Conditions for Dual Certificate Existence in Semidefinite Rank-1 Matrix Recovery

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***Abstract*—We study the existence of dual certificates in convex minimization problems where a rank-1 matrix X_0 is to be recovered under semidefinite and linear constraints. We provide an example where such a dual certificate does not exist. We prove that dual certificates are guaranteed to exist if the linear measurement matrices can not be recombined to form something positive and orthogonal to X_0. If the measurements can be recombined in this way, the problem is equivalent to one with additional linear constraints. That augmented problem is guaranteed to have a dual certificate at the minimizer, providing the form of an optimality certificate for the original problem.**

I. Introduction

We consider the problem of showing that $X_0 = x_0 x_0^*$ is a minimizer to the semidefinite program

$$\min f(X) \text{ subject to } X \succeq 0, \mathcal{A}(X) = b. \tag{1}$$

for $x_0 \in \mathbb{R}^n$, $X \in \mathcal{S}_n$ is a symmetric real $n \times n$ matrix, f is convex and continuous everywhere, and $\mathcal{A}$ is linear, and $\mathcal{A}(X_0) = b \in \mathbb{R}^n$. Let $\langle X, Y \rangle = \operatorname{tr}(Y^* X)$ be the Hilbert-Schmidt inner product. Matrix orthogonality is understood to be with respect to this inner product. The linear measurements $\mathcal{A}(X) = b$ can be written as

$$\mathcal{A}(X)_i = \langle X, A_i \rangle = b_i \text{ for } i = 1, \dots, m$$

for certain symmetric matrices A_i. Note that the adjoint of $\mathcal{A}$ is given by $\mathcal{A}^* \lambda = \sum_i \lambda_i A_i$.

One problem of this form is phase retrieval via PhaseLift, where $f(X) = \operatorname{tr}(X)$ and $A_i = z_i z_i^*$ for vectors z_i [3]. Another example is the corresponding sparse recovery problem with $f(X) = \|X\|_1 + c \operatorname{tr}(X)$, where the first term is the entry-wise ℓ^1 norm of X [6].

In these matrix recovery problems, a recovery result that X_0 is a minimizer is often proved by constructing a dual certificate (or approximation thereof) at X_0. Similar to [5] and [2], we call $Y \in \mathcal{S}_n$ a dual certificate at X_0 if

$$\begin{cases} Y = \mathcal{A}^* \lambda + Q \in -\partial f(X_0) \\ Q \preceq 0 \\ Q \perp X_0. \end{cases} \tag{2}$$

If a dual certificate exists at X_0 then X_0 is a minimizer of (1). Further, it is straightforward to prove that existence of a dual certificate at X_0 is equivalent to (1) satisfying strong duality with dual attainment by (λ, Q).

In the development of convex programs for matrix recovery, it is desirable to know if strong duality holds. Without guarantees of existence, attempting to analytically construct dual certificates in particular problems may be futile. Under strong duality, negative results guaranteeing that X_0 is not a minimizer could be proven by showing no dual certificate exists, as done in [6].

The perspective of this note is to ease proofs of new semidefinite relaxations, rather than easing their computation. In particular, we are concerned with conditions on A_i under which problem (1) has a dual certificate at the minimizer X_0 or can be augmented into an equivalent problem that does.

A. *Counterexample*

Though sufficient, existence of a dual certificate (2) is not necessary for X_0 to minimize (1). Consider the following problem:

$$\min \frac{1}{2}\|X\|^2 \text{ subject to } X \succeq 0, \left\langle X, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = 0, \tag{3}$$
$$\left\langle X, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle = 1,$$

where $\|\cdot\|$ is the Frobenius norm. To analyze this problem, we recall the fact that

$$X \succeq 0 \text{ and } \langle X, qq^* \rangle = 0 \text{ for } q \in \mathbb{R}^n \Rightarrow Xq = 0$$
$$\Rightarrow \langle X, y \otimes q \rangle = 0 \text{ for any } y, \tag{4}$$

where $y \otimes q = yq^* + qy^*$ is the symmetric tensor product. Using (4), we can see that any feasible X satisfies

$$\left\langle X, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = 0. \tag{5}$$

Hence, the minimizer and only feasible point of (3) is

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this example, the subdifferential of $f(X) = \frac{1}{2}\|X\|^2$ contains only the single element $\partial f(X_0) = \{X_0\}$. Again using (4), we note that the dual certificate conditions (2) can not be satisfied because there is no (Q, λ) such that

$$-\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + Q.$$

for $Q \preceq 0, Q \perp X_0$. If we were to supplement (3) with the constraint (5), the conditions (2) could be satisfied for some (Q, λ).

B. Constraint Qualifications

It is well known that semidefinite programs of form (1) can have a nonzero duality gap or can have a Lagrangian dual problem for which the dual optimum is not attained [10], [12]. A constraint qualification (CQ) is a condition such that strong duality and dual attainment is ensured. For example, the presence of a strictly feasible $X \succ 0$ such that $\mathcal{A}(X) = b$, is a constraint qualification and is known as Slater's condition [1].

Slater's condition can be insufficient for low-rank matrix recovery problems. As in the counterexample, if a linear combination of the A_i are nonnegative and orthogonal to X_0, then there is no strictly feasible point. Additional constraint qualifications can be found in [11], [12].

The work in this paper will be based of the following constraint qualification. The Rockafellar-Pshenichnyi condition [4], [7], [12], [13] in the present context is that X_0 minimizes (1) if and only if there exists a $Y \in (-\partial f(X_0)) \cap \partial I_{X\succeq 0, \mathcal{A}(X)=b}(X_0)$, where $I_{X\succeq 0, \mathcal{A}(X)=b}$ is the indicator function of the feasible set. Let the cone of candidate dual certificates be

$$S := \left\{ \sum_i \lambda_i A_i + Q \mid Q \preceq 0, Q \perp X_0 \right\}, \quad (6)$$

$$= \partial I_{X\succeq 0}(X_0) + \partial I_{\mathcal{A}(X)=b}(X_0). \quad (7)$$

A constraint qualification is thus that

$$\partial I_{X\succeq 0}(X_0) + \partial I_{\mathcal{A}(X)=b}(X_0) = \partial I_{X\succeq 0, \mathcal{A}(X)=b}(X_0). \quad (8)$$

This constraint qualification is a weakest constraint qualification because it is independent of f [12].

One way to interpret this CQ is in terms of extremal directions. We say that A is an extremal direction of X_0 relative to the feasible set if $\langle A, X\rangle \leq \langle A, X_0\rangle$ for all feasible X. Any element of S is an extremal direction of X_0, but S does not necessarily contain all directions in which X_0 is extreme. The set of all such directions is the subdifferential $\partial I_{X\succeq 0, \mathcal{A}(X)=b}(X_0)$. The CQ (8) is that S contains all directions in which X_0 is extreme. Note that $\partial I_{X\succeq 0, \mathcal{A}(X)=b}(X_0)$ is the negative polar cone of the tangent cone of the feasible set at X_0.

C. Sufficient condition for dual certificate existence

Avoiding the pathology of the counterexample, we present a condition for which dual certificates are guaranteed to exist.

Theorem 1. *Let X_0 minimize* (1)*. If $\nexists A \in span\{A_i\}$ such that $A \succeq 0$ and $A \perp X_0$, then strong duality holds and a dual certificate exists at X_0.*

That is, the pathology of the counterexample arrives because there is a linear combination of A_i that is positive semi-definite and orthogonal to X_0. If this case is excluded, a dual certificate necessarily exists at the rank-one solution X_0.

D. Weaker sufficient condition for dual certificate existence

If there is a positive semi-definite measurement matrix A that is orthogonal to X_0, then (4) provides additional constraints on X that may or may not be implied by the linear constraints $\mathcal{A}(X) = b$ alone. For any $q \in \text{Range}(A)$, and for any y, all feasible X satisfy $\langle X, y \otimes q\rangle = 0$. Hence $y \otimes q$ is an extremal direction of X_0, and must be in S in order for strong duality to hold. We say that S is complete at X_0 if the following condition holds:

$$\text{If } A = \mathcal{A}^*\lambda \succeq 0, A \perp X_0, \text{ then}$$
$$y \otimes q \in S \text{ for all } y \text{ and for all } q \in \text{Range}(A). \quad (9)$$

Theorem 2. *Let X_0 minimize* (1)*. If S satisfies the completeness condition* (9) *then strong duality holds and a dual certificate exists at X_0.*

E. General certificate form

As the counterexample illustrates, the problem (1) may not contain the linear equations $\langle X, y \otimes q\rangle = 0$ for the q described in section I-D. In this case, the optimality certificate for (1) can be expressed as a dual certificate for the problem augmented with linear constraints implied by $X \succeq 0$ and $\mathcal{A}(X) = b$. This augmented problem is equivalent to (1) and satisfies the conditions of Theorem 2. Hence, its dual contains the form of the optimality certificate for (1).

The following procedure outlines a process for augmenting the measurement matrices $\{A_i\}$ in such a way that there exists a dual certificate of the form $\sum_i \lambda_i A_i + Q$ for $Q \preceq 0, Q \perp X_0$:

1) Consider all $A \succeq 0, A \in \text{span}\{A_i\}, \langle A, X_0\rangle = 0$.
2) Write each $A = \sum_k c_k q_k q_k^*$ with $c_k > 0$.
3) For each coordinate basis element e_j, if $e_j \otimes q_k \notin \text{span}\{A_i\}$, append $\langle X, e_j \otimes q_k\rangle = 0$ to $\mathcal{A}(X) = b$.
4) Repeat until $\mathcal{A}$ remains unchanged.

This process will produce a set S satisfying (9), and it will terminate after finitely many repetitions because $\text{rank}(\text{span}\{A_i\})$ increases each time. Each added measurement is implied by the constraints of (1) and does not change the underlying problem.

This process can be viewed as a regularization of the convex problem (1). It differs from a minimal cone regularization because the positive semidefinite cone constraint is kept [10], [12]. Another regularization approach in the literature is the extended Lagrange-Slater Dual (ELSD), which is an alternative to the Langrangian dual that guarantees strong duality at the expensive of polynomially many additional variables [9], [10]. The regularization procedure above is different from ELSD because it get strong duality while keeping the standard Lagrangian dual. The dual variables can hence be viewed as Lagrange multipliers for direct or implied measurements of the matrix X_0. Unfortunately, the procedure above can not be written down mechanically, whereas the ELSD can. Hence, it is less useful for performing the semidefinite optimization than it is as a theoretical process for ensuring that a dual certificate exists.

II. Proofs

A. Notation

For a subspace $V \subset \mathbb{R}^n$, let $V^\perp$ be the orthogonal complement with respect to the ordinary inner product. Let $I_{V^\perp}$ be the matrix corresponding to orthogonal projection of vectors onto $V^\perp$. Let $\mathcal{P}_{V^\perp} X = I_{V^\perp} X I_{V^\perp}$ be the projection of symmetric matrices onto symmetric matrices with row and column spans in $V^\perp$. Let $\mathcal{P}_{x_0^\perp}$ be the special case in the instance where $V = \text{span}\{x_0\}$. In the special case where x_0 is the coordinate basis element e_1, $\mathcal{P}_{x_0^\perp} X$ is the restriction of X to the lower-right $n-1 \times n-1$ block. Let the indicator function for the set Ω be $I_\Omega(X)$, which is zero on Ω and infinity otherwise.

B. Proof of Theorems

Under the assumptions of Theorem 1, the set S trivially satisfies the completeness condition (9). The theorem is thus a special case of Theorem 2, and we will prove them together. As per the constraint qualification (8), it suffices to prove the following technical lemma. This main technical lemma establishes additivity of subgradients of a class of indicator functions. The primary direction uses a separating hyperplane argument to build an item in the subgradient. That argument requires S be closed, as proven in Lemma 2. It also hinges on Lemma 4 which classifies when perturbations from X_0 remain positive semidefinite.

Lemma 1. *Let $X_0 = x_0 x_0^*$ and $\mathcal{A}(X_0) = b$. S satisfies the completeness condition* (9) *if and only if*

$$\partial I_{X\succeq 0, \mathcal{A}(X)=b}(X_0) = \partial I_{X\succeq 0}(X_0) + \partial I_{\mathcal{A}(X)=b}(X_0). \quad (10)$$

Proof of Lemma 1: We omit the proof that $\neg(9) \Rightarrow \neg(10)$.

Now, we show (9) $\Rightarrow$ (10). One inclusion in (10) is automatic:

$$\partial I_{X\succeq 0, \mathcal{A}(X)=b}(X_0) = \partial(I_{X\succeq 0} + I_{\mathcal{A}(X)=b})(X_0) \quad (11)$$
$$\supset \partial I_{X\succeq 0}(X_0) + \partial I_{\mathcal{A}(X)=b}(X_0). \quad (12)$$

To prove the other inclusion, we let $Y \notin S = \partial I_{X\succeq 0}(X_0) + \partial I_{\mathcal{A}(X)=b}(X_0)$. We will show that $Y \notin \partial I_{X\succeq 0,\mathcal{A}(X)=b}(X_0)$ by exhibiting a feasible X such that $\langle Y, X - X_0\rangle > 0$.

As we will prove in Lemma 2, (9) implies that S is closed. By the separating hyperplane theorem, for any $Z \notin S$, there exists a Λ_Z such that

$$\mathcal{A}(\Lambda_Z) = 0, \quad (13)$$
$$\langle \Lambda_Z, Q\rangle \le 0 \text{ for all } Q \preceq 0, Q \perp X_0, \quad (14)$$
$$\langle \Lambda_Z, M\rangle = 0 \text{ if } \pm M \in S, \quad (15)$$
$$\langle \Lambda_Z, Z\rangle > 0. \quad (16)$$

We observe that (14) implies $\mathcal{P}_{x_0^\perp}\Lambda_Z \succeq 0$.

Let $B = \{qq^* \mid qq^* \perp X_0, qq^* \notin S\}$. We will build a $\tilde\Lambda$ satisfying the following properties:

$$\mathcal{A}(\tilde\Lambda) = 0, \quad (17)$$
$$\langle \tilde\Lambda, Q\rangle \le 0 \text{ for all } Q \preceq 0, Q \perp X_0, \quad (18)$$
$$\langle \tilde\Lambda, M\rangle = 0 \text{ if } \pm M \in S, \quad (19)$$
$$\langle \tilde\Lambda, qq^*\rangle > 0 \text{ for all } qq^* \in B. \quad (20)$$

We build $\tilde\Lambda$ through the following process. Choose a $q_1q_1^* \in B$ and find a corresponding $\Lambda_{q_1q_1^*}$. Restrict B to a set $\tilde B$ containing only the elements that are orthogonal to $\Lambda_{q_1q_1^*}$. All elements in $B \setminus \tilde B$ have a positive inner product with $\Lambda_{q_1q_1^*}$. Choose $q_2q_2^* \in \tilde B$ and find $\Lambda_{q_2q_2^*}$. Further restrict $\tilde B$ to only the elements that are orthogonal to $\Lambda_{q_2q_2^*}$. Now, all elements in $B\setminus\tilde B$ have a positive inner product with $\Lambda_{q_1q_1^*}$ or $\Lambda_{q_2q_2^*}$. Repeat this process until B is empty. The process will complete after a finite number of repetitions because the set $\tilde B$ is restricted to a space of strictly decreasing dimension at each step. Let $\tilde\Lambda = \sum_i \Lambda_{q_iq_i^*}$. We observe (17)–(19) hold due to (13)–(15). Every element of B has a positive inner product with $\Lambda_{q_iq_i^*}$ for some i. Hence, we have (20).

Let $\Lambda = \Lambda_Y + \varepsilon\tilde\Lambda$, where ε is small enough that $\langle\Lambda, Y\rangle > 0$. By Lemma 4, if (a) $\mathcal{P}_{x_0^\perp}\Lambda \succeq 0$ and (b) $\Lambda \perp qq^*$ and $qq^* \perp X_0 \Rightarrow \Lambda \perp x_0 \otimes q$, then there exists $\delta > 0$ such that $X_0 + \delta\Lambda \succeq 0$. By (14) and (18), (a) holds. To show (b) holds, we consider a $qq^* \perp \Lambda$, $qq^* \perp X_0$. By (20) and the definition of Λ, qq^* must be in S. By (9), $\pm x_0 \otimes q \in S$. Hence, by (15) and (19), $\Lambda \perp x_0 \otimes q$, and (b) holds.

As given by Lemma 4, let $X = X_0 + \delta\Lambda$. Because $X \succeq 0$ and $\mathcal{A}(\Lambda) = 0$, X is feasible. Additionally, $\langle Y, X - X_0\rangle > 0$ because $\langle\Lambda, Y\rangle > 0$. Hence, $Y \notin \partial I_{X\succeq 0,\mathcal{A}(X)=b}(X_0)$.

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The hyperplane separation argument above requires that S be closed. The following lemma reduces the closedness of $S \subset \mathcal{S}_n$ to an $n-1 \times n-1$ case without the orthogonality constraint, which is proved in Lemma 3.

Lemma 2. *If $S = \{\sum_i \lambda_i A_i + Q \mid Q \preceq 0, Q \perp X_0\}$ satisfies the completeness condition* (9) *then S is closed.*

Proof of Lemma 2: Without loss of generality let $X_0 = e_1e_1^*$. This can be seen by letting V be an orthogonal matrix with $x_0/\|x_0\|$ in the first column, and by considering the set V^*SV. If necessary, linearly recombine the A_i such that the first columns of $A_1, \ldots, A_\ell$ are independent and the first columns of the remaining $A_{\ell+1}, \ldots, A_m$ are zero.

Consider a Cauchy sequence $A^{(k)} + Q^{(k)} \to X$, where $A^{(k)} = \sum_{i=1}^m \lambda_i^{(k)} A_i$. We will establish that $X \in S$. Because $Q^{(k)} \preceq 0$ and $Q^{(k)} \perp e_1e_1^*$, it is zero in the first row and column. Hence the first column of $\sum_{i=1}^\ell \lambda_i^{(k)} A_i$ converges to the first column of X. By independence, we obtain that $\lambda_i^{(k)}$ converges to some $\lambda_i^{(\infty)}$ for each $1 \le i \le \ell$. As a result,

$$\sum_{i=\ell+1}^m \lambda_i^{(k)} A_i + Q^{(k)} \to \overline{X},$$

where $\overline{X} = X - \sum_{i=1}^{\ell} \lambda_i^{(\infty)} A_i$, and $\overline{X}$ is zero in the first row and column.

The problem has now been reduced to one of size $n-1 \times n-1$ without an orthogonality constraint, and Lemma 3 completes the proof. Let $\tilde{A}_i$ be the lower-right $n-1 \times n-1$ sub matrix of A_i. Let $\tilde{S} = \{\sum_{i=\ell+1}^{m} \lambda_i \tilde{A}_i + \tilde{Q} \mid \tilde{Q} \preceq 0\} \in \mathcal{S}_{n-1}$. If $\tilde{q}\tilde{q}^* \in \tilde{S}$ then $\begin{pmatrix} 0 \\ \tilde{q} \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{q} \end{pmatrix}^* \in S$. By (9), $\begin{pmatrix} 0 \\ \tilde{y} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \tilde{q} \end{pmatrix} \in S \ \forall y \in \mathbb{R}^{n-1}$. By independence of the first columns of $A_1, \ldots, A_\ell$, $\tilde{y} \otimes \tilde{q} \in \tilde{S}$. The conditions of Lemma 3 are met. Hence, $\overline{X} = \sum_{i=\ell+1}^{m} \lambda_i^{(\infty)} A_i + Q^{(\infty)}$ with $Q^{(\infty)} \preceq 0, Q^{(\infty)} \perp e_1 e_1^*$. We conclude $X \in S$ and S is closed.

■

The closedness of S above relies on the closedness of a lower dimensional $\tilde{S}$ without the orthogonality constraint.

Lemma 3. *The set* $\tilde{S} = \{\sum_i \lambda_i A_i + Q \mid Q \preceq 0\} \subset \mathcal{S}_n$ *is closed if*

$$qq^* \in \tilde{S} \Rightarrow y \otimes q \in \tilde{S} \ \forall y. \tag{21}$$

Proof of Lemma 3: Consider a Cauchy sequence $A^{(k)} + Q^{(k)} \to X$, where $A^{(k)} = \sum_i \lambda_i^{(k)} A_i$. Let $V = \text{span}\{q \mid qq^* \in \tilde{S}\}$. For each $q \in V$, (21) gives that $y \otimes q \in \tilde{S} \ \forall y$. Because $\mathcal{P}_{V^\perp}$ is the projection of matrices onto matrices with row and column spaces living in $V^\perp$,

$$\pm(X - \mathcal{P}_{V^\perp} X) \in \tilde{S} \text{ for any } X. \tag{22}$$

The Cauchy sequence satisfies

$$\mathcal{P}_{V^\perp} A^{(k)} + \mathcal{P}_{V^\perp} Q^{(k)} \to \mathcal{P}_{V^\perp} X. \tag{23}$$

If $\|\mathcal{P}_{V^\perp} A^{(k)}\|_{\text{F}} \to \infty$, then $\frac{\|\mathcal{P}_{V^\perp} A^{(k)}\|_{\text{F}}}{\|\mathcal{P}_{V^\perp} Q^{(k)}\|_{\text{F}}} \to 1$ and $\left\langle \frac{\mathcal{P}_{V^\perp} A^{(k)}}{\|\mathcal{P}_{V^\perp} A^{(k)}\|_{\text{F}}}, \frac{\mathcal{P}_{V^\perp} Q^{(k)}}{\|\mathcal{P}_{V^\perp} Q^{(k)}\|_{\text{F}}} \right\rangle \to -1$ as $k \to \infty$. The sets $\{A \in \mathcal{P}_{V^\perp} \text{span } A_i\} \cap \{\|A\|_{\text{F}} = 1\}$ and $\{Q \preceq 0\} \cap \{\|Q\|_{\text{F}} = 1\}$ are compact. Hence $\langle A, Q \rangle$ achieves its minimum. That minimum value must be -1, which implies that there exists a nonzero, positive semidefinite matrix $-Q \in \mathcal{P}_{V^\perp} \text{span } A_i$. This is is impossible by the construction of V. Suppose $\mathcal{P}_{V^\perp} \mathcal{A}^* \lambda \succeq 0$. By (22), we see $\mathcal{P}_{V^\perp} \mathcal{A}^* \lambda \in \tilde{S}$. Hence every rank-1 component qq^* of $\mathcal{P}_{V^\perp} \mathcal{A}^* \lambda \succeq 0$ belongs to $\tilde{S}$. We reach a contradiction because q would belong to V and can not be in the column space of $\mathcal{P}_{V^\perp} \mathcal{A}^* \lambda$.

Hence, $\mathcal{P}_{V^\perp} A^{(k)}$ has a bounded subsequence. Thus, there is a further subsequence that converges and $\mathcal{P}_{V^\perp} X$ is of the form $\mathcal{P}_{V^\perp}(\sum_i \lambda_i^{(\infty)} A_i + Q^{(\infty)})$. By (22), we conclude $X = \sum_{i=1}^{m} \lambda_i^{(\infty)} A_i + Q^{(\infty)}$ with $Q^{(\infty)} \preceq 0$.

■

The following lemma establishes a necessary and sufficient condition for when a symmetric perturbation from a positive rank 1 matrix remains positive.

Lemma 4. *Let* $X_0 = x_0 x_0^* \in \mathbb{R}^{n \times n}$. $X_0 + \delta\Lambda \succeq 0$ *for some* $\delta > 0$ *if and only if (a)* $\mathcal{P}_{x_0^\perp} \Lambda \succeq 0$ *and (b)* $\Lambda \perp qq^*$ *and* $q \perp x_0 \Rightarrow \Lambda \perp x_0 \otimes q$.

Proof: Without loss of generality, assume $X_0 = e_1 e_1^*$. In this case $\mathcal{P}_{x_0^\perp}$ is the restriction to the lower-right $n-1 \times n-1$ block. Let $\Lambda_{x_0^\perp} \in \mathcal{S}_{n-1}$ be that lower-right block of Λ. Write the block form

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \rho^* \\ \rho & \Lambda_{x_0^\perp} \end{pmatrix}.$$

First we prove $X_0 + \delta\Lambda \succeq 0 \Rightarrow$ (a) and (b). We immediately have (a) because X_0 is zero on the lower-right subblock. Using a Schur complement, if $1 + \delta\Lambda_{11} > 0$, then

$$X_0 + \delta\Lambda \succeq 0 \Leftrightarrow \Lambda_{x_0^\perp} - \frac{\delta}{1 + \delta\Lambda_{11}} \rho\rho^* \succeq 0. \tag{24}$$

If necessary, δ can be reduced to enforce $1 + \delta\Lambda_{11} > 0$. If (b) does not hold, then there is $\xi \in \mathbb{R}^{n-1}$ such that $\Lambda_{x_0^\perp} \perp \xi\xi^*$ and $\rho \not\perp \xi$. By testing against ξ, we see $\Lambda_{x_0^\perp} - \frac{\delta}{1+\delta\Lambda_{11}} \rho\rho^* \not\succeq 0$

Second, we prove (a) and (b) $\Rightarrow X_0 + \delta\Lambda$ for some $\delta > 0$. Assume (a) and (b) hold. Using the property (24) about Schur complements, it suffices to show

$$\Lambda_{x_0^\perp} - \frac{\delta}{1 + \delta\Lambda_{11}} \rho\rho^* \succeq 0. \tag{25}$$

Let $V = \text{span } \{q \mid \Lambda_{x_0^\perp} \perp qq^*\} \subset \mathcal{S}_{n-1}$. There is some ϵ such that $\Lambda_{x_0^\perp} \succeq \epsilon I_{V^\perp}$. If not, there would be a sequence of $x^{(\varepsilon)} \in V^\perp$ such that $\|x^{(\varepsilon)}\| = 1$ and $0 < x^{(\varepsilon)} \Lambda_{x_0^\perp} x^{(\varepsilon)*} < \varepsilon$. Such $x^{(\varepsilon)}$ would have a convergent subsequence to some $x^{(0)} \in V^\perp$ such that $x^{(0)} \Lambda_{x_0^\perp} x^{(0)*} = 0$, which is impossible.

We note that for any $q \in V$, (b) guarantees $\rho \perp q$. Hence $\rho \in V^\perp$ and there is a sufficiently small δ such that $\frac{\delta}{1+\delta\Lambda_{11}} \rho\rho^* \preceq \varepsilon I_{V^\perp}$. We conclude that (25) holds, and hence $\exists\delta > 0$ such that $X_0 + \delta\Lambda \succeq 0$. ■

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References

[1] S. Boyd, L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[2] E. J. Candes, Y. Eldar, T. Strohmer,V. Voroninski. *Phase retrieval via matrix completion.* SIAM J. on Imaging Sciences 6(1), 2011. 199–225.

[3] E. J. Candes, T. Strohmer, V. Voroninski. *PhaseLift: Exact and Stable Signal Recovery from Magnitude Measurements via Convex Programming* To appear in Comm. Pure Appl. Math., 2012.

[4] M. Guignard. *Generalized Kuhn-Tucker conditions for mathematical programming problems in a Banach space.* SIAM J. Control, 7(2):232-241, 1969.

[5] D. Gross. *Recovering Low-Rank Matrices From Few Coefficients In Any Basis* IEEE Trans. Inf. Theory., 57(3), 2011.

[6] X. Li, V. Voroninski. *Sparse Signal Recovery from Quadratic Measurements via Convex Programming.* arXiv preprint 1209.4785., 2012.

[7] B.N. Pshenichniyi. *Necessary conditions for an extremum.* Pure and Applied Mathematics, vol. 4. Dekker, New York (1971).

[8] R. Rockafellar. *Convex Analysis* Princeton University Press, 1970.

[9] M. V. Ramana, *An Exact Duality Theory for Semidefinite Programming and its Complexity Implications*, DIMACS Technical report 95-02R, RUTCOR, Rutgers University, New Brunswick, NJ, 1995.

[10] M. V. Ramana, L. Tuncel, H. Wolkowicz. *Strong Duality for Semidefinite Programming* SIAM J. Optim. 7(3), 641–662, 1997.

[11] A. Shapiro, K. Scheinberg. *Duality and optimality conditions*. In: *Handbook of Semidefinite Programming*. Internat. Ser. Oper. Res. Management Sci., 27, 67–110. Kluwer Academic Boston, 2000.

[12] L. Tuncel, H. Wolkowicz. *Strong duality and minimal representations for cone optimization.* Comput. Optim. Appl 53, 619-648, 2012.

[13] H. Wolkowicz. *Geometry of optimality conditions and constraint qualifications: the convex case.* Math. Programming, 19(1):32-60, 1980.