

On Optimal Sampling Trajectories for Mobile Sensing

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Abstract—We study the design of sampling trajectories for stable sampling and reconstruction of bandlimited spatial fields using mobile sensors. As a performance metric we use the path density of a set of sampling trajectories, defined as the total distance traveled by the moving sensors per unit spatial volume of the spatial region being monitored. We obtain new results for the problem of designing stable sampling trajectories with minimal path density, that admit perfect reconstruction of bandlimited fields. In particular, we identify the set of parallel lines with minimal path density that contains a set of stable sampling for isotropic fields.

I. INTRODUCTION

Let the square-integrable mapping $f : \mathbb{R}^d \mapsto \mathbb{C}$ denote a d -dimensional time-invariant spatial field, with $f(r)$ representing the field value at a location r in d -dimensional space. The Fourier transform of f is defined as

$$F(\omega) = \int_{\mathbb{R}^d} f(r) \exp(-i\langle \omega, r \rangle) dr, \quad \omega \in \mathbb{R}^d \quad (1)$$

where i denotes the imaginary unit, and $\langle u, v \rangle$ denotes the scalar product between vectors u and v in \mathbb{R}^d . We say that f is bandlimited to some set $\Omega \subset \mathbb{R}^d$, if the Fourier transform F of f is supported on Ω . In this case we write $f \in \mathcal{B}_\Omega$ where \mathcal{B}_Ω denotes the collection of fields with finite energy bandlimited to Ω , i.e.,

$$\mathcal{B}_\Omega := \{f \in L^2(\mathbb{R}^d) : F(\omega) = 0 \text{ for } \omega \notin \Omega\}. \quad (2)$$

The classical theory of sampling and reconstructing such high-dimensional bandlimited fields dates back to Petersen and Middleton [1] who identified conditions for reconstructing such fields from their measurements on a lattice of points in space. Further research on non-uniform sampling generated more results on conditions for perfect reconstruction from samples taken at non-uniformly distributed spatial locations [2], [3], [4], [5], [6], [7]. Such works primarily deal with the problem of reconstructing the field from measurements taken by a collection of static sensors distributed in space, like that shown in Figure 1(a), and hence the performance metric usually used to quantify the efficiency of a sampling scheme is the spatial density of samples which is exactly equal to the number of sensors required for sampling per unit volume of the spatial region being monitored.

In this paper we consider the problem of reconstructing a bandlimited spatial field (where $d = 2$ or 3) using its samples taken by a mobile sensor that moves along a continuous path

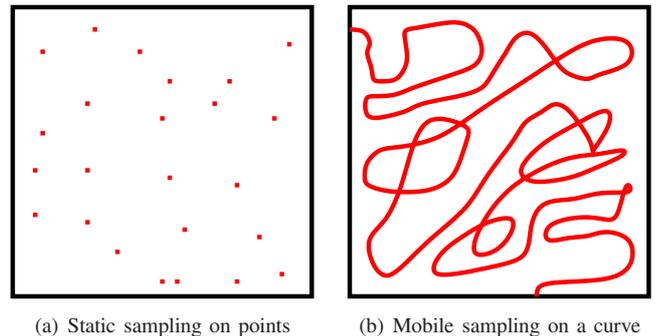


Fig. 1. Two approaches for sampling a field in \mathbb{R}^2

through space taking measurements along its path, as shown in Figure 1(b). In such cases it is often relatively inexpensive to increase the spatial sampling rate along the sensor's path while the main cost of the sampling scheme comes from the total distance that needs to be traveled by the moving sensor. Hence it is reasonable to assume that the sensor can record the field values at an arbitrarily high but finite resolution on its path. Furthermore, for such a sampling application, the density of the sampling points in \mathbb{R}^d used in classical sampling theory is not a relevant performance metric. Instead, as we argued in our previous work [8] [9], a more relevant metric is the average distance that needs to be traveled by the sensor per unit spatial volume (or area, for $d = 2$). We call this metric the *path density*. Such a metric is relevant in applications like environmental monitoring using moving sensors [10], [11], where the path density directly measures the distance moved by the sensor per unit area. This metric is also useful in designing k -space trajectories for Magnetic Resonance Imaging (MRI) [5], where the path density captures the total length of the trajectories per unit area in k -space which can be used as a proxy for the total scanning time per unit area in k -space.

In [8] and [9] we introduced the problem of designing sampling trajectories for bandlimited fields that are minimal in path density. We obtained conditions on unions of uniformly spaced straight line trajectories that admit perfect reconstruction of bandlimited fields. From this class of trajectories, we identified those with minimal path density. In this paper we extend our past work to arbitrary configurations of parallel line trajectories. We introduce the notion of trajectories that admit

stable sampling. We identify new designs of trajectories for fields in \mathbb{R}^d , $d \geq 3$ that are strictly better in path density than those identified in [9].

The paper is organised as follows. In Section II we describe the formal problem statement, in Section III we present our new results and we conclude with some discussion in Section IV. Below we introduce notations we use frequently in the paper.

Notation: We use $\langle \cdot, \cdot \rangle$ to denote the canonical inner product, and e_k to denote the unit vector along the k -th coordinate axis. For $u \in \mathbb{R}^d$ we denote the hyperplane orthogonal to u through the origin by $u^\perp = \{x \in \mathbb{R}^d : \langle x, u \rangle = 0\}$. For a set $S \subset \mathbb{R}^d$ we use $|S|$ to denote the volume of S relative to its affine hull, $\text{relint}(S)$ to denote the relative interior of S , $S(x)$ to denote its shifted version $S(x) = \{y + x : y \in S\}$, and $\mathcal{P}_{u^\perp} S$ to denote the orthogonal projection of S onto the hyperplane u^\perp . We use B_a^d and $B_a^d(x)$ for denoting spherical balls of radius a centered at the origin and $x \in \mathbb{R}^d$ respectively. For a discrete set Λ we use $\#(\Lambda)$ to denote its cardinality.

II. PROBLEM STATEMENT

A trajectory p_i in \mathbb{R}^d refers to a curve in \mathbb{R}^d . We represent a trajectory by a continuous function $p(\cdot)$ of a real variable taking values on \mathbb{R}^d :

$$p : \mathbb{R} \mapsto \mathbb{R}^d.$$

A trajectory set P is defined as a countable collection of trajectories:

$$P = \{p_i : i \in \mathbb{I}\} \quad (3)$$

where \mathbb{I} is a countable set of indices and for each $i \in \mathbb{I}$, p_i is a trajectory in the trajectory set P . For any given trajectory set P we denote its *path density* by $\ell(P)$ defined as follows:

$$\ell(P) := \limsup_{a \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}^d} \mathcal{D}^P(a, x)}{\text{Vol}_d(a)} \quad (4)$$

where $\mathcal{D}^P(a, x)$ represents the total arc-length of trajectories from P located within the ball $B_a^d(x)$ and $\text{Vol}_d(a)$ represents the volume of the d -dimensional ball. A simple example of a trajectory set in \mathbb{R}^2 is a doubly infinite sequence of equispaced parallel lines through \mathbb{R}^2 (e.g., see Figure 2(a)). We call such a trajectory set a *uniform set in \mathbb{R}^2* . Such a uniform set has a path density equal to $\frac{1}{\Delta}$ (see [9, Lem 2.2]) where Δ is the spacing between the lines. Similarly a *uniform set in \mathbb{R}^d* is defined as a collection of parallel lines in \mathbb{R}^d such that the cross-section forms a $(d-1)$ -dimensional lattice, as shown in Figure 2(b).

We say that a set of points $\Lambda \subset \mathbb{R}^d$ is *uniformly discrete* if we have $\inf\{\|x - y\| : x, y \in \Lambda, x \neq y\} > 0$, i.e., there exists $r > 0$ such that for any two distinct points $x, y \in \Lambda$ we have $\|x - y\| > r$.¹ We say that Λ forms a *set of stable sampling*

¹For example lattices in \mathbb{R}^d are uniformly discrete, but a sequence in \mathbb{R}^d converging to a point in \mathbb{R}^d is not.

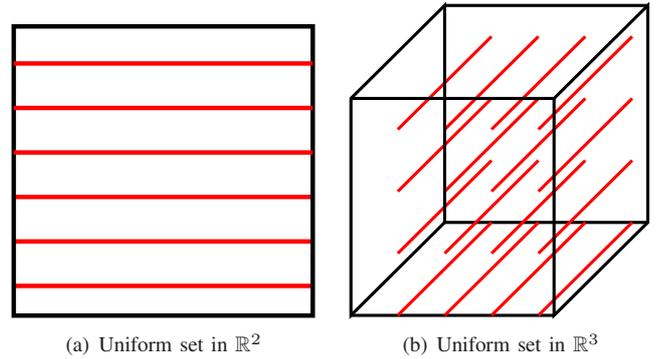


Fig. 2. Examples of uniform sets in \mathbb{R}^2 and \mathbb{R}^3 .

[4][3] for a set $\Omega \subset \mathbb{R}^d$ if there exists positive scalars A and B such that

$$A\|f\|^2 \leq \sum_{x \in \Lambda} |f(x)|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{B}_\Omega. \quad (5)$$

Further, let \mathcal{A}_Ω denote the collection of all uniformly discrete sets $\Lambda \subset \mathbb{R}^d$ that form sets of stable sampling for Ω . Classical sampling theory is primarily concerned with the elements of \mathcal{A}_Ω , e.g., Nyquist sampling lattices [1].

The following are some desirable properties of sampling trajectory sets.

Definition 2.1: A trajectory set P of the form (3) is called a *stable Nyquist trajectory set* for $\Omega \subset \mathbb{R}^d$ if it satisfies the following conditions:

- (C1) [Nyquist] There exists a uniformly discrete set Λ of points on the trajectories in P such that Λ forms a stable sampling set for \mathcal{B}_Ω , i.e., $\Lambda \subset \{p_i(t) : i \in \mathbb{I}, t \in \mathbb{R}\}$ and $\Lambda \in \mathcal{A}_\Omega$.
- (C2) [Non-degeneracy] For any $x \in \mathbb{R}^d$, there is a continuous curve of length no more than $\mathcal{D}^P(a, x) + o(a^d)$ that contains the portion of the trajectory set P that is located within $B_a^d(x)$.

Here condition (C2) is a regularity condition to ensure that the path density metric does indeed capture the total distance traveled per unit area by a single moving sensor using the trajectories in P . We also introduce a special notation for the collection of all stable Nyquist trajectory sets:

Definition 2.2: We use \mathcal{N}_Ω to denote the collection of all stable Nyquist trajectory sets for Ω , i.e., \mathcal{N}_Ω is the collection of all trajectory sets P of the form (3) that satisfy conditions (C1) and (C2).

Sampling theory for mobile sensing is primarily concerned with identifying trajectory sets in \mathcal{N}_Ω . The key optimization problem that we seek to solve is to identify trajectory sets in \mathcal{N}_Ω with minimal path density:

$$\min_{P \in \mathcal{N}_\Omega} \ell(P). \quad (6)$$

In [9] and [12] we identified various examples of trajectory sets in \mathcal{N}_Ω , and obtained partial solutions to (6) optimizing over

specific restricted classes of trajectories, such as uniform sets and unions of uniform sets. In this paper we present optimality results from broader classes of trajectory sets.

III. NEW OPTIMALITY RESULTS FOR PARALLEL LINES

Let P denote a trajectory set composed of parallel lines in \mathbb{R}^d . For any $x \in \mathbb{R}^d$ let $N_a^x(P)$ denote the number of lines in P that intersect the d -dimensional ball $B_a^d(x)$ of radius a centered at x . We restrict our attention to trajectory sets that are homogenous in the sense defined below.

Definition 3.1: We say that P is a *homogenous parallel set* if

$$\lim_{a \rightarrow \infty} \frac{N_a^x(P)}{|B_a^{d-1}|} \text{ exists and is equal for all } x \in \mathbb{R}^d.$$

Most practically useful parallel trajectory sets such as uniform sets, approximately uniform sets (e.g., with bounded offsets) and their finite unions are homogenous. For $\Omega \subset \mathbb{R}^d$ we use \mathcal{H}_Ω to denote homogenous parallel sets in \mathcal{N}_Ω . Below, we characterize the path density of homogenous parallel sets.

Lemma 3.1: Any homogenous parallel set P in \mathbb{R}^d satisfies

$$\ell(P) = \lim_{a \rightarrow \infty} \frac{N_a^0(P)}{|B_a^{d-1}|}. \quad \square \quad (7)$$

We provide a proof in the appendix. We now tackle (6) for trajectory sets in \mathcal{H}_Ω and compact convex symmetric sets Ω . We first establish a lower bound on the path density.

Proposition 3.2: Let $\Omega \subset \mathbb{R}^d$ be a compact convex set with non-empty interior. Assume further that Ω has a point of symmetry at the origin. Let $Q \in \mathcal{H}_\Omega$ be a trajectory set composed of lines parallel to $q \in \mathbb{R}^d$. Then $\ell(Q) \geq \frac{|\Omega \cap q^\perp|}{(2\pi)^{d-1}}$.

Proof: Assume without loss of generality that $q = e_1$, the unit vector along the first coordinate axis. Consider a field of the form $f(x) = \text{sinc}(\epsilon x_1)g(x_2, x_3, \dots, x_d)$ and g is bandlimited to a closed set Ω^g where $\Omega^g \subset \text{relint}(\Omega \cap q^\perp)$. For ϵ small enough, $f \in \mathcal{B}_\Omega$. For stably recovering f from samples on Q , the non-uniform collection of points at which the lines in Q intersect the hyperplane e_1^\perp must form a set of stable sampling for Ω^g . We know from Landau's result [2] (see also [4, Cor. 1]) that the sampling density of such a set must necessarily be greater than or equal to $\frac{|\Omega^g|}{(2\pi)^{d-1}}$.

Thus, by Lemma 3.1 it follows that $\ell(Q) \geq \frac{|\Omega^g|}{(2\pi)^{d-1}}$ for all $\Omega^g \subset \text{relint}(\Omega \cap q^\perp)$. Hence $\ell(Q) \geq \frac{|\Omega \cap q^\perp|}{(2\pi)^{d-1}}$. \blacksquare

Although the result of Proposition 3.2 only provides a lower-bound on the path density, we believe that the techniques used in [13] can be used to construct trajectory sets in \mathcal{H}_Ω that achieve arbitrarily close to this bound for convex and symmetric Ω . However, in this paper, we only establish the following achievability result, which is tight for some specific choices of Ω as we discuss below.

Proposition 3.3: Let $\Omega \subset \mathbb{R}^d$ be a compact convex set with non-empty interior and a point of symmetry at the origin. Let $\mathcal{S}(\Omega)$ denote the volume of the smallest projection of Ω onto a hyperplane defined as

$$\mathcal{S}(\Omega) := \min_{u \in \mathbb{R}^d: \|u\|=1} |\mathcal{P}_{u^\perp} \Omega|. \quad (8)$$

Let u^* be the minimizer in (8). Then for any $\epsilon > 0$ there exists $P \in \mathcal{H}_\Omega$ such that the lines in P are parallel to u^* and

$$\ell(P) \leq \frac{\mathcal{S}(\Omega)}{(2\pi)^{d-1}} + \epsilon.$$

Sketch of proof: We do not provide a complete proof due to lack of space. The optimal trajectory set is obtained by choosing the lines in P parallel to u^* such that their points of intersection with $(u^*)^\perp$ approximates an optimal set of stable sampling for $\mathcal{P}_{(u^*)^\perp} \Omega$. Such an optimal set can be designed using the results of [13, Cor 4.5]. In this case, the path density of this trajectory set matches the sampling density of the optimal set of sampling which is equal to $|\mathcal{P}_{(u^*)^\perp} \Omega| + \epsilon$. \blacksquare

The following corollary is immediate from the above two results.

Corollary 3.3.1: Let $\Omega \subset \mathbb{R}^d$ be a compact convex set with non-empty interior and a point of symmetry at the origin. Suppose that Ω satisfies the condition

$$\min_{u \in \mathbb{R}^d: \|u\|=1} |\Omega \cap u^\perp| = \mathcal{S}(\Omega). \quad (9)$$

Then

$$\inf_{Q \in \mathcal{H}_\Omega} \ell(Q) = \frac{\mathcal{S}(\Omega)}{(2\pi)^{d-1}}. \quad \square \quad (10)$$

In words, condition (9) is the requirement that the volume of the smallest section of Ω through the origin is equal to the volume of the smallest projection of Ω onto a hyperplane. This condition holds in the following practically relevant cases:

- $\Omega \subset \mathbb{R}^2$ such that Ω is convex and compact [14, Thm 12.18].
- $\Omega \subset \mathbb{R}^d$ such that Ω is a spherical ball (obvious), or an n -cube [15], or an ellipsoid (can be shown).

However, this condition does not hold in general, a simple counter-example being the regular octahedron in \mathbb{R}^3 : $\Omega = \{\omega \in \mathbb{R}^3 : \|\omega\|_1 \leq 1\}$. Nevertheless for Ω 's that satisfy condition (9), the trajectory set of Proposition 3.3 gives the optimal configuration of parallel lines for sampling fields in \mathcal{B}_Ω . In particular, when Ω is a spherical ball in \mathbb{R}^d , the trajectory set of Proposition 3.3 gives the optimal configuration of parallel lines for sampling isotropic fields in \mathbb{R}^d . Similarly, for convex and compact sets $\Omega \subset \mathbb{R}^2$, we showed in [9] that the optimum configuration of parallel lines given by Proposition 3.3 is a uniform set in \mathcal{H}_Ω . For general \mathbb{R}^d , the result of Proposition 3.3 gives the best known solution to the minimum path density problem of (6). In Section IV we discuss the possibility of extending this result to all of \mathcal{N}_Ω .

IV. DISCUSSION

This work opens up several possible research directions. An obvious question is to check if under the conditions of Proposition 3.3 it is possible to design a trajectory set in \mathcal{H}_Ω that achieves a path density arbitrarily close to the lower bound. Another direction of interest is to extend the results on parallel lines obtained in this paper to parallel sampling manifolds of higher dimensions, like those considered in [9].

Although we have obtained various optimality results on parallel line trajectories in this paper, our original task of identifying minimal length trajectories for sampling spatial bandlimited fields still remains open. A first case to analyze is the necessary condition on a trajectory set in \mathcal{N}_Ω composed of arbitrary (not necessarily parallel) straight lines. A generalization of the notion of Fourier frames [4] [5] may be a possible approach towards such a result.

A different question of interest is to examine the definition of \mathcal{N}_Ω . In the current version of this work, while defining the set \mathcal{N}_Ω we have placed the restriction that a sampling trajectory set in \mathcal{N}_Ω must contain a uniformly discrete set of points that form a set of stable sampling for Ω . In addition we have the requirement of Condition (C2). Nevertheless, it has recently come to our knowledge that under this definition of \mathcal{N}_Ω it is possible to design sampling trajectories in \mathcal{N}_Ω that have arbitrarily small path density. However, this leads to the stability ratio $\frac{B}{A}$ of parameters A and B in the definition of (5) to be arbitrarily high. It is of interest to examine whether a constraint on the ratio $\frac{B}{A}$ can be incorporated in the definition of \mathcal{N}_Ω to obtain a non-trivial lower bound on the path density of all trajectory sets in \mathcal{N}_Ω . However, it is to be noted that if we restrict ourselves to trajectory sets in \mathcal{H}_Ω , then the problem is still well-posed as evidenced by Proposition 3.2. It would be of interest to examine whether such a non-trivial lower bound on the path density continues to hold if we expand \mathcal{H}_Ω to all trajectory sets composed of straight lines.

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APPENDIX

A. Proof of Lemma 3.1

For simplicity, we prove the result only for $d = 2$, since the same proof idea works for higher dimensions. Without loss of generality assume that the lines in P are parallel to e_2 . Since the lines are homogenous we just need to evaluate (4) when x is the origin. We number the lines in P such that for each $i \in \mathbb{Z}^+(\mathbb{Z}^-)$, $\ell_{a,i}$ denotes the length of the portion of the i -th line to the right (left) of the origin that is contained within a disc of radius a centered at the origin. Without affecting the value of the computation we assume that the line indexed by 0 passes through the origin. Let $d_i = \sum_{j=0}^i \Delta_j$ where Δ_j denotes the spacing between lines indexed by j and $j + 1$. Now let $I_{a,f} = \{i \in \mathbb{Z} : f\epsilon a \leq d_i < (f + 1)\epsilon a\}$ for $-\frac{1}{\epsilon} \leq f \leq \frac{1}{\epsilon}$. Let $L_{a,f} = \sum_{i \in I_{a,f}} \ell_{a,i}$ and $N_{a,f} = \#(I_{a,f})$. Clearly $\lim_{a \rightarrow \infty} \frac{N_{a,f}}{a\epsilon} = \rho$ where ρ is the right hand side expression in (7). Further, for $f \in [0, \frac{1}{\epsilon}]$,

$$2a(1 - (f + 1)^2\epsilon^2)^{\frac{1}{2}}N_{a,f} \leq L_{a,f} \leq 2a(1 - f^2\epsilon^2)^{\frac{1}{2}}N_{a,f}.$$

Hence

$$\frac{2\epsilon\rho}{\pi}(1 - (f + 1)^2\epsilon^2)^{\frac{1}{2}} \leq \lim_{a \rightarrow \infty} \frac{L_{a,f}}{\pi a^2} \leq \frac{2\epsilon\rho}{\pi}(1 - f^2\epsilon^2)^{\frac{1}{2}}.$$

For $f < 0$ the above relation holds with the signs reversed. Thus we see that $\sum_{f=0}^{\frac{1}{\epsilon}} \lim_{a \rightarrow \infty} \frac{L_{a,f}}{\pi a^2}$ is bounded between the right hand and left hand Riemann sums that approximate the Riemann integral $\int_0^1 \frac{2\rho}{\pi}(1 - x^2)^{\frac{1}{2}} dx$. Since this holds for all ϵ it follows that as we let $\epsilon \rightarrow 0$, we get $\lim_{a \rightarrow \infty} \frac{\sum_{i \in \mathbb{Z}^+} \ell_{a,i}}{\pi a^2} = \int_0^1 \frac{2\rho}{\pi}(1 - x^2)^{\frac{1}{2}} dx$. Following the same steps for negative indices and combining, we get

$$\lim_{a \rightarrow \infty} \frac{\sum_{i \in \mathbb{Z}} \ell_{a,i}}{\pi a^2} = \int_{-1}^1 \frac{2\rho}{\pi}(1 - x^2)^{\frac{1}{2}} dx = \rho.$$

□

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