Signal Adaptive Frame Theory

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***Abstract*—The projection method is an atomic signal decomposition designed for adaptive frequency band (AFB) and ultra-wide-band (UWB) systems. The method first windows the signal and then decomposes the signal into a basis via a continuous-time inner product operation, computing the basis coefficients in parallel. The windowing systems are key, and we develop systems that have variable partitioning length, variable roll-off and variable smoothness. These include systems developed to preserve orthogonality of any orthonormal systems between adjacent blocks, and *almost orthogonal* windowing systems that are more computable/constructible than the orthogonality preserving systems. The projection method is, in effect, an adaptive Gabor system for signal analysis. The natural language to express this structure is frame theory.**

I. Introduction

Adaptive frequency band (AFB) and ultra-wide-band (UWB) systems present challenges to current methods of signal processing. Despite extensive advances, wideband problems continue to hit barriers in sampling architectures and analog-to-digital conversion (ADC). ADC signal-to-noise and distortion ratio (the effective number of resolution bits) declines with sampling rate due to timing jitter, circuit imperfections, and electronic noise. ADC performance (speed and total integrated noise) can be improved to some extent, e.g., by cooling. However, the energy cost may be significant, and this presents a major hurdle for implementation in miniaturized devices. Digital circuitry has provided dramatically enhanced DSP operation speeds, but there has not been a corresponding dramatic energy capacity increase in batteries to operate these circuits. Moore's Law for chips is slowing down, and there is no Moore's Law for batteries or ADCs.

A growing number of applications face this challenge, such as miniature and hand-held devices for communications, robotics, and micro aerial vehicles (MAVs). Very wideband sensor bandwidths are desired for dynamic spectrum access and cognitive radio, radar, and ultra-wideband systems. Multi-channel and multi-sensor systems compound the issue, such as MIMO, array processing and beamforming, multi-spectral imaging, and vision systems. All of these rely on analog sensing and a digital interface, perhaps with feedback. This motivates mixed-signal circuit designs that tightly couple the analog and digital portions, and operate with parallel reduced bandwidth paths to relax ADC requirements. The goal of such wideband integrated circuit designs is to achieve good tradeoffs in dynamic range, bandwidth, and parallelization, while maintaining low energy consumption.

From a signal processing perspective, we can approach this problem by implementing an appropriate signal decomposition in the analog portion that provides parallel outputs for integrated digital conversion and processing. This naturally leads to an architecture with windowed time segmentation and parallel analog basis expansion. In this paper we view this from the sampling theory perspective, including segmentation and window design, achieving orthogonality between segments, basis expansion and choice of basis, signal filtering, and reconstruction. Definitions and computations for the paper follow those given in Benedetto [1].

II. Windowing

We first construct smooth bounded adaptive partitions of unity, or *BAPU Systems*. These are generalizations of bounded uniform partitions of unity (*BUPU Systems*) in that they allow for signal adaptive windowing. These systems give a flexible adaptive partition of unity of variable smoothness and are useful whenever a partition of unity is used, such as in compressed sensing. The construction elements for this system are B-splines. The second type of system we develop preserves orthogonality of any orthonormal (ON) system between adjacent blocks. The construction here uses any orthonormal basis for $L^2(\mathbb{R})$ and is created by solving a Hermite interpolation problem with constraints. These ON preserving window systems were the motivation for the methods in this paper. They allow us to create a method of time-frequency analysis for a wide class of signals. The third type of system we develop uses the concept of *almost orthogonality* developed by Cotlar, Knapp and Stein. It employs our B-spline techniques to create almost orthogonal windowing systems that are more computable/constructible than the orthogonality preserving systems.

The windowing systems for the partition of unity $\{\mathbb{B}_k(t)\}$ satisfy $\sum_k \mathbb{B}_k(t) \equiv 1$. The key difference between the partition of unity systems and (ON) systems is that the second preserves orthogonality. Preserving orthogonality requires that the windowing systems $\{\mathbb{W}_k(t)\}$ satisfy $\sum_k [\mathbb{W}_k(t)]^2 \equiv 1$. The almost orthogonal systems require that there exists a δ, $0 \leq \delta \leq 1/2$ such that for all k

$$1 - \delta \leq [\mathbb{A}_k(t)]^2 + [\mathbb{A}_{k+1}(t)]^2 \leq 1 + \delta$$

for $t \in [kT, (k+1)T]$.

A. Partition of Unity Systems

The theory of B-splines gives us the tools to create smooth partition of unity systems.

Definition 1 (Bounded Adaptive Partition of Unity): A *Bounded Adaptive Partition of Unity* is a set of functions $\{\mathbb{B}_k(t)\}$ such that

$$\begin{aligned} &(i.) \quad \operatorname{supp}(\mathbb{B}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\ &(ii.) \quad \mathbb{B}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r]\,, \\ &(iii.) \quad \sum_k \mathbb{B}_k(t) \equiv 1\,, \\ &(iv.) \quad \{\widehat{\mathbb{B}_k{}^{\circ}}[n]\} \in l^1\,. \end{aligned} \tag{1}$$

Conditions $(i.), (ii.)$ and $(iii.)$ make $\{\mathbb{B}_k(t)\}$ a bounded partition of unity. Condition $(iii.)$ means that these systems do not preserve orthogonality between blocks. We will generate our systems by translations and dilations of a given window $\mathbb{B}_I$, where $\operatorname{supp}(\mathbb{B}_I) = [(-T/2 - r), (T/2 + r)]$.

Our general window function $\mathbb{W}_I$ is k-times differentiable, has $\operatorname{supp}(\mathbb{B}_I) = [(-T/2 - r), (T/2 + r)]$ and has values

$$\mathbb{B}_I = \begin{cases} 0 & |t| \geq T/2 + r \\ 1 & |t| \leq T/2 - r \\ \rho(\pm t) & T/2 - r < |t| < T/2 + r \end{cases} \tag{2}$$

We solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{cases} (a.) & \rho(T/2 - r) = 1 \\ (b.) & \rho^{(n)}(T/2 - r) = 0\,,\ n = 1, 2, \ldots, k \\ (c.) & \rho^{(n)}(T/2 + r) = 0\,,\ n = 0, 1, 2, \ldots, k\,, \end{cases}$$

with the conditions that $\rho \in C^k$ and

$$[\rho(t)] + [\rho(-t)] = 1 \text{ for } t \in [T/2 - r, T/2 + r]\,. \tag{3}$$

We use B-splines as our cardinal functions. Let $0 < \alpha \ll \beta$ and consider $\chi_{[-\alpha,\alpha]}$. We want the n-fold convolution of $\chi_{[\alpha,\alpha]}$ to fit in the interval $[-\beta, \beta]$. Then we choose α so that $0 < n\alpha < \beta$ and let

$$\Psi(t) = \underbrace{\chi_{[-\alpha,\alpha]} * \chi_{[-\alpha,\alpha]} * \cdots * \chi_{[-\alpha,\alpha]}(t)}_{n-times}\,.$$

The β-periodic continuation of this function, $\Psi^{\circ}(t)$ has the Fourier series expansion

$$\sum_{k \neq 0} \frac{\alpha}{n\beta} \left[\frac{\sin(\pi k\alpha/n\beta)}{2\pi k\alpha/n\beta} \right]^n \exp(\pi i k t/\beta)\,.$$

The C^k solution for ρ is given by a theorem of Schoenberg (see [7], pp. 7-8). Schoenberg solved the Hermite interpolation problem

$$\begin{cases} (a.) & S^{(n)}(-1) = 0\,,\ n = 0, 1, 2, \ldots, k\,, \\ (b.) & S(1) = 1\,, \\ (b.) & S^{(n)}(1) = 0\,,\ n = 1, 2, \ldots, k\,. \end{cases}$$

An interpolant that minimizes the Chebyshev norm is called the *perfect spline*. The perfect spline $S(t)$ for Hermite problem above is given by the integral of the function

$$M(x) = (-1)^n \sum_{j=0}^{k} \frac{\Psi(t - t_j)}{\phi'(t_j)}\,,$$

where Ψ is the $(k+1)$ convolution of characteristic functions, the knot points are $t_j = -\cos(\frac{\pi j}{k})$ and $\phi(t) \prod_{j=0}^{k} (t - t_j)$. We then have that $\rho(t) = S \circ \ell(t)$, where $\ell(t) = \frac{1}{r}t - \frac{2T}{2r}$. For this ρ, and for

$$\mathbb{B}_I = \begin{cases} 0 & |t| \geq T/2 + r \\ 1 & |t| \leq T/2 - r \\ \rho(\pm t) & T/2 - r < |t| < T/2 + r \end{cases}$$

we have that $\widehat{\mathbb{B}_I}(\omega)$ is given by the antiderivative of a linear combination of functions of the form $[\sin(\omega)/\omega]^{k+1}$, and therefore has decay $1/\omega^{k+2}$ in frequency.

B. Orthogonality Preserving Systems

Our first system of signal segmentation uses sine, cosine and linear functions. This was created because it is relatively easy to implement, cuts down on frequency error and preserves orthogonality. Consider a signal block of length $T + 2r$ centered at the origin. Let $0 < r \ll T$. Ideally, we would like to make r as small as possible. Define $\operatorname{Cap}(t)$ as follows.

$$\begin{cases} 0 & |t| \geq \frac{T}{2} + r\,, \\ 1 & |t| \leq \frac{T}{2} - r\,, \\ \sin(\pi/(4r)(t + (T/2 + r))) & \frac{-T}{2} - r < t < \frac{-T}{2} + r\,, \\ \cos(\pi/(4r)(t - (T/2 - r))) & \frac{T}{2} - r < t < \frac{T}{2} + r\,. \end{cases} \tag{4}$$

Given Cap, we form a tiling system $\{\operatorname{Cap}_k(t)\}$ such that $\operatorname{supp}(\operatorname{Cap}_k(t)) \subseteq [kT - r, (k+1)T + r]$ for all k. Note that the Cap window has several properties that make it a good window for our purposes. It has a partition property in that it windows the signal in $[\frac{-T}{2} - r, \frac{T}{2} + r]$ and is identically 1 on $[\frac{-T}{2} + r, \frac{T}{2} - r]$. It has a continuous roll-off at the endpoints. Finally, it has the property that for all $t \in \mathbb{R}$

$$[\operatorname{Cap}_k(t)]^2 + [\operatorname{Cap}_{k+1}(t)]^2 = 1\,.$$

This last condition is needed to preserve the orthogonality of basis elements between adjacent blocks. Additionally, it has $1/\omega^2$ decay in frequency space, and, when one time block is ramping down, the adjacent block is ramping up at exactly the same rate. If we had a signal f with an absolutely convergent Fourier series,

$$(f \cdot \operatorname{Cap})_k\hat{\ }[n] = \sum_m f[n - m] \operatorname{Cap}\hat{\ }[m] = \hat{f} * \operatorname{Cap}\hat{\ }[n]\,.$$

The Fourier transform of Cap is a linear combination of $\operatorname{sinc}(\omega)$ and $\sin(\omega)$ functions and has an asymptotic $1/\omega^2$ decay.

The theory of splines gives us the tools to generalize this system. The idea is to cut up the time domain into perfectly aligned segments so that there is no loss of information.

We also want the systems to be smooth, so as to provide control over decay in frequency, and adaptive, so as to adjust accordingly to changes in frequency band. Finally, we develop our systems so that the orthogonality of bases in adjacent and possible overlapping blocks is preserved.

Definition 2 (ON Window System): An *ON Window System* is a set of functions $\{\mathbb{W}_k(t)\}$ such that for all $k \in \mathbb{Z}$

$$\begin{array}{ll} (i.) & \text{supp}(\mathbb{W}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\ (ii.) & \mathbb{W}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r]\,, \\ (iii.) & \mathbb{W}_k \text{ is symmetric about its midpoint }, \\ (iv.) & [\mathbb{W}_k(t)]^2 + [\mathbb{W}_{k+1}(t)]^2 = 1\,, \\ (v.) & \{\widehat{\mathbb{W}_k{}^\circ}[n]\} \in l^1\,. \end{array} \tag{5}$$

Conditions $(i.)$ and $(ii.)$ are partition properties, in that they give an exact snapshot of the input function f on $[kT + r, (k+1)T - r]$ with smooth roll-off at the edges. Conditions $(iii.)$ and $(iv.)$ are needed to preserve orthogonality between adjacent blocks. Condition $(v.)$ is needed for the computation of Fourier coefficients. We generate our systems by translations and dilations of a given window $\mathbb{W}_I$, where $\text{supp}(\mathbb{W}_I) = [-T/2 - r, T/2 + r]$. Our next proposition shows the need for the condition $(v.)$. Let $I = T + 2r$ and let $\mathbb{PW}_\Omega$ denote the Paley-Wiener space for bandlimit Ω.

Proposition 1: Let $f \in \mathbb{PW}_\Omega$ and let $\{\mathbb{W}_k(t)\}$ be an ON Window System with generating window $\mathbb{W}_I$. Then

$$\frac{1}{I}\int_{-T/2-r}^{T/2-r} [f \cdot \mathbb{W}_I]^\circ(t) \exp(-2\pi i n t/[I])\, dt = \widehat{f} * \widehat{\mathbb{W}_I}[n]\,. \tag{6}$$

Our general window function $\mathbb{W}_I$ is k-times differentiable, has $\text{supp}(\mathbb{W}_I) = [-T/2 - r, T/2 + r]$ and has values

$$\mathbb{W}_I = \begin{cases} 0 & |t| \geq T/2 + r \\ 1 & |t| \leq T/2 - r \\ \rho(\pm t) & T/2 - r < |t| < T/2 + r \end{cases} \tag{7}$$

We solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{cases} (a.) & \rho(T/2 - r) = 1 \\ (b.) & \rho^{(n)}(T/2 - r) = 0\,,\ n = 1, 2, \ldots, k \\ (c.) & \rho^{(n)}(T/2 + r) = 0\,,\ n = 0, 1, 2, \ldots, k\,, \end{cases}$$

with the conditions that $\rho \in C^k$ and

$$[\rho(t)]^2 + [\rho(-t)]^2 = 1 \text{ for } t \in [\pm(\frac{T}{2} - r), \pm(\frac{T}{2} + r)]\,. \tag{8}$$

The constraint (8) directs us to get solutions expressed in terms of $\sin(t)$ and $\cos(t)$. Solving for ρ so that the window in C^1, we get that $\rho(t)$ equals

$$\begin{cases} \sqrt{\left[1 - \frac{1}{2}\left[1 - \sin(\frac{\pi}{2r}(\frac{T}{2} - t))\right]^2\right]} & \frac{T}{2} - r \leq t \leq \frac{T}{2} \\ \frac{1}{\sqrt{2}}\left[1 - \sin(\frac{\pi}{2r}(t - \frac{T}{2}))\right] & \frac{T}{2} \leq t \leq \frac{T}{2} + r\,. \end{cases} \tag{9}$$

With each degree of smoothness, we get an additional degree of decay in frequency.

C. Orthogonality Between Blocks

We designed the ON Window Systems $\{\mathbb{W}_k(t)\}$ so that they would preserve orthogonality of basis element of overlapping blocks. Because of the partition properties of these systems, we need only check orthogonality of adjacent overlapping blocks. The best way to think about the construction is to visualize how one would do the extension for a system of sines and cosines. We would extend the odd reflections about the left endpoint and the even reflections about the right. Let $\{\varphi_j(t)\}$ be an orthonormal basis for $L^2[-T/2, T/2]$. Define

$$\widetilde{\varphi_j}(t) = \begin{cases} 0 & |t| \geq T/2 + r \\ \varphi_j(t) & |t| \leq T/2 - r \\ -\varphi_j(-T - t) & -T/2 - r < t < -T/2 \\ \varphi_j(T - t) & T/2 < t < T/2 + r\,. \end{cases} \tag{10}$$

Theorem 1: $\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi_j}(t)\}$ is an ON basis for $L^2(\mathbb{R})$.

Proof : See [3]. □

D. Almost Orthogonal Systems

The Partition of Unity Systems do *not* preserve orthogonality between blocks. However, they are easier to compute in both time and frequency. Therefore, these systems can be used to approximate the Cap system with B-splines. We get windowing systems that nearly preserve orthogonality. Each added degree of smoothness in time adds to the degree of decay in frequency.

Cotlar, Knapp and Stein introduced *almost orthogonality* via operator inequalities. The concept allows us to create windowing systems that are more computable/constructible such as the Bounded Adaptive Partition of Unity Systems $\{\mathbb{B}_k(t)\}$ with the orthogonality preservation of the ON Window Systems $\{\mathbb{W}_k(t)\}$.

Definition 3 (Almost ON System): Let $0 < r \ll T$. An **Almost ON System** for adaptive and ultra-wide band sampling is a set of functions $\{\mathbb{A}_k(t)\}$ for which there exists δ, $0 \leq \delta < 1/2$, such that

$$\begin{array}{ll} (i.) & \text{supp}(\mathbb{A}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\ (ii.) & \mathbb{A}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r]\,, \\ (iii.) & \mathbb{A}_k((kT + T/2) - t) = \mathbb{A}_k(t - (kT + T/2))\,, \\ (iv.) & 1 - \delta \leq [\mathbb{A}_k(t))]^2 + [\mathbb{A}_{k+1}(t))]^2 \leq 1 + \delta\,, \\ (v.) & \{\widehat{\mathbb{A}_k{}^\circ}[n]\} \in l^1\,. \end{array}$$

Starting with Cap(t), let $\Delta_{(T,r)} = \frac{T+2r}{m}$. By placing equidistant knot points $-T/2 - r = x_0, -T/2 - r + \Delta_{(T,r)} = x_1, \ldots, T/2 + r = x_m$, we can construct C^{m-1} polynomial splines S_{m+1} approximating Cap(t) in $[(-T/2 - r), (T/2 + r)]$. A theorem of Curry and Schoenberg gives that the set of B-splines $\{B^{(m+1)}_{-(m+1)}, \ldots, B^{(m+1)}_k\}$ forms a basis for S_{m+1}. Therefore, Cap$(t) \approx \sum_{i=-(m+1)}^{k} a_i B_i^{(m+1)}$. Let

$$\delta = \left\| \sum_{i=-(m+1)}^{k} a_i B_i^{(m+1)} - \text{Cap}(t) \right\|_\infty .$$

Then, $\delta < 1/2$, with the largest value for the piecewise linear spline approximation. Moreover, $\delta \longrightarrow 0$ as m and k increase. Thus we get computable windowing systems that nearly preserve orthogonality. Each added degree of smoothness in time adds to the degree of decay in frequency.

III. Signal Expansions

Given characteristics of the class of input signals, the choice of basis functions used can be tailored to optimal representation of the signal or a desired characteristic in the signal.

Theorem 2 (The Projection Formula for ON Windowing): Let $\{\mathbb{W}_k(t)\}$ be an ON Window System, and let $\{\Psi_{k,j}\}$ be an orthonormal basis that preserves orthogonality between adjacent windows. Let $f \in \mathbb{PW}_\Omega$ and $N = N(T, \Omega)$ be such that $\langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle = 0$ for all $n > N$ and all k. Then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n=-N}^{N} \langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle \Psi_{k,n}(t)\right]. \quad (11)$$

This theorem gives a new method for A-D conversion. Unlike the Shannon method which examined the function at specific points, then used those individual points to recreate the curve, the projection method breaks the signal into time blocks and then approximates their respective periodic expansions with a Fourier series. This process allows the system to individually evaluate each piece and base its calculation on the needed bandwidth. The individual Fourier series are then summed, recreating a close approximation of the original signal. It is important to note that instead of fixing T, the method allows us to fix any of the three while allowing the other two to fluctuate. From the design point of view, the easiest and most practical parameter to fix is N. For situations in which the bandwidth does not need flexibility, it is possible to fix Ω and T by the equation $N = \lceil T \cdot \Omega \rceil$. However, if greater bandwidth Ω is need, choose shorter time blocks T.

The windowing systems above allow us to develop *Signal Adaptive Frame Theory*. The idea is as follows. If we work with an ON Windowing System $\{\mathbb{W}_k(t)\}$, let $\{\Psi_{k,j}\}$ be an orthonormal basis that preserves orthogonality between adjacent windows. Let $f \in \mathbb{PW}_\Omega$ and $N = N(T, \Omega)$ be such that $\langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle = 0$ for all $n > N$ and all k. Then

$$f(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} \langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle \Psi_{k,n}(t)\right]. \quad (12)$$

This also gives

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} |\langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle|^2\right]. \quad (13)$$

Given that $\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi_j}(t)\}$ is an orthonormal basis for $L^2(\mathbb{R})$, we have a representation of a given function f in L^2. The set $\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi_j}(t)\}$ is an exact normalized tight frame for L^2. The restriction that these basis elements present is computability. They become increasing difficult to compute as the smoothness in time/decay in frequency increases.

A way around this is to connect the Bounded Adaptive Partition of Unity Systems $\{\mathbb{B}_k(t)\}$ to frame theory. The ideas behind this connection go back to the curvelet work of Candès and Donoho. The paper of Borup and Neilsen [2] gives a nice overview of this connection, and we will refer to that paper for the background from which we develop our approach. The set $\{\mathbb{B}_k(t)\}$ form an *admissible* cover, in that they form a partition of unity and have overlap with only their immediate neighbors.

For each window $\mathbb{B}_k(t)$, let $\phi_{n,k}(t)$ be the shifted $\exp[\pi i t T/n]$ centered in the window. Then define

$$\Phi_{k,n} = \mathbb{B}_k(t)\phi_{k,n}(t) .$$

Given and $f \in L^2$ we can write

$$f(t) \approx \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} \langle f \cdot \mathbb{B}_k, \Phi_{k,n}\rangle \Phi_{k,n}(t)\right]. \quad (14)$$

For this system we can compute

$$A\|f\| \le \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} |\langle f \cdot \mathbb{B}_k, \Phi_{k,n}\rangle|^2\right] \le B\|f\| . \quad (15)$$

The bounds are a function of how much of the signal is concentrated in the overlap regions and will be tightened for the almost orthogonal windowing systems. The closer the approximation, the better the frame bounds. Developing these signal adaptive frames, their bounds and the associated frame operators will be a major point of emphasis in future work. We will additionally develop biorthogonal adaptive frames using our B-spline constructions. We conjecture the following:

$$\mathcal{A}_{1-\delta}\|f\|^2 \le \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} |\langle f \cdot \mathbb{A}_k, \Psi_{k,n}\rangle|^2\right] \le \mathcal{A}_{1+\delta}\|f\|^2 . \quad (16)$$

Moreover, this $\longrightarrow$ Normalized Tight Frame as $\delta \longrightarrow 0$.

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