist samples, in order to perform signal detection. In [9], the authors propose a method to estimate finite resolution approximations to the true spectrum exploiting multicoset sampling. Spectrum reconstruction is also considered in [10] both in the time and frequency domains. However, no analysis on the minimal sampling rate ensuring perfect reconstruction of the spectrum was performed.

We consider the class of wide-sense stationary multiband signals, whose frequency support lies within several continuous intervals (bands). We will consider three different scenarii: (1) the signal is not assumed to be sparse, (2) the signal is assumed to be sparse and the carrier frequencies of the narrowband transmissions are known, (3) the signal is sparse but we do not assume carrier knowledge. We consider the sampling methods proposed in [6]–[8] and use a similar recovery technique to those derived in [9], [10] in order to reconstruct the signal spectrum from the sub-Nyquist samples. Our main contribution is deriving the minimal sampling rate allowing for perfect reconstruction of the spectrum in a noise-free environment, for each one of the above three cases. We show that the rate required for spectrum reconstruction is half the rate that allows for perfect signal reconstruction, for each one of the scenarii, namely the Nyquist rate, the Landau rate [11] and twice the Landau rate [7].

This paper is organized as follows. In Section II, we present the stationary multiband model and formulate the problem. Section III describes the sub-Nyquist sampling stage and the spectrum reconstruction. In Section IV, we derive the minimal sampling rate for each one of the three scenarii described above. Numerical experiments are presented in Section V.

II. System Model and Goal

A. System Model

Let $x(t)$ be a real-valued continuous-time signal, supported on $\mathcal{F} = [-T_{\rm Nyq}/2, +T_{\rm Nyq}/2]$. Formally, the Fourier transform of $x(t)$ defined by

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \mathrm{d}t \tag{1}$$

is zero for every $f \notin \mathcal{F}$. We denote by $f_{\text{Nyq}} = 1/T_{\text{Nyq}}$ the Nyquist rate of $x(t)$. We assume that $x(t)$ is composed of up to N_{sig} uncorrelated stationary transmissions with disjoint frequency supports. The bandwidth of each signal does not exceed $2B$ (where we consider both positive and negative frequency bands together) [6]. We consider three different scenarii.

1) No sparsity assumption: In the first scenario, we assume no *a priori* knowledge on the signal and we do not suppose that $x(t)$ is sparse, namely $2N_{sig}B \leq f_{Nyq}$.

2) Sparsity assumption and non blind detection: Here, we assume that $x(t)$ is sparse, namely $2N_{sig}B \ll f_{Nyq}$. Moreover, the support of the potentially active transmissions is known and correspond to the frequency support of licensed users defined by the communication standard. However, since the PUs' activity can vary over time, we wish to develop a detection algorithm that is independent of a specific known signal support.

3) Sparsity assumption and blind detection: In the last scenario, we assume that $x(t)$ is sparse but we do not assume any *a priori* knowledge on the carrier frequencies.

B. Problem Formulation

In each one of the scenarii defined in the previous section, our goal is to assess which of the N_{sig} transmissions are active from sub-Nyquist samples of $x(t)$. For each signal, we define the hypothesis $\mathcal{H}_{i,0}$ and $\mathcal{H}_{i,1}$, namely the ith transmission is absent and active, respectively.

In order to assess which of the N_{sig} transmissions are active, we will first reconstruct the spectrum of $x(t)$. In our first and third scenarii, we fully reconstruct the spectrum. In the second one, we exploit our prior knowledge and reconstruct it only at the potentially occupied locations. We can then perform detection on the fully or partially reconstructed spectrum. Note that, to do so, we do not sample $x(t)$ at its Nyquist rate, nor compute its Nyquist rate samples. For each one of the scenarii, we derive the minimal sampling rate enabling perfect spectrum reconstruction in a noise-free environment.

III. Sub-Nyquist Sampling and Spectrum Reconstruction

We consider two different sampling schemes: multicoset sampling [7] and the modulated wideband converter (MWC) [6] which were previously proposed for sparse multiband signals. We show that the reconstruction stage is identical for both schemes. In this section, we reconstruct the whole spectrum. In Section IV-B, we show how we can reconstruct the spectrum only at potentially occupied locations when we have *a priori* knowledge on the carrier frequencies.

1) Multicoset sampling: Multicoset sampling [12] can be described as the selection of certain samples from the uniform grid. More precisely, the uniform grid is divided into blocks of N consecutive samples, from which only M are kept. The ith sampling sequence is defined as

$$x_{c_i}[n] = \begin{cases} x(nT_{\text{Nyq}}), & n = mN + c_i, m \in \mathbb{Z} \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $0 < c_1 < c_2 < \cdots < c_M < N-1$. Let $f_s = \frac{1}{NT_{\text{Nyq}}} \geq B$ be the sampling rate of each channel and $\mathcal{F}_s = [-f_s/2, f_s/2]$. Following the derivations from multicoset sampling [7], we obtain

$$\mathbf{z}(f) = \mathbf{A}\mathbf{x}(f), \qquad f \in \mathcal{F}_s, \tag{3}$$

where $\mathbf{z}_i(f) = X_{c_i}(e^{j2\pi f T_{\text{Nyq}}}), 0 \leq i \leq M-1$ is the DTFT of the multicoset samples and

$$\mathbf{x}_k(f) = X\left(f + \frac{K_k}{NT_{\text{Nyq}}}\right), \quad 1 \leq k \leq N, \tag{4}$$

where $K = \{(-\frac{N-1}{2}, \ldots, \frac{N-1}{2})\}$ for odd N (see [7] for even N). Each entry of $\mathbf{x}(f)$ is referred to as bin since it consists of a slice of the spectrum of $x(t)$. The ikth element of the $M \times N$ matrix $\mathbf{A}$ is given by

$$\mathbf{A}_{ik} = \frac{1}{NT_{\text{Nyq}}} e^{j\frac{2\pi}{N} c_i K_k}. \tag{5}$$

2) MWC sampling: The MWC [6] is composed of M parallel channels. In each channel, an analog mixing front-end, where $x(t)$ is multiplied by a mixing function $p_i(t)$, aliases the spectrum, such that each band appears in baseband. The mixing functions $p_i(t)$ are required to be periodic. We denote by T_p their period and we require $f_p = 1/T_p \geq B$. The function $p_i(t)$ has a Fourier expansion

$$p_i(t) = \sum_{l=-\infty}^{\infty} c_{il} e^{j\frac{2\pi}{T_p} lt}. \tag{6}$$

In each channel, the signal goes through a lowpass filter with cut-off frequency $f_s/2$ and is sampled at the rate $f_s \geq f_p$. For the sake of simplicity, we choose $f_s = f_p$. The overall sampling rate is Mf_s where $M \leq N = f_{\text{Nyq}}/f_s$. Repeating the calculations in [6], we derive the relation between the known DTFTs of the samples $z_i[n]$ and the unknown $X(f)$

$$\mathbf{z}(f) = \mathbf{A}\mathbf{x}(f), \qquad f \in \mathcal{F}_s, \tag{7}$$

where $\mathbf{z}(f)$ is a vector of length N with ith element $\mathbf{z}_i(f) = Z_i(e^{j2\pi f T_s})$. The unknown vector $\mathbf{x}(f)$ is given by (4). The $M \times N$ matrix $\mathbf{A}$ contains the coefficients c_{il}:

$$\mathbf{A}_{il} = c_{i,-l} = c^*_{il}. \tag{8}$$

For both sampling schemes, the overall sampling rate is

$$f_{tot} = Mf_s = \frac{M}{N} f_{\text{Nyq}}. \tag{9}$$

A. Spectrum Reconstruction

We note that the systems are identical for both sampling schemes. The only difference is the sampling matrix $\mathbf{A}$. We assume that $\mathbf{A}$ is full spark in both cases [6], [7]. We thus can derive a method for spectrum reconstruction for both sampling schemes together. We define the autocorrelation matrices $\mathbf{R_z} = \mathbb{E}[\mathbf{z}(f)\mathbf{z}^H(f)]$ and $\mathbf{R_x} = \mathbb{E}[\mathbf{x}(f)\mathbf{x}^H(f)]$. Then from (3), we have

$$\mathbf{R_z} = \mathbf{A}\mathbf{R_x}\mathbf{A^H}. \tag{10}$$

Here the exposant H denotes the Hermitian operation. Our goal is to recover $\mathbf{R_x}$ from $\mathbf{R_z}$.

Since $x(t)$ is a wide-sense stationary process, we have [13]

$$\mathbb{E}[X(\omega)X^*(\nu)] = 2\pi P_x(\omega)\delta(\omega - \nu) \tag{11}$$

where $P_x(\omega)$ denotes the spectrum of $x(t)$. Therefore $\mathbf{R_x}$ is a diagonal matrix with $\mathbf{R_x}(i,i) = P_x(f + \frac{i}{NT_s})$ [9]. It follows that

$$\mathbf{r_z} = (\mathbf{A}^* \otimes \mathbf{A})\text{vec}(\mathbf{R_x}) = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{B}\mathbf{r_x} \triangleq \mathbf{\Phi}\mathbf{r_x}, \tag{12}$$

where $\mathbf{\Phi} = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{B}$. Here the exposant $*$ denotes the conjugate operation. Here $\otimes$ is the Kronecker product, $\mathbf{r_z} = \text{vec}(\mathbf{R_z})$, $\mathbf{B}$ is a $N^2 \times N$ selection matrix that has a "1" at the jth column and the $[(j-1)N+j]$th row, $1 \leq j \leq N$ and zeros elsewhere.

We wish to recover $\mathbf{r_x}$ from $\mathbf{r_z}$. In the next section, we will derive the conditions on the sampling rate for (12) to have a unique solution.

IV. Minimal Sampling Rate

A. No sparsity Constraints

The system defined in (12) is overdetermined for $M^2 \geq N$, if $\mathbf{\Phi}$ is full column rank. The following proposition provides the condition for the system defined in (12) to have a unique solution. Due to lack of space, the proofs of the following two propositions are omitted here and will be found in a future paper.

Proposition 1. *Let $\mathbf{A}$ be a full spark $M \times N$ matrix ($M \leq N$) and $\mathbf{B}$ be a $N^2 \times N$ selection matrix that has a "1" at the jth column and the $[(j-1)N+j]$th row, $1 \leq j \leq N$ and zeros elsewhere. The matrix $\mathbf{C} = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{B}$ is full column rank if $M^2 \geq N$ and $2M > N$.*

From Proposition 1, (12) has a unique solution if $M^2 \geq N$ and $2M > N$. This can happen even for $M < N$ which is our basic assumption. If $M \geq 2$, we have $M^2 \geq 2M$. Thus, in this case, the values of M for which we obtain a unique solution are $N/2 < M < N$.

In this case, the minimal sampling rate is

$$f_{(1)} = Mf_s > \frac{N}{2}B = \frac{f_{\text{Nyq}}}{2}. \tag{13}$$

This means that even without any sparsity constraints on the signal, we can retrieve its spectrum by exploiting its stationary property, whereas the measurement vector $\mathbf{z}$ exhibits no stationary constraints in general.

B. Sparsity Constraints - Non-Blind Detection

We now consider the second scheme, where we have *a priori* knowledge on the frequency support of $x(t)$ and we assume that it is sparse. Instead of reconstructing the entire spectrum, we propose to exploit our knowledge of the signal's potential frequencies in order to further reduce the reconstruction problem and only reconstruct the potentially occupied bands.

In this scenario, the only non zero elements of $\mathbf{R_x}$ are $K_f \ll N$ diagonal elements. The reduced dimensionality spectrum is defined as

$$\hat{\mathbf{r}}_\mathbf{x} = \mathbf{M}_f \mathbf{r_x}. \tag{14}$$

Here $\mathbf{M}_f \in \mathbb{R}^{K_f \times N}$ is a matrix with elements equal to 1 at the indices of potential non-zero entries and $\hat{\mathbf{r}}_\mathbf{x} \in \mathbb{C}^{K_f \times 1}$. Furthermore, we also define $\mathbf{G}$ to be the $N \times K_f$ matrix that selects the corresponding K_f columns of $\mathbf{\Phi}$ and $\hat{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{G}$. The reduced problem can then be expressed as

$$\mathbf{r_z} = \hat{\mathbf{\Phi}}\hat{\mathbf{r}}_\mathbf{x}. \tag{15}$$

The following proposition provides the condition for the system defined in (12) to have a unique solution.

Proposition 2. *Let $\mathbf{A}$ be a full spark $M \times N$ matrix ($M \leq N$) and $\mathbf{B}$ be defined as in Proposition 1. Let $\mathbf{C} = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{B}$ and $\mathbf{G}$ be the $N \times K_f$ that selects any $K_f < N$ columns of $\mathbf{C}$. The matrix $\mathbf{D} = \mathbf{C}\mathbf{G}$ is full column rank if $M^2 \geq K_f$ and $2M > K_f$.*

In this case, the minimal sampling rate is

$$f_{(2)} = Mf_s > \frac{K_f}{2}B = N_{sig}B. \tag{16}$$

Landau [11] developed a minimal rate requirement for perfect signal reconstruction in the non-blind setting, which corresponds to the actual band occupancy. Here, we find that the minimal sampling rate for perfect spectrum recovery is half the Landau rate.

C. Sparsity Constraints - Blind Detection

We now consider the second scheme, namely $x(t)$ is sparse, without any *a priori* knowledge on the support. In the previous section, we showed that $\hat{\mathbf{\Phi}}$ is full column rank, for any choice of K_f columns of $\mathbf{\Phi}$ (that correspond to K_f columns of $\mathbf{A}$), if $M^2 \geq K_f$ and $2M > K_f$. Therefore, for $M \geq 2$, if $\mathbf{r_x}$ is M-sparse, it is the unique sparsest solution of (12).

In this case, the minimal sampling rate is

$$f_{(3)} = Mf_s > K_f B = 2N_{sig}B. \tag{17}$$

As expected, this is twice the rate obtained in the previous case. As in signal recovery, the minimal rate for blind reconstruction is twice the minimal rate for non-blind reconstruction [7].

V. Simulation Results

We now demonstrate spectrum reconstruction from sub-Nyquist samples obtained close to the minimal sampling rate for the first and third scenarii, respectively. We use the MWC analog front-end [6] for the sampling stage.

It is interesting to notice that (12), which is written in the frequency domain, is valid in the time domain as well. We can therefore estimate $\mathbf{r_z}(f)$ and reconstruct $\mathbf{r_x}(f)$ in the frequency domain, or alternatively, we can estimate $\mathbf{r_z}[n]$ and reconstruct $\mathbf{r_x}[n]$ in the time domain. In order to estimate the autocorrelation matrix $\mathbf{R_z}(f)$, we first compute the estimates of $\mathbf{z}_i(f), 1 \leq i \leq M$, $\hat{\mathbf{z}}_i(f)$, using FFT on the samples $z_i[n]$ over a finite time window. We then estimate the elements of $\mathbf{R}_{\mathbf{z}}(f)$ as

$$\hat{\mathbf{R}}_{\mathbf{z}}(i,j,f) = \frac{1}{P}\sum_{p=1}^{P} \hat{\mathbf{z}}^p(i,f)\hat{\mathbf{z}}^p(j,f), \quad f \in \mathcal{F}_s, \tag{18}$$

where P is the number of frames for the averaging of the spectrum and $\hat{\mathbf{z}}^p(i,f)$ is the value of the FFT of the samples $\mathbf{z}_i[n]$ at the frequency f and the pth frame. In order to estimate the autocorrelation matrix $\mathbf{R}_{\mathbf{z}}[n]$ in the time domain, we perform a convolution between the samples $z_i[n]$ over a finite time window as

$$\hat{\mathbf{R}}_{\mathbf{z}}[i,j,n] = \frac{1}{P}\sum_{p=1}^{P} z_i^p[n] * z_j^p[n], \quad n \in [0, T/T_{\text{Nyq}}]. \tag{19}$$

We first consider the spectrum reconstruction of a non sparse signal. Let $x(t)$ be white Gaussian noise with variance 100, and Nyquist rate $f_{\text{Nyq}} = 10GHz$ with two stop bands. We consider $N = 65$ spectral bands and $M = 33$ analog channels, each with sampling rate $f_s = 154MHz$ and with $N_s = 131$ samples each. The overall sampling rate is therefore equal to 50.77% of the Nyquist rate. Figure 1 shows the original and the reconstructed spectrum at half the Nyquist rate (both with averaging over $P = 1000$).

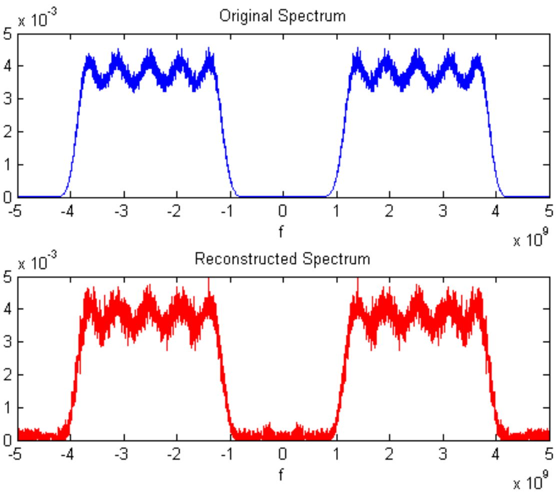

Fig. 1. Original and reconstructed spectrum of a non sparse signal at half the Nyquist rate.

We now consider the blind reconstruction of a sparse signal. Let the number of potentially active transmissions $N_{\text{sig}} = 6$ and the actual number of active transmissions be 3. Each transmission is white Gaussian noise with variance 100 and Nyquist rate $f_{\text{Nyq}} = 10GHz$, filtered by a bandpass filter whose central frequency is drawn uniformly at random and whose bandwidth is $B = 120Mhz$. We consider $N = 75$ spectral bands and $M = 13$ analog channels, each with sampling rate $f_s = 133MHz$ and with $N_s = 131$ samples each. The overall sampling rate is equal to 110% of the minimal rate (17). Figure 2 shows the original and the reconstructed spectrum at 17.3% of the Nyquist rate (both with averaging over $P = 1000$ frames).

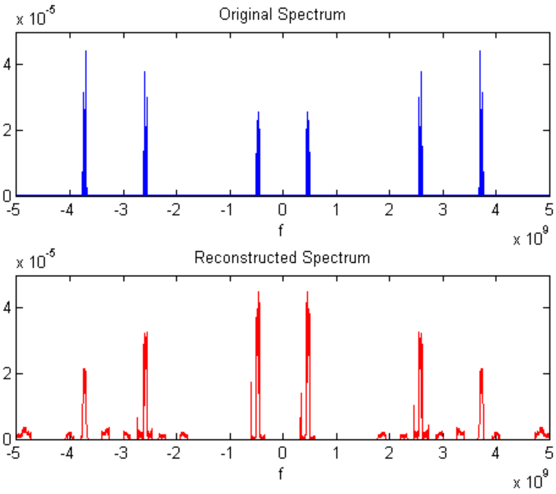

Fig. 2. Original and reconstructed spectrum of a non sparse signal at 17.3% of the Nyquist rate.

We note that the difference between the original and the reconstructed spectrum comes from the fact that the matrix $\mathbf{R}_{\mathbf{x}}(f)$ is not perfectly diagonal.

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