

Fourier-Laguerre transform, Convolution and Wavelets on the Ball

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Abstract—We review the Fourier-Laguerre transform, an alternative harmonic analysis on the three-dimensional ball to the usual Fourier-Bessel transform. The Fourier-Laguerre transform exhibits an exact quadrature rule and thus leads to a sampling theorem on the ball. We study the definition of convolution on the ball in this context, showing explicitly how translation on the radial line may be viewed as convolution with a shifted Dirac delta function. We review the exact Fourier-Laguerre wavelet transform on the ball, coined *flaglets*, and show that flaglets constitute a tight frame.

Index Terms—Harmonic analysis, sampling, wavelets, three-dimensional ball.

I. INTRODUCTION

Data often live naturally on the three-dimensional ball. For example, in cosmology the distribution of galaxies that traces the large-scale structure of the Universe is observed on the celestial sphere (e.g. [1]), augmented with depth information given by redshift. A spherical shell at a given redshift represents a given epoch in the history of our Universe; thus, such data live naturally on the three-dimensional ball (hereafter referred to as simply the *ball*).

One would like to analyse such data-sets on the ball to study the physics responsible for them. Since many physical processes are manifest on different physical scales, while also spatially localised, wavelet analysis is a power method for this purpose. Recently, two wavelet transforms have been derived on the ball [4], [5]. The former [4] is based on an undecimated wavelet construction, built on the Fourier-Bessel transform. The latter [5] is based on a tiling of harmonic space, built on a Fourier-Laguerre transform, and developed by the authors of the current article. Our approach [5]: (i) yields wavelets that are not isotropic but rather exhibit an angular opening that is invariant under radial translation; (ii) is theoretically exact; and (iii) leads to a fast multiresolution algorithm.

In this article we review our recent work [5] where we consider the Fourier-Laguerre transform and construct wavelets (which we coin *flaglets*) on the ball. Furthermore, we illuminate the translation operator on the radial line, showing how this may be viewed as convolution with a shifted Dirac delta function. We also show that flaglets constitute a tight frame.

II. FOURIER-LAGUERRE TRANSFORM

The canonical harmonic transform on the ball is the Fourier-Bessel transform, where the basis functions are the eigenfunctions of the Laplacian on the ball. The Fourier-Bessel basis functions separate into the usual spherical harmonic functions

on the sphere and the spherical Bessel functions on the radial line. However, the Fourier-Bessel transform suffers from a serious shortcoming. To the best of our knowledge there does not exist a sampling theorem for the Fourier-Bessel transform, since there does not exist an exact quadrature rule for the evaluation of the spherical Bessel transform (the radial part of the Fourier-Bessel transform).

To overcome this limitation we consider the Fourier-Laguerre transform, for which we developed a sampling theorem [5]. The Fourier-Laguerre transform follows by adopting the Laguerre polynomials (the standard orthogonal polynomials on \mathbb{R}^+) as the radial basis functions, while keeping the spherical harmonics as the spherical basis functions. We define the Fourier-Laguerre basis functions on the ball $\mathbb{B}^3 = \mathbb{R}^+ \times \mathbb{S}^2$ by

$$Z_{\ell mp}(\mathbf{r}) = K_p(r)Y_{\ell m}(\theta, \varphi), \quad (1)$$

with spherical coordinates $\mathbf{r} = (r, \theta, \varphi) \in \mathbb{B}^3$, where $r \in \mathbb{R}^+$ denotes radius, $\theta \in [0, \pi]$ colatitude and $\varphi \in [0, 2\pi)$ longitude, and where $\ell, p \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ such that $|m| \leq \ell$. The standard spherical harmonics are denoted by $Y_{\ell m}$ and the normalised spherical Laguerre basis functions are defined on the radial line by

$$K_p(r) \equiv \sqrt{\frac{p!}{(p+2)!}} \frac{e^{-r/2\tau}}{\sqrt{\tau^3}} L_p^{(2)}(r/\tau), \quad (2)$$

where $L_p^{(2)}$ is the p -th generalised Laguerre polynomial of order two and $\tau \in \mathbb{R}^+$ is a radial scale factor.

A square-integrable signal $f \in L^2(\mathbb{B}^3)$ can then be decomposed as

$$f(\mathbf{r}) = \sum_{p=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell mp} Z_{\ell mp}(\mathbf{r}), \quad (3)$$

where the harmonic coefficients are given by the usual projection

$$f_{\ell mp} = \langle f | Z_{\ell mp} \rangle_{\mathbb{B}^3} = \int_{\mathbb{B}^3} d^3\mathbf{r} f(\mathbf{r}) Z_{\ell mp}^*(\mathbf{r}), \quad (4)$$

where $d^3\mathbf{r} = r^2 \sin \theta dr d\theta d\varphi$ is the usual rotation invariant measure in spherical coordinates.¹ We consider band-limited

¹This measure is a natural choice since it allows the Fourier-Laguerre transform to be related directly to the Fourier-Bessel transform, such that the Fourier-Bessel coefficients can be computed exactly from Fourier-Laguerre coefficients (see [5] for further details).

signals, with angular and radial band-limits L and P , respectively, *i.e.* signals f such that $f_{\ell mp} = 0, \forall \ell \geq L, \forall p \geq P$. In this case the summations in Eqn. (3) over ℓ and p may be truncated to $L - 1$ and $P - 1$ respectively.

In practice, computing the Fourier-Laguerre transform involves the evaluation of the integral of Eqn. (4). An exact quadrature rule for the evaluation of this integral for a band-limited function f naturally gives rise to a sampling theorem. Since the Fourier-Laguerre transform is separable in angular and radial coordinates, we may appeal to separate sampling theorems on the sphere and radial line. For the angular part, we adopt the equiangular sampling theorem on the sphere developed recently by one of the authors [7]. Other sampling theorems on the sphere could alternatively be adopted (*e.g.* [2]), however we select the sampling theorem developed by [7] since it leads to the most efficient sampling of the sphere (*i.e.* the fewest number of samples to represent a band-limited signal exactly). For the radial part, we appeal to Gaussian quadrature to develop an exact quadrature rule and, consequently, a sampling theorem [5]. Combining these results we recover a sampling theorem and, equivalently, an exact Fourier-Laguerre transform on \mathbb{B}^3 . For a band-limited signal all of the information content of the signal is captured in $N = P[(2L - 1)(L - 1) + 1] \sim 2PL^2$ samples on the ball [5].

We have developed the public FLAG² code [5] to compute the Fourier-Laguerre transform. The FLAG code computes exact forward and inverse Fourier-Laguerre transforms at machine precision and is stable to extremely large band-limits, relying on the public SSHT³ code [7] developed by one of the authors for the angular part, which in turn relies on FFTW⁴. FLAG supports both the C and Matlab programming languages.

III. CONVOLUTION ON THE BALL

We review the definition of convolution on the ball [5], highlighting how the translation operator defined on the radial line may be viewed as convolution with a Dirac delta function. By the angular and radial separability of the Fourier-Laguerre transform, we construct a convolution operator on the ball from convolution operators on the sphere and radial line (*e.g.* [3]).

On the sphere, we adopt the usual convolution of $f \in L^2(\mathbb{S}^2)$ with an axisymmetric kernel $h \in L^2(\mathbb{S}^2)$ given by the inner product (*e.g.* [8])

$$\begin{aligned} (f \star h)(\theta, \varphi) &\equiv \langle f | \mathcal{R}_{(\theta, \varphi)} h \rangle_{\mathbb{S}^2} \\ &= \int_{\mathbb{S}^2} d\Omega(\theta', \varphi') f(\theta', \varphi') (\mathcal{R}_{(\theta, \varphi)} h)^*(\theta', \varphi'), \end{aligned} \quad (5)$$

where $d\Omega(\theta, \varphi) = \sin \theta d\theta d\varphi$ is the usual rotation invariant measure on the sphere. The translation operator on the sphere is given by the standard three-dimensional rotation: $(\mathcal{R}_{(\alpha, \beta, \gamma)} h)(\theta, \varphi) = h(\mathcal{R}_{(\alpha, \beta, \gamma)}^{-1}(\theta, \varphi))$, with $(\alpha, \beta, \gamma) \in$

$\text{SO}(3)$, where $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$. We make the association $\theta = \beta$ and $\varphi = \alpha$, *i.e.* $\mathcal{R}_{(\theta, \varphi)} \equiv \mathcal{R}_{(\alpha, \beta, 0)}$, and restrict our attention to convolution with axisymmetric functions that are invariant under azimuthal rotation, *i.e.* $\mathcal{R}_{(0, 0, \gamma)} h = h$, so that we recover a convolved function $f \star h$ defined on the sphere. In harmonic space, axisymmetric convolution may be written

$$(f \star h)_{\ell m} = \langle f \star h | Y_{\ell m} \rangle_{\mathbb{S}^2} = \sqrt{\frac{4\pi}{2\ell + 1}} f_{\ell m} h_{\ell 0}^*, \quad (6)$$

with $f_{\ell m} = \langle f | Y_{\ell m} \rangle_{\mathbb{S}^2}$ and $h_{\ell 0} \delta_{m0} = \langle h | Y_{\ell m} \rangle_{\mathbb{S}^2}$. The generalisation to directional convolution on the sphere is straightforward (see *e.g.* [8]), however we do not present it here since we consider axisymmetric wavelets subsequently.

On the radial line, we consider a convolution operator appropriate for the spherical Laguerre basis. We adopt a convolution similar to that considered by [3] and others (see additional references contained in [3]), although we recover this operator in an alternative manner. Firstly, we define a translation operator \mathcal{T} on the radial line, which is constructed by analogy with the case for the infinite line, for which the standard orthogonal basis is given by the complex exponentials $\phi_\omega(x) = \exp(i\omega x)$, with $x, \omega \in \mathbb{R}$. Translation of the basis functions on the infinite line is simply defined by the shift of coordinates: $(\mathcal{T}_u^\mathbb{R} \phi_\omega)(x) \equiv \phi_\omega(x - u) = \phi_\omega^*(u) \phi_\omega(x)$, with $u \in \mathbb{R}$ and where the final equality follows by the standard rules for exponents. We define translation of the spherical Laguerre basis functions on the radial line by analogy:

$$(\mathcal{T}_s K_p)(r) \equiv K_p(s) K_p(r), \quad (7)$$

where $s \in \mathbb{R}^+$ (since K_p is real we drop the complex conjugation). This leads to a natural harmonic expression for the translation of a radial function $f \in L^2(\mathbb{R}^+)$:

$$(\mathcal{T}_s f)(r) = \sum_{p=0}^{\infty} f_p K_p(s) K_p(r), \quad (8)$$

implying

$$(\mathcal{T}_s f)_p = K_p(s) f_p, \quad (9)$$

where $f_p = \langle f | K_p \rangle_{\mathbb{R}^+}$.

With a translation operator to hand, we may define convolution on the radial line of $f, h \in L^2(\mathbb{R}^+)$ by the inner product

$$(f \star h)(r) \equiv \langle f | \mathcal{T}_r h \rangle_{\mathbb{R}^+} = \int_{\mathbb{R}^+} ds s^2 f(s) (\mathcal{T}_r h)(s), \quad (10)$$

from which it follows that radial convolution in harmonic space is given by the product

$$(f \star h)_p = \langle f \star h | K_p \rangle_{\mathbb{R}^+} = f_p h_p, \quad (11)$$

where $h_p = \langle h | K_p \rangle_{\mathbb{R}^+}$.

Although the definition of the convolution operator on the radial line is complete, we would like to gain further intuition. The action of the translation operator is described in harmonic space through Eqn. (9), which remains somewhat opaque. We would also like to view the translation operator that we have constructed on the radial line in real space.

²<http://www.flaglets.org/>

³<http://www.jasonmccewen.org/>

⁴<http://www.fftw.org/>

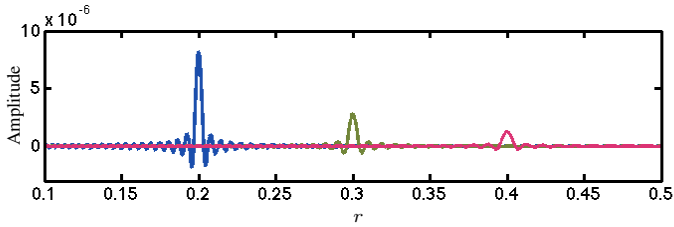


Fig. 1. Band-limited Dirac delta functions plotted on the radial line at positions $s = \{0.2, 0.3, 0.4\}$ (plotted in blue, green and red, respectively). Oscillations are caused by the finite band-limit (here $P = 256$); as $P \rightarrow \infty$ oscillations vanish as the band-limited Delta converges to $\delta_s(r) = r^{-2}\delta^{\mathbb{R}}(r-s)$.

In order to recover a real space representation of the radial translation operator we must first consider the Dirac delta function on the radial line. We define the Dirac delta on the radial line at position s by $\delta_s(r) \equiv r^{-2}\delta^{\mathbb{R}}(r-s)$, where $\delta^{\mathbb{R}}$ is the usual Dirac delta defined on the infinite line \mathbb{R} . The Dirac delta on the radial line satisfies the following normalisation and sifting properties, respectively:

$$\int_{\mathbb{R}^+} dr r^2 \delta_s(r) = 1; \quad (12)$$

$$\int_{\mathbb{R}^+} dr r^2 f(r) \delta_s(r) = f(s). \quad (13)$$

The harmonic expansion of the Dirac delta is given by

$$\delta_s(r) = \sum_{p=0}^{\infty} K_p(s) K_p(r), \quad (14)$$

which follows trivially by the sifting property. For the analysis of band-limited functions, it is sufficient to consider the band-limited Dirac delta (see Fig. 1), where the summation of Eqn. (14) is truncated to $P-1$.

With the Dirac delta function now defined on the radial line, we show that the radial translation operator defined above is simply the convolution of a function with the shifted Dirac delta function:

$$(f \star \delta_s)(r) = \sum_{p=0}^{\infty} f_p K_p(s) K_p(r) = (\mathcal{T}_s f)(r), \quad (15)$$

where the final equality follows by Eqn. (8). Radial convolution and translation are thus the natural analogues of the respective operators defined on the infinite line.

We define the translation operator on the ball by combining the angular and radial translation operators, giving

$$\mathcal{T}_{\mathbf{r}} \equiv \mathcal{T}_r \mathcal{R}_{(\theta, \varphi)}. \quad (16)$$

The action of the radial translation operator on functions defined on the ball is shown in Fig. 2. The convolution on the ball of $f \in L^2(\mathbb{B}^3)$ with an axisymmetric kernel $h \in L^2(\mathbb{B}^3)$ is then defined by the inner product

$$(f \star h)(\mathbf{r}) \equiv \langle f | \mathcal{T}_{\mathbf{r}} h \rangle_{\mathbb{B}^3} = \int_{\mathbb{B}^3} d^3 \mathbf{s} f(\mathbf{s}) (\mathcal{T}_{\mathbf{r}} h)^*(\mathbf{s}), \quad (17)$$

where $\mathbf{s} \in \mathbb{B}^3$. In harmonic space, axisymmetric convolution on the ball may be written

$$(f \star h)_{\ell m p} = \langle f \star h | Z_{\ell m p} \rangle_{\mathbb{B}^3} = \sqrt{\frac{4\pi}{2\ell+1}} f_{\ell m p} h_{\ell 0 p}^*, \quad (18)$$

with $f_{\ell m p} = \langle f | Z_{\ell m p} \rangle_{\mathbb{B}^3}$ and $h_{\ell 0 p} \delta_{m0} = \langle h | Z_{\ell m p} \rangle_{\mathbb{B}^3}$.

IV. FLAGLETS ON THE BALL

With an exact harmonic transform and a convolution operator defined on the ball in hand, we are now in a position to construct our exact wavelet transform on the ball, which we call the flaglet transform (for Fourier-LAGuerre wavelet transform) [5].

For a function of interest $f \in L^2(\mathbb{B}^3)$, we define its jj' -th wavelet coefficient $W^{\Psi^{jj'}} \in L^2(\mathbb{B}^3)$ by the convolution of f with the axisymmetric wavelet, or flaglet, $\Psi^{jj'} \in L^2(\mathbb{B}^3)$:

$$W^{\Psi^{jj'}}(\mathbf{r}) \equiv (f \star \Psi^{jj'}) (\mathbf{r}) = \langle f | \mathcal{T}_{\mathbf{r}} \Psi^{jj'} \rangle_{\mathbb{B}^3}. \quad (19)$$

The scales $j, j' \in \mathbb{N}_0^+$ respectively relate to angular and radial spaces. The wavelet coefficients contain the detail information of the signal only; a scaling function and corresponding scaling coefficients must be introduced to represent the low-frequency, approximate information of the signal. The scaling coefficients $W^{\Phi} \in L^2(\mathbb{B}^3)$ are defined by the convolution of f with the scaling function $\Phi \in L^2(\mathbb{B}^3)$:

$$W^{\Phi}(\mathbf{r}) \equiv (f \star \Phi)(\mathbf{r}) = \langle f | \mathcal{T}_{\mathbf{r}} \Phi \rangle_{\mathbb{B}^3}. \quad (20)$$

Provided the flaglets and scaling function satisfy an admissibility property (defined below), the function f may be reconstructed exactly from its wavelet and scaling coefficients by

$$f(\mathbf{r}) = \int_{\mathbb{B}^3} d^3 \mathbf{r}' W^{\Phi}(\mathbf{r}') (\mathcal{T}_{\mathbf{r}} \Phi)(\mathbf{r}') + \sum_{j=J_0}^J \sum_{j'=J'_0}^{J'} \int_{\mathbb{B}^3} d^3 \mathbf{r}' W^{\Psi^{jj'}}(\mathbf{r}') (\mathcal{T}_{\mathbf{r}} \Psi^{jj'}) (\mathbf{r}'). \quad (21)$$

The parameters J_0 and J (J'_0 and J') define the minimum and maximum wavelet scales considered respectively for the angular (radial) space and depend on the band-limit of f and the specific definition of the wavelets and scaling function (see [5]).

The admissibility condition under which a band-limited function f can be reconstructed exactly is given by the following resolution of the identity:

$$\frac{4\pi}{2\ell+1} \left(|\Phi_{\ell 0 p}|^2 + \sum_{j=J_0}^J \sum_{j'=J'_0}^{J'} |\Psi_{\ell 0 p}^{jj'}|^2 \right) = 1, \quad \forall \ell, p, \quad (22)$$

where $\Phi_{\ell 0 p} \delta_{m0} = \langle \Phi | Z_{\ell m p} \rangle_{\mathbb{B}^3}$ and $\Psi_{\ell 0 p}^{jj'} \delta_{m0} = \langle \Psi^{jj'} | Z_{\ell m p} \rangle_{\mathbb{B}^3}$. We refer the reader to our previous article [5] for an example of the construction of specific wavelets and scaling functions that satisfy the admissibility condition, where we construct suitable wavelets by tiling the ℓ - p harmonic plane. The resulting wavelets are plotted in Fig. 2.

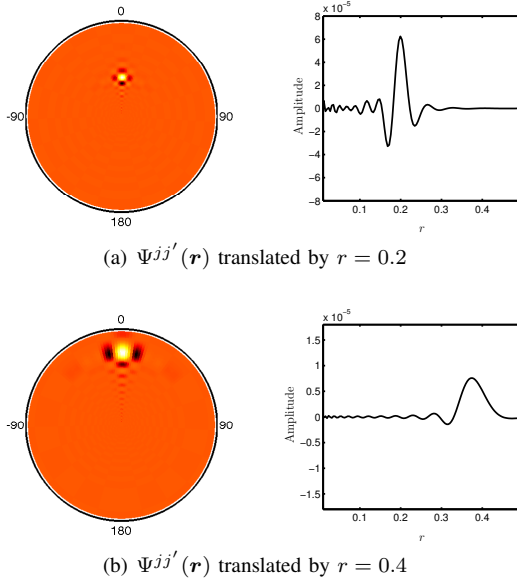


Fig. 2. Slices of the flaglet $\Psi^{jj'}(\mathbf{r})$ with $j = j' = 5$ constructed on the ball of radius $R = 1$ at resolution $P = L = 64$. The three-dimensional flaglets can be visualised by rotating the slices of the left panel (zoomed on a ball of radius $r = 0.5$ for clarity) around the vertical axis passing through the origin. The radial profiles are shown in the right panels. Flaglets are well localised in both real and Fourier-Laguerre spaces. Furthermore, their angular aperture is invariant under radial translation.

We prove that flaglets are a tight frame by showing they satisfy

$$A\|f\|_{\mathbb{B}^3}^2 \leq \int_{\mathbb{B}^3} d^3\mathbf{r} |\langle f | \mathcal{T}_{\mathbf{r}} \Phi \rangle_{\mathbb{B}^3}|^2 + \sum_{j=J_0}^J \sum_{j'=J'_0}^{J'} \int_{\mathbb{B}^3} d^3\mathbf{r} |\langle f | \mathcal{T}_{\mathbf{r}} \Psi^{jj'} \rangle_{\mathbb{B}^3}|^2 \leq B\|f\|_{\mathbb{B}^3}^2, \quad (23)$$

with $A = B \in \mathbb{R}_*^+$, for any band-limited $f \in L^2(\mathbb{B}^3)$, and where $\|\cdot\|_{\mathbb{B}^3}^2 \equiv \langle \cdot | \cdot \rangle_{\mathbb{B}^3}$. We adopt a shorthand integral notation in Eqn. (23), although by appealing to our exact quadrature rule these integrals may be replaced by finite sums. Noting the harmonic expression for axisymmetric convolution given by Eqn. (18) and the orthogonality of the Fourier-Laguerre basis functions, it is straightforward to show that the term of Eqn. (23) bounded between inequalities may be written

$$\sum_{p=0}^{P-1} \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \left(|\Phi_{\ell 0 p}|^2 |f_{\ell m p}|^2 + \sum_{j=J_0}^J \sum_{j'=J'_0}^{J'} |\Psi_{\ell 0 p}^{jj'}|^2 |f_{\ell m p}|^2 \right) = \sum_{p=0}^{P-1} \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} |f_{\ell m p}|^2 = \int_{\mathbb{B}^3} d^3\mathbf{r} |f(\mathbf{r})|^2 = \|f\|_{\mathbb{B}^3}^2, \quad (24)$$

where the second line follows from the admissibility property Eqn. (22). Thus, we find flaglets indeed constitute a tight frame

with $A = B = 1$, implying the energy of f is conserved in flaglet space.

We have developed the public FLAGLET⁵ code [5] to compute the flaglet transform. The FLAGLET code computes the exact forward and inverse flaglet transform at machine precision, exploiting a fast multiresolution algorithm, and is stable to extremely large band-limits (the computation time and numerical precision of the FLAGLET code is evaluated in detail in [5], where a toy application is also presented). FLAGLET relies on the public code S2LET⁶ [6] (to compute wavelet transforms on the sphere), FLAG⁵ [5], SSHT⁷ [7] and FFTW⁸, and supports both the C and Matlab programming languages.

To summarise, flaglets live naturally on the ball (with an angular opening that is invariant under radial translation), yield a theoretically exact wavelet transform on the ball (in both the continuous and discrete settings), and exhibit a fast multiresolution algorithm. It is our hope that flaglets will prove useful for analysing data defined on the ball. Indeed, in the near future we intend to apply flaglets to study the large-scale structure of the Universe traced by the distribution of galaxies.

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⁵<http://www.flaglets.org/>

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