Signal Analysis with Frame Theory and Persistent Homology

Holger Boche
TU München
Theresienstr. 90/IV (LTI)
80333 München, Germany
boche@tum.de

Mijail Guillemard
TU Berlin
Institut für Mathematik
10623 Berlin, Germany
guillemard@math.tu-berlin.de

Gitta Kutyniok
TU Berlin
Institut für Mathematik
10623 Berlin, Germany
kutyniok@math.tu-berlin.de

Friedrich Philipp
TU Berlin
Institut für Mathematik
10623 Berlin, Germany
philipp@math.tu-berlin.de

***Abstract*—A basic task in signal analysis is to characterize data in a meaningful way for analysis and classification purposes. Time-frequency transforms are powerful strategies for signal decomposition, and important recent generalizations have been achieved in the setting of frame theory. In parallel recent developments, tools from algebraic topology, traditionally developed in purely abstract settings, have provided new insights in applications to data analysis. In this report, we investigate some interactions of these tools, both theoretically and with numerical experiments, in order to characterize signals and their frame transforms. We explain basic concepts in persistent homology as an important new subfield of computational topology, as well as formulations of time-frequency analysis in frame theory. Our objective is to use persistent homology for constructing topological signatures of signals in the context of frame theory. The motivation is to design new classification and analysis methods by combining the strength of frame theory as a fundamental signal processing methodology, with persistent homology as a new tool in data analysis.**

I. Introduction

Modern developments in signal processing have triggered important interactions between pure and applied mathematics. A basic example is given by new advances in time-frequency analysis and its generalizations to frame theory [2, 8], but another recent and major development illustrating a rich interplay between abstract ideas and practical applications is persistent homology [1, 7], which in the last few years has become an important subfield of computational topology. These developments in persistent homology have been applied in different situations, and particular results relevant in our setting are recent results in sensor networks [10, 11]. This report is a natural continuation of our previous work [12] which introduced a strategy for integrating time-frequency analysis with persistent homology. Our contribution now is to further understand and improve these interactions by combining frame theory with the stability of persistent homology.

The outline of this report is as follows. We begin with a short overview of time-frequency analysis and frame theory, with a particular focus on voice transformations and how this setting is generalized in (continuous) frame theory by considering analysis operators $V : \mathcal{H} \to L^2(\mathcal{X})$. Here, $\mathcal{X}$ is a locally compact group for the case of voice transformations, and a locally compact Hausdorff space in frame theory.

We then shortly present elements of persistent homology as a new important branch in data analysis which, given a point cloud data $X = \{x_i\}_{i=1}^m$, (or more generally, a family of simplicial complexes $K_1 \subset K_2 \subset \cdots \subset K_r = \mathcal{X}$) constructs a diagram that encodes a topological features of X (resp. $\mathcal{X}$). We then proof a property combining the basic stability of persistent diagrams with frame theory, and illustrate this concept with computational experiments.

II. Time-frequency Analysis and Frame Theory

Given a Hilbert space $\mathcal{H}$ as, for instance, a functional space of signals $L^2(\mathbb{R})$, the basic strategy in time-frequency analysis is to segment a signal $f \in \mathcal{H}$ in smaller chunks $x_b = fg_b$, for g a window function, and $g_b(t) = g(t - b)$. This segmentation procedure is the basis of *Gabor analysis* and the short term Fourier transform (STFT), and it allows to locally analyze the frequency behavior of f and its evolution in time. A generalization of this method can be described using a locally compact group G acting in a Hilbert space $\mathcal{H}$ (see [8]). This action is an irreducible and square integrable group representation, $\pi : G \to U(\mathcal{H})$, defined as a group homomorphism between G and $U(\mathcal{H})$, the group of unitary operators in $\mathcal{H}$. The basic transformation that is constructed with π is the *analysis operator* or the *voice transform*:

$$V_\psi : \mathcal{H} \to L^2(G), \qquad V_\psi(f)(x) = \langle f, \pi(x)\psi \rangle,$$

which maps each $f \in \mathcal{H}$ to a square integrable function $V_\psi f$ that "unfolds" the content of f in the setting provided by G. We remark that a fundamental property of V is to be a quasi isometry, which allows to perform not only analysis but also synthesis procedures.

A. Continuous and Discrete Frames

Despite the major role of the voice transform and its group representation background, in some applications it is too restrictive to assume the existence of a group G that parametrizes the family of dictionary vectors $\{\pi(x)\psi\}_{x\in G}$. An important generalization of these procedures is frame theory which considers a family of vectors $\{\psi_x\}_{x\in\mathcal{X}}$ in a Hilbert space $\mathcal{H}$, where $\mathcal{X}$ is a locally compact Hausdorff space with a positive Radon measure μ (see [9]). When $\mathcal{X}$ is finite or discrete (e.g. $\mathcal{X} = \mathbb{N}$), we will consider a counting measure

μ, and the resulting concept will be a generalization of an orthogonal basis, and it provides powerful mechanisms for the analysis and synthesis of a signal $f \in \mathcal{H}$.

The main property required by a frame $\{\psi_x\}_{x\in\mathcal{X}} \subset \mathcal{H}$ is the stabilization of the analysis operator.

Definition 1. A set of vectors $\{\psi_x\}_{x\in\mathcal{X}} \subset \mathcal{H}$ in a Hilbert space $\mathcal{H}$ is a *frame*, if

$$A||f||^2 \leq ||Vf||^2 \leq B||f||^2, \quad \forall f \in \mathcal{H}$$

for $0 < A \leq B < \infty$, the *lower and upper frame bounds*, and $V : \mathcal{H} \to L^2(\mathcal{X}), (Vf)(x) = \langle f, \psi_x \rangle$ is the *analysis operator.*

Reducing the difference between A and B improves the stability of V, and for the case of $A = B$, or $A = B = 1$, the resulting frame is denominated *tight frame* and *Parseval frame*, respectively. The corresponding synthesis operator $V^* : L^2(\mathcal{X}) \to V$, with $V^*((a_x)_{x\in\mathcal{X}}) = \int_{\mathcal{X}} a_x\psi_x \, d\mu(x)$ is defined with an adequate positive Radon measure μ, when $\mathcal{X}$ is a locally compact Hausdorff space (see [9]). The maps V^* and V are combined in the frame operator

$$S = V^*V : \mathcal{H} \to \mathcal{H}, Sf = \int_{\mathcal{X}} \langle f, \psi_x \rangle \psi_x d\mu(x),$$

which plays an important role due to the fact that the operator norm of S can be bounded by A and B, namely:

$$A \leq ||S||_{op} \leq B. \tag{1}$$

III. Persistent Homology

In order to shortly introduce the basic concepts in persistent homology, we recall elementary ideas in simplicial homology. One of the simplest homology theories available is simplicial homology which translates topological data into an algebraic formulation. The fundamental objective is to compute qualitative properties of a topological space $\mathcal{X}$, as the number of n-dimensional holes $\mathcal{X}$ has. The basic object to analyze is a (finite) *abstract simplicial complex* K, defined as a nonempty family of subsets of a vertex set $V = \{v_i\}_{i=1}^m$ with $\{v\} \in K$ if $v \in V$, and if $\alpha \in K, \beta \subseteq \alpha$, then $\beta \in K$. We define *faces* (or *simplices*) to be the elements of K, and their corresponding *dimension* will be their cardinality minus one.

In order to compute the number of holes of a given simplicial complex K, we translate its topological or combinatorial properties in the language of linear algebra. There are three basic steps in this procedure: first, we construct a family of free groups C_n, the *group of n-chains* defined as the formal combinations of k-dimensional faces with coefficients in a given group (or rings and fields in more specific cases). Secondly, we construct the *boundary operators* ∂_n, defined as homomorphisms (or more specifically linear maps) between the group of k-chains C_k. The homomorphism maps a face $\sigma = [p_0, \cdots p_n] \in C_n$ into C_{n-1} by

$$\partial_n\sigma = \sum_{k=0}^{n}(-1)^k[p_0, \cdots , p_{k-1}, p_{k+1}, \cdots p_n].$$

Finally, in the third step, we construct the *homology groups* defined as the quotients $H_k := \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$. The main property is now the computation of the *Betti numbers*, which represent the amount of k-dimensional holes, and it corresponds to the rank of the homology groups, $\beta_k = \mathrm{rank}(H_k)$.

The fundamental ideas of persistent homology have been introduced at the end of the last century (see [6]) where the estimation of topological properties of finite sets arises as an important problem in many applications. An important scenario is the analysis of a point cloud data $X = \{x_i\}_{i=1}^m$ which represents the challenging situation that no simplicial structure is given a priori. The strategy is to consider special type of simplicial complexes (e.g. Čech complexes, Vietoris Rips complexes) arising by considering the set $R_\epsilon(X)$, defined with X as the vertex set, and setting the vertices $\sigma = \{x_0, \ldots , x_k\}$ to span a k-simplex of $R_\epsilon(X)$ if $d(x_i, x_j) \leq \epsilon$ for all $x_i, x_j \in \sigma$. The fundamental remark is to notice that for a finite point cloud data $X = \{x_i\}_{i=1}^m$ there is only a finite number of simplicial complexes that fully characterize the family $\{R_\epsilon(X)\}_{\epsilon>0}$. Namely, there is only a finite number of non-homeomorphic simplicial complexes $K_1 \subset K_2 \subset \cdots \subset K_r$ (a so called *filtration*) that fully describe the topology of the sets $\{R_\epsilon(X)\}_{\epsilon>0}$. The power of persistent homology lies in efficient algorithms that compute homology information for the filtration $K_1 \subset K_2 \subset \cdots \subset K_r$.

Definition 2 (Persistent homology)**.** A *filtration* is the basic input of persistent homology, and it is defined for a topological space $\mathcal{X}$, as a family of non-homeomorphic simplicial complexes $K_1 \subset K_2 \subset \cdots \subset K_r = \mathcal{X}$. We define a *persistent homology group* (at the level n) of a filtration as the image of a group homomorphisms $f_n^{ij} : H_n(K_i) \to H_n(K_{i+j})$. The maps f_n^{ij} are induced from the continuous inclusions $K_i \subset K_j$ by the functorial properties of homology. The images of f_n^{ij} represent the homology classes born at i and still alive at $i + j$. The rank of theses images $\beta_n^{ij} = \mathrm{rank}(\mathrm{Im} f_n^{ij})$ is the *persistent Betti number* (at the homology level n). The *persistent diagram* $\mathrm{dgm}(\mathcal{X})$ (at the level n) of $\mathcal{X}$ is constructed by associating the value β_n^{ij} to the pairs (i, j), $1 \leq i \leq j \leq r$.

A. Stability Properties

We now present an important component in the persistent homology toolbox denominated the *stability of persistent diagrams* [5]. In order to explain this concept, we first introduce some preliminary notions.

Definition 3 (Homological critical values and tame functions)**.** Let $\mathcal{X}$ be a topological space, and $\alpha : \mathcal{X} \to \mathbb{R}$ a continuous function. A *homological critical value* (or HCV) is a number $a \in \mathbb{R}$ for which the map induced by α

$$H_n(\alpha^{-1}(] - \infty, a - \epsilon[)) \to H_n(\alpha^{-1}(] - \infty, a + \epsilon[))$$

is not an isomorphism for all $\epsilon > 0$. Remember that each $\alpha^{-1}(] - \infty, a[)$ is a *level sets* of α, and it plays a crucial role in Morse theory, as well as in our current setting. A *tame function* is now defined to be a function $\alpha : \mathcal{X} \to \mathbb{R}$ that has only a finite number of HCV.

Typical examples of tame functions are *Morse functions* on compact manifolds, and piecewise linear functions on finite simplicial complexes [5].

Definition 4. For a tame function $\alpha : \mathcal{X} \to \mathbb{R}$, we define its *persistent diagram* $\mathrm{dgm}(\alpha)$ as the persistent diagram of the filtration $K_1 \subset K_2 \subset \cdots \subset K_r = \mathcal{X}$ where we define $K_i = f^{-1}(]-\infty, a_i])$, and $a_1 < a_2 < \cdots < a_r$ are the critical values of α (see [4]).

Definition 5 (Bottleneck and Hausdorff distances)**.** For two nonempty sets $X, Y \subset \mathbb{R}^2$ the *Hausdorff distance* and *bottleneck distances* are defined as

$$d_H(X,Y) = \max\left\{ \sup_{x\in X}\inf_{y\in Y} ||x-y||_\infty, \sup_{y\in Y}\inf_{x\in X} ||y-x||_\infty \right\}$$

$$d_B(X,Y) = \inf_{\gamma}\sup_{x\in X} ||x-\gamma(x)||_\infty,$$

where we consider all possible bijections $\gamma : X \to Y$. Here, we use $||p-q||_\infty = \max\{|p_1-q_1|, |p_2-q_2|\}$ for $p, q \in \mathbb{R}^2$. We also remark the following inequality between these distances: $d_H(X,Y) \le d_B(X,Y)$ (see [5]).

Theorem 1 (Stability of persistent diagrams [3, 4, 5])**.** Let $\mathcal{X}$ be a topological space with tame functions $\alpha, \beta : \mathcal{X} \to \mathbb{R}$. Then, the following stability property holds:

$$d_B(\mathrm{dgm}(\alpha), \mathrm{dgm}(\beta)) \le ||\alpha - \beta||_\infty. \qquad (2)$$

IV. Frames Analysis and Persistent Homology

Our objective is now to combine the core concepts in frame theory with persistent diagrams in order to combine the strength and features of these different analysis tools. Our theorem provides stability properties of persistent diagrams of frame transforms $|Vf|$, when considering a frame decomposition $Vf \in L^2(\mathcal{X})$. We assume the frame parametrization space $\mathcal{X}$ to have a counting measure, which is anyway the case when considering discrete structures for applications.

Theorem 2. Let $f, g \in \mathcal{H}$ and $|Vf|$, $|Vg|$ tame functions with $V : \mathcal{H} \to L^2(\mathcal{X})$ a frame analysis operator with upper bound B. We consider a discrete topological space $\mathcal{X}$ with a counting measure. Then, the following stability property holds:

$$d_B(\mathrm{dgm}(|Vf|), \mathrm{dgm}(|Vg|)) \le \sqrt{B}||f-g||_{\mathcal{H}}.$$

Proof: This is a consequence of the inequality (1) (the bounding of the norm of the frame operator) and the stability of the persistent diagrams described in the inequality (2):

$$\begin{aligned} d_B(\mathrm{dgm}(|Vf|), &\mathrm{dgm}(|Vg|)) \\ &\le ||\,|Vf| - |Vg|\,||_\infty \\ &\le ||Vf - Vg||_2 \\ &\le ||V||\,||f-g||_{\mathcal{H}} \\ &= \sqrt{||V^*V||}||f-g||_{\mathcal{H}} \\ &= \sqrt{||S||}\,||f-g||_{\mathcal{H}} \\ &\le \sqrt{B}||f-g||_{\mathcal{H}}, \end{aligned}$$

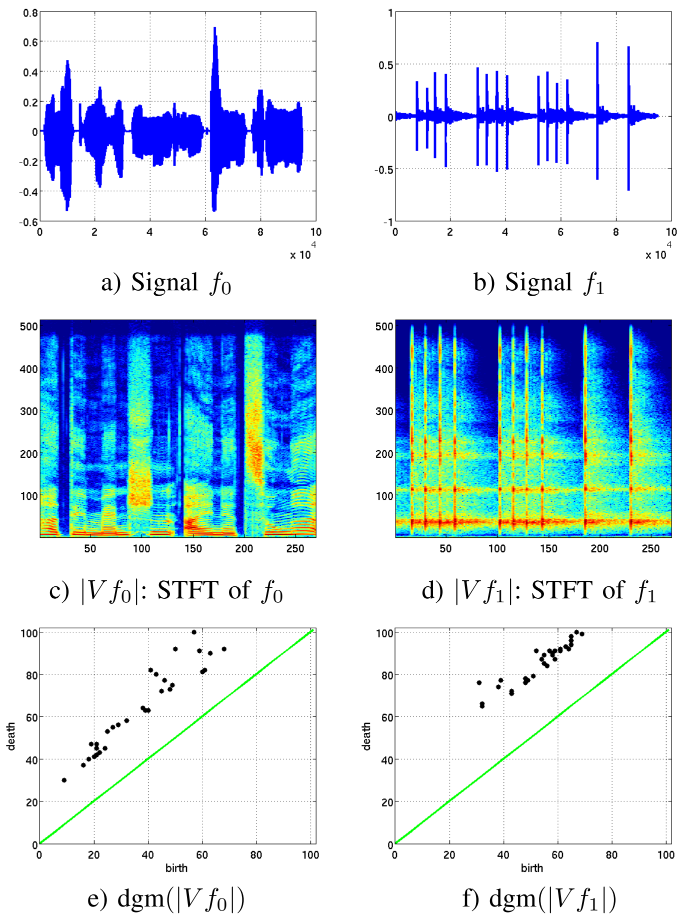
where we use $||V||^2 = ||V^*V||$. ∎

This proposition is an initial step towards the integration of frame theory and persistent stability. We remark that important developments have been achieved in generalizing the work in [5], and the inequality (2), by avoiding the restrictions imposed by the functional setting and expressing the stability in a purely algebraic language (see [1, 3, 4]). The usage of these more flexible and general stability properties is a natural future step in our program.

A. Experiments

We now experiment with acoustic signals the interaction between the components in our framework (frame transformations and persistent diagrams). A main objective is to study both the stability and the discriminative power of persistent diagrams in the setting of frame theory. We consider two signals f_0 and f_1 together with a process transforming f_0 into f_1 encoded with a family of signals $\{f_t\}_{0\le t\le 1}$ defined as:

$$f_t = (1-t)f_0 + tf_1, \quad t \in [0,1].$$



a) Signal f_0 b) Signal f_1
c) $|Vf_0|$: STFT of f_0 d) $|Vf_1|$: STFT of f_1
e) $\mathrm{dgm}(|Vf_0|)$ f) $\mathrm{dgm}(|Vf_1|)$

Fig. 1. Time-frequency plots and discriminative properties of persistence

In Fig. 1 (a) and Fig. 1 (b), the plots of f_0 and f_1 are shown, and they represent a female speech recording and a castanet signal respectively. In Fig. 1(c) and Fig. 1(d) the corresponding spectrograms (STFT) $|Vf_0|$ and $|Vf_1|$ are displayed, indicating different frequency characteristics. Here, the horizontal axis refers to the time domain, and the vertical axis corresponds to the frequency domain. The case of the speech signal f_0 presents a mixture of harmonic and transitory effects originated by vocal and consonants features of a speech signal. In the case of the castanets signal f_1, a sequence of transients are displayed indicating the complex frequency behavior of the rapid series of clicks.

The spectrograms $|Vf_0|$ and $|Vf_1|$ are then fed to a persistent homology algorithm by considering its level sets as indicated in Definitions 3 and 4. We use a Morse-theory based algorithm that analyzes a quantized version of an input function, and feeds the resulting level sets to an efficient persistent homology implementation, see [13]. In our persistent diagrams of Fig. 1(e) and Fig. 1(f), we have selected only the 30 most prominent 1-dimensional homological structures, displayed by the 30 dots with the largest distance to the diagonal in the persistent diagram. We are therefore not considering topological unstable (noisy) components represented by dots, or 1-homology features, located in closer regions to the diagonal in Figures 1(e) and 1(f). These persistent diagrams are homological fingerprints characterizing the shape of the corresponding spectrograms. Notice that these homological structures are clearly identifying and discriminating these spectrograms using a limited set of homological components. This description represents a new type of topological characterization of time-frequency data.

As indicated in Theorem 2, the persistent diagram $\mathrm{dgm}(|Vf|)$ has the crucial property to be robust with respect to perturbations of the signal f. This important feature can be used to illustrate the discriminative power of persistent homology by studying the distances between persistent diagrams $\mathrm{dgm}(|Vf_0|)$ and $\mathrm{dgm}(|Vf_t|)$, for $t \in [0, 1]$. In Fig. 2, we display the function $d(t) := d_H(\mathrm{dgm}(|Vf_0|), \mathrm{dgm}(|Vf_t|))$ using the Hausdorff distance, whose implementation is simpler and it does not interfere with the stabilty property, due to the inequality $d_H(X, Y) \le d_B(X, Y)$ (see Definition 5). Notice that when the parameter t increases from 0 to 1, the Hausdorff distance between $\mathrm{dgm}(|Vf_0|)$ and $\mathrm{dgm}(|Vf_t|)$ increases, which indeed resonates with the discriminative properties of persistent homology in the setting of frame analysis.

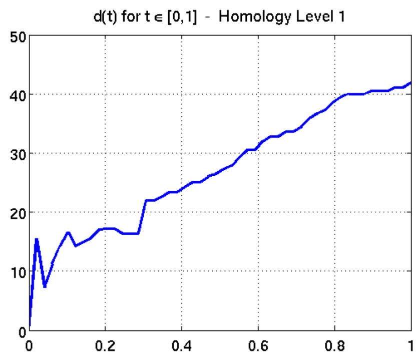


Fig. 2. $d(t) := d_H(\mathrm{dgm}(|Vf_0|), \mathrm{dgm}(|Vf_t|)),\ t \in [0, 1]$

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