Joint Signal Sampling and Detection

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***Abstract*—In this paper, we examine the joint signal sampling and detection problem when noisy samples of a signal are collected in the sequential fashion. In such a scheme, at the each observation time point we wish to make a decision that the observed data record represents a signal of the assumed target form. Moreover, we are able simultaneously to recover a signal when it departs from the target class. For such a joint signal detection and recovery setup, we introduce a novel algorithm relying on the smooth correction of linear sampling schemes. Given a finite frame of noisy samples of the signal we design a detector being able to test a departure from a target signal as quickly as possible. Our detector is represented as a continuous time normalized partial-sum stochastic process, for which we obtain a functional central limit theorem under weak assumptions on the correlation structure of the noise. The established limit theorems allow us to design monitoring algorithms with the desirable level of the probability of false alarm and able to detect a change with probability approaching one.**

***Index Terms*—joint sampling-detection, parametric signals, nonparametric alternatives**

I. Introduction

The problem of reconstructing an analog signal from its discrete samples plays a critical role in the modern technology of digital data transmission and storage. In fact, the theory of signal sampling and recovery has attracted a great deal of research activities lately, see [8], [9] and the references cited therein. In particular, the problem of signal sampling and recovery from imperfect data has been addressed in a number of recent works [5], [1], [2], [6]. The efficiency of sampling schemes depends strongly on the *a priori* knowledge of an assumed class of signals. For a class of bandlimited signals the signal sampling and recovery theory builds upon the celebrated Whittaker-Shannon interpolation scheme. On the other hand, there exists a class of nonbandlimited signals which can be recovered using the frequency rate below the Nyquist threshold. This is possible since this class is completely specified by a finite dimensional parameter. This parametric class of functions is often referred to as *finite rate innovation* signals [4], [3]. In practice, when only random samples are available it is difficult to verify whether a signal is bandlimited, parametric or belongs to some general function space. This calls for a joint nonparametric detection-reconstruction scheme to verify a type of the signal and simultaneously able to recover it. In fact, the problem of automatic rapid detection of signals differing from a reference (target) signal is important in many fields of signal processing and communication, e.g., in the analysis of radar signals and synchronization procedures the joint detection and reconstruction provides the basis to design effective receivers. The additional difficulty of designing detection/reconstruction procedures comes from the fact that samples are inherently noisy and observed sequentially within a fixed time frame. Hence, at the current frame we have a noisy data set $\{y_i : i \leq k\}$, and a detector should be applied immediately when a new observation y_{k+1} is available to the system. Hence, suppose we are given noisy measurements

$$y_k = f(k\tau) + \epsilon_k, \tag{1}$$

where τ is the sampling period, $\{\epsilon_k\}$ is a zero mean noise process, and $f(\bullet)$ is an unspecified signal which belongs to some signal space. In this paper we are interested in the following on-line detection problem. We are given a reference (target) parametric class of signals $\mathcal{S} = \{f(t;\theta) : \theta \in \Theta\}$, where Θ is a subset of a finite dimensional space, and wish to test the null hypothesis $H_0 : f \in \mathcal{S}$ against an arbitrary alternative $H_1 : f \notin \mathcal{S}$. Throughout the paper, we assume that the signal $f(t)$ of interest is observed over a finite time frame, i.e., $t \in [0, T]$, for some $0 < T < \infty$. Indeed, in practice we can only process a part of the signal which can be otherwise defined over an arbitrary interval. As a result, we are interested in methods relying on a finite data set $\{y_k : k = 0, \ldots, n\}$ obtained from model (1). Concerning the noise process in (1), we admit a wide class of correlated error processes. Our assumption is nonparametric and specifies a certain asymptotic behavior of the noise process. Specifically we assume that $\{\epsilon_k\}$ satisfies the so-called *invariance principle* or *functional central limit theorem* also often referred to as the Donsker's property, see [10] for further details. Hence, the condition on the error process employed in this paper is as follows.

Assumption 1 *Let $\{\epsilon_k\}$ be a weakly stationary stochastic process with zero mean which satisfies a functional central limit theorem, i.e.,*

$$n^{-1/2} \sum_{k=0}^{\lfloor ns \rfloor} \epsilon_k \Rightarrow \sqrt{\eta} B(s),$$

as $n \to \infty$, for some finite constant η.

Here $B(t)$ denotes a standard Brownian motion and $\Rightarrow$ stands for the convergence in distribution. Also $\lfloor x \rfloor$ denotes the greatest integer less or equal to x. It is worth mentioning that the validity of the functional central limit theorem, i.e., of Assumption 1 is not limited to the $i.i.d.$ case but also holds for many dependent stationary processes with summable auto-covariances. For instance, it holds for linear processes and mixing processes [10]. The dependence structure of the measured data is controlled by the parameter η appearing in Assumption 1. This parameter is identified with the limit $\lim_{n\to\infty} Var(n^{-1/2}\sum_{k=0}^{n}\epsilon_k)$ and is often referred to as the time-average (long-run) variance of $\{\epsilon_k\}$.

Our test statistic builds upon the signal recovery methods developed in [5], [6], where it has been proved that they possess the consistency property, i.e., they are able to converge to a large class of signals not necessarily being bandlimited. A generic form of such estimates is given by

$$\hat{f}_n(t) = \tau \sum_{j=0}^{n} y_j \mathcal{K}_\Omega(j\tau, t), \tag{2}$$

where $\mathcal{K}_\Omega(u,t)$ is the reconstruction kernel parameterized by the parameter Ω. For the consistency results the parameter Ω and the sampling period τ should depend on the data size n and be selected appropriately. In fact, we need that $\Omega \to \infty$ and $\tau \to 0$ as $n \to \infty$ with the controlled rate. For example, the choice $\Omega = n^{1/3}$ and $\tau = n^{-1}$ would be sufficient to assure the consistency for a wide nonparametric class of signals defined on the a finite interval. Our detection algorithm uses the data observed over the interval $[0,T]$ and therefore we select $\tau = T/n$ and fix Ω to some large number. The kernel $\mathcal{K}_\Omega(u,t) = \sin(\Omega(t-u))/\pi(t-u)$ is particularly important since it is the reproducing kernel for bandlimited signals with the bandwidth Ω. For a broader class of signals we can use generalized kernels $\mathcal{K}_\Omega(u,t) = \sum_k \phi(\Omega(u-k))\psi(\Omega(t-k))$, where $\phi(t)$, $\psi(t)$ can be specified as biorthogonal functions [8].

II. Reconstruction and Detection Algorithms

Our detection technique is relying on the consistent reconstruction method defined in (2). Our asymptotic results assume $n \to \infty$ but we will also provide useful approximations for finite n. Note that n can be regarded as the planed maximum number of observations in the time interval $[0,T]$. In this paper we address the following question: how long do we have to sample the signal, until the available data provide enough evidence to reject the null hypothesis $H_0 : f \in \mathcal{S}$? A specific example of the null hypothesis class $\mathcal{S}$ is a class of signals that are a superposition of shifted versions of a known pulse $h(t)$, i.e., $f(t;\theta) = \sum_{k=1}^{L} a_k h(t-t_k)$. Here θ is a $2L$ dimensional vector of unknown parameters.

Our goal in this paper is to decide whether the null hypothesis is true or not, given a sequentially observed data set drawn from the observation model (1), where $f(t)$ is an unknown signal from a large nonparametric function space. Hence, if the alternative signal is unknown, we propose a detector which can be computed without specifying this signal. We will use a sequential version of the nonparametric estimator (2), which automatically adapts to the unknown alternative signal as sampling proceeds. Specifically, we use $\hat{f}_n(t)$ as a basic building block of our detection method, i.e., we stop our detection process at the first time point $t = k\tau$ if a certain distance measure between $\hat{f}_k(t)$ and the target parametric signal $f(t;\theta_0)$ from $\mathcal{S}$ is is too large. Here θ_0 denotes the "true" parameter if the null hypothesis holds. Since the parameter θ_0 is unknown we replace it in our test statistic with its consistent estimate $\hat{\theta}$, see [7] for an extensive overview of estimation algorithms and their performance for specific classes of parametric signals. In [3] the estimation problem associated with the class of finite rate innovation signals has been also examined. To define our detection scheme, let us introduce the following sequential partial sum process, which represents the sequence of the estimators as a step function

$$\mathcal{F}_n(s,t) = \sqrt{\tau} \sum_{0\le l \le \lfloor ns \rfloor} [y_l - f(l\tau;\hat{\theta})]\mathcal{K}_\Omega(l\tau, t), \tag{3}$$

for $0 < s_0 \le s \le 1,\;\; t \in [0,T]$. The condition $s_0 \le s$ ensures that at least the first $n_0 = \lfloor ns_0 \rfloor$ observations are used ensuring a certain degree of precision in the reconstruction. This allows us in our asymptotic analysis to replace $\hat{\theta}$ in (3) by θ_0. Then, for $s = k/n$ the value $\mathcal{F}_n(k/n, t)$ can be interpreted as the deviation of $\tau^{-1/2}(\hat{f}_k(t) - E_0\hat{f}_k(t))$, where throughout the paper E_0 and P_0 denote that the expectation and probability are taken under the null hypothesis, i.e., that $f(t) = f(t;\theta_0)$. The interpretation of $\mathcal{F}_n(s,t)$ as a function of one variable is as follows:

- For fixed t the step function $s \mapsto \mathcal{F}_n(s,t)$ describes the sequence of deviations of $\hat{f}_{\lfloor ns \rfloor}(t)$ from $f(t;\hat{\theta})$ as sampling proceeds.
- For fixed s the function $t \mapsto \mathcal{F}_n(s,t)$ is the current estimate of the whole signal, using $\lfloor ns \rfloor$ sampled values.

The sequential nonparametric decision problem for rejecting the hypothesis $H_0 : f \in \mathcal{S}$ can now be handled by the following detector statistics. A *global maximum detector* is defined as follows

$$M_n = \min\left\{ n_0 \le k \le n : \max_{0\le t \le Tk/n} |\mathcal{F}_n(k/n,t)| > c_M \right\}$$

for some appropriately chosen control limit c_M. The detector M_n looks at the largest absolute value of the deviation process. Notice that when calculating the maximum at a candidate time point Tk/n, the maximum is determined for time points t between 0 and Tk/n. That interval corresponds to the time frame where observations are present. For $t > Tk/n$ the estimator $\hat{f}_k(t)$ can be considered as an extrapolation scheme. Alternatively, one can consider a *global integrated detector*

$$I_n = \min\left\{ n_0 \le k \le n : \int_0^{Tk/n} |\mathcal{F}_n(k/n,t)|^2\, dt > c_I \right\}$$

for some appropriately chosen control limit c_I. Without loss of generality, however, we confine our investigation to the detector M_n, which is easy to calculate and interpret. In order to assess the statistical accuracy of the detector M_n we need to establish the limiting distribution of the process $\mathcal{F}_n(s,t)$. This is shown in the next section.

III. Limit Distributions

The statistical accuracy of the aforementioned detection scheme M_n depends critically on the the choice of the threshold parameter c_M. The asymptotic choice of this parameter can be obtained from the limiting distribution of $\mathcal{F}_n(s,t)$. Below we establish that the limiting distribution is a locally stationary Gaussian process $\mathcal{F}(s,t)$ with mean 0 and a certain covariance function.

Theorem 1: Suppose the noise process $\{\epsilon_k\}$ meets Assumption 1. Then under the hypothesis H_0 we have

$$\mathcal{F}_n(s,t) \Rightarrow \mathcal{F}(s,t), \qquad n \to \infty,$$

where the limit stochastic process, $\mathcal{F}(s,t)$ is given by

$$\mathcal{F}(s,t) = \sqrt{T\eta}\int_0^s \mathcal{K}_\Omega(Tz,t)dB(z).$$

As a result, the process $\mathcal{F}(s,t)$ is a locally stationary Gaussian process with the following covariance function

$$cov(\mathcal{F}(s_1,t_1),\mathcal{F}(s_2,t_2))$$
$$= T\eta\int_0^{\min(s_1,s_2)} \mathcal{K}_\Omega(Tz,t_1)\mathcal{K}_\Omega(Tz,t_2)dz.$$

The smoothness of the sample paths of the Gaussian process $\mathcal{F}(s,t)$ is determined by smoothness of its variance, i.e., the function $T\eta\int_0^s \mathcal{K}_\Omega^2(Tz,t)dz$. The above result allows us to establish the limit of our detector statistic. Under the conditions of Theorem 1 the following central limit theorem also holds true.

$$M_n/n \Rightarrow \mathcal{M} = \inf\{s \in [s_0,1] : \sup_{0\le t\le sT} |\mathcal{F}(s,t)| > c_M\}.$$

These results allow us to specify the control limit c_M in such a way that the probability of a false alarm in the time frame $[s_0,1]$ is not greater than $\alpha < 1$. For our detector M_n one can proceed as follows. The detection error (under the hypothesis H_0) occurs if $M_n/n < 1$ and $P_0(M_n/n < 1) \to P(\mathcal{M} < 1)$ by the aforementioned result. Since the event $\{\mathcal{M} > z\}$ is equivalent to following one

$$\{\sup_{s_0\le s\le z}\sup_{0\le t\le sT} |\mathcal{F}(s,t)| \le c_M\}, \tag{4}$$

we can obtain a procedure for selecting c_M with an asymptotic detection error being equal to α. In fact, we choose c_M as the $1-\alpha$ quantile of the distribution of the complement of the event in (4) with $z = 1$, i.e., the constant c_M is found as the smallest c being the solution of the following inequality

$$P\left(\sup_{s\in[s_0,1]}\sup_{t\in[0,sT]} |\mathcal{F}(s,t)| > c\right) \le \alpha, \tag{5}$$

where the probability is taken with respect to the extrema of the absolute value of the Gaussian process $\mathcal{F}(s,t)$.

The question arises how the above results can be applied in practice. The distribution of the random variable $X = \sup_{s_0\le s\le 1}\sup_{0\le t\le sT} |\mathcal{F}(s,t)|$ required to evaluate the false alarm error can be simulated by Monte Carlo methods using the following algorithm.

1) Generate trajectories of the Gaussian process $\mathcal{F}(s,t)$ on a grid $\{(s_i,t_j) : i = 1,\dots,N, j = 1,\dots,N\}$ where $0 \le s_1 < \cdots < s_N \le 1$ and $0 \le t_1 < \cdots < t_N \le T$.
2) Return X by calculating the maximum of the values $|\mathcal{F}(s_i,t_j)|$ for all (i,j) such that the constraints $s_0 \le s_i \le 1$ and $0 \le t_j \le s_i T$ are satisfied.
3) Repetitions of Step 1 and Step 2 produce realizations of X that can be utilized for estimating $c_M(\alpha)$.

Simulating the process $\mathcal{F}(s,t)$ in Step 1 is feasible, since the covariance function can be evaluated numerically provided that T and η are known. The choice of η is critical for the accuracy of our detectors. We wish to estimate η without assuming which hypothesis holds, i.e., to estimate η using only the available data $\{y_0,\dots,y_k\}$ without the knowledge of the signal shape. Here we can utilize the discrepancies of local means. One of such estimates takes the form

$$\widetilde{\eta}_k = \frac{b_k}{2(L-1)}\sum_{j=1}^{L-1}(A_j - A_{j-1})^2, \tag{6}$$

where $A_j = \sum_{l=jb_k}^{jb_k+b_k-1} y_l/b_k$ is the local mean, $j = 0,1,\dots,L$ and $L+1 = \lfloor (k+1)/b_k \rfloor$ denotes the number of data groups. It can be demonstrated [11] that this estimate can converge to the true η with the rate $O_P(k^{-1/3})$ with virtually no assumptions on the form of the underlying signals.

Having established the asymptotic distributions under the null hypothesis, it remains to see how our detection method behaves when $f \notin \mathcal{S}$, i.e., when the true signal differs from the target parametric signal. We can consider a class of local alternatives for modeling this situation, i.e., let

$$f(t) = f(t;\theta_0) + a_n g(t), \tag{7}$$

where a_n is the sequence tending to zero as $n \to \infty$ and $g(t)$ is a fixed function assumed to be piecewise continuous and bounded. Under this condition and Assumption 1 we can show that under the alternative local hypothesis and the choice $a_n = n^{-1/2}$ the process $\mathcal{F}_n(s,t)$ has the following limit

$$\mathcal{F}^A(s,t) = \mathcal{F}(s,t) + T^{-1/2}\int_0^{sT} \mathcal{K}_\Omega(z,t)g(z)dz, \tag{8}$$

where $\mathcal{F}(s,t)$ is the locally stationary Gaussian process found in Theorem 1. It is worth noting that if the departure from the reference signal $f(t;\theta_0)$ in the local alternative in (7) is of order $a_n = O(n^{-\beta})$, for $\beta > 1/2$, then there is no visible effect on the asymptotic distribution, i.e., $\mathcal{F}_n(s,t) \Rightarrow \mathcal{F}(s,t)$. Thus, even in large samples there is no chance to detect such small departures from the target signal. The rate $\beta = 1/2$ is the right order for getting a non-trivial limit distribution. The result in (8) allows us to evaluate the power $\mathcal{P}_n = P_1(TM_n/n < T)$ of our detector. In fact, the limit in (8) yields

$$\lim_{n\to\infty} \mathcal{P}_n = P\left(\sup_{s_0\le s\le 1}\sup_{0\le t\le sT} |\mathcal{F}^A(s,t)| > c_M\right). \tag{9}$$

This holds for any c_M but the proper value of c_M can be obtained by satisfying the bound for the probability of false alarm in (5). In practise, the probability in (9) can be evaluated by the aforementioned Monte Carlo algorithm.

τ	0.01	0.015	0.02	0.025	0.03
r_τ	0.0924	0.0640	0.0571	0.0480	0.0432

TABLE I
SIMULATED REJECTION RATE FOR VARIOUS SAMPLING INTERVALS τ

IV. SIMULATION STUDIES

In our simulation studies we will focus on the issues related to the choice of the proper control limit and the resulting detector rejection rate and power. This is studied in the context of the length of the sampling interval τ and the problem of the influence that selection of the filter bandwidth Ω has on the detector power. We assume that the target signal is $f_0(t) = \sin(4t)$ on $[0, 2]$. This signal undergoes the jump-point distortion to produce the alternative signal $f_1(t) = f_0(t) + 0.2\mathbf{1}(t \geq 1)$. Taking into account the global maximum norm detector M_n we follow the proposed Monte Carlo algorithm to estimate the proper control limit c_M being the sample 95%-quantile of 50000 simulation replicates. Our base reconstruction algorithm is the post-filtering method [5] utilizing the kernel function $\mathcal{K}_\Omega(u, t) = \sin(\Omega(t-u))/\pi(t-u)$, where Ω is the bandwidth of a low-pass filter. To study the influence of the sampling interval τ on c_M, we applied the above procedure with $s_0 = 0.1$, $\sqrt{\eta} = 0.2$, $\Omega = 10$, $n = 100$. The true rejection rate (the probability of rejection under the null hypothesis) denoted by r_τ was estimated by a Monte Carlo simulation with 50000 repetitions for each given $\tau \in \{0.01, \ldots, 0.03\}$. Since $n\tau = T$ therefore this corresponds to the design intervals ranging from $[0, 1]$ to $[0, 3]$. Note that the fixed value $\tau = 0.02$ was used in the illustrative example. Table I provides the results. It can be seen that there is some influence of the sampling interval on the accuracy of the approximation, but it is still moderate for a rather large range of values of τ. There is an evident drop in the value of the rejection rate for τ larger than 0.01 corresponding to design intervals larger than $[0, 1]$.

Next, we studied the influence of the filter bandwidth Ω of our reconstruction algorithm $\hat{f}_n(t)$ on the detection power (defined in (9)) using the corrected control limit. The parameter η was estimated by the method mentioned in Section III. We employed the fixed alternative $f_1(t) = f_0(t) + 0.1\sin\left(8(t-1) + \frac{\pi}{2}\right), t \in [0, 2]$. This alternative is characterized by the frequency and phase deformation, although the difference between $f_0(t)$ and $f_1(t)$ is small. The results (shown in Figure 1) indicate that there is an optimal value $\Omega^* \in [8, 12]$ that maximizes the detector power. The value of Ω^* is about 10.5 for n ranging from 500 to 1000. The corresponding power for the optimal values of Ω is above 95% ($n = 750$) and 99% ($n = 1000$). This is a quite remarkable fact noting that the L_2 norm of $f_1(t) - f_0(t)$ is as small as 0.0098.

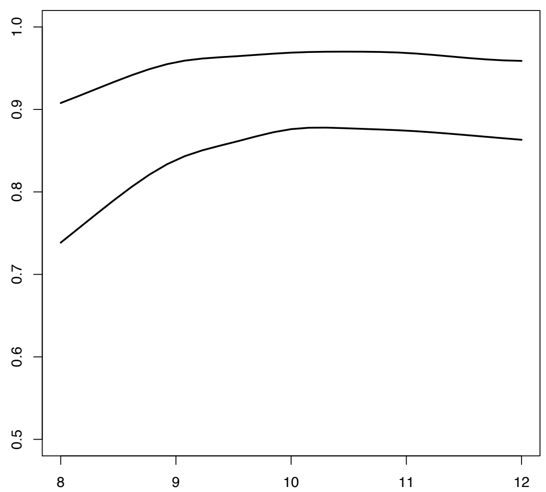

Fig. 1. Simulated power in (9) to detect a change in frequency and phase as a function of $\Omega \in [8, 12]$ for the sample sizes $n = 750$ (bottom curve) and $n = 1000$.

V. CONCLUDING REMARKS

We investigated a new joint sampling-detection procedure for testing the parametric form of a signal observed in the presence of correlated noise. Our detection methods are based on sequentially applied reconstruction algorithms which are related to linear sampling schemes. The asymptotic distribution of our detectors is established via functional central limit theorems and Donsker's invariance principle. This allows us to evaluate the probability of false alarm and the corresponding control limit. The asymptotic performance under local alternatives is also examined.

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