

Shannon Sampling and Parseval Frames on Compact Manifolds

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I. INTRODUCTION

The problem of representation and analysis of functions defined on manifolds (signals, images, and data in general) is ubiquitous in many fields ranging from statistics and cosmology to neuroscience and biology. It is very common to consider input signals as points in a high-dimensional measurement space, however, meaningful structures lay on a manifold embedded in this space.

In the last decades, the importance of these applications triggered the development of various generalized wavelet bases suitable for the unit spheres S^2 and S^3 and the rotation group of \mathbf{R}^3 . The goal of the present study is to describe a general approach to bandlimited localized Parseval frames in a space $L_2(\mathbf{M})$, where \mathbf{M} is a compact homogeneous Riemannian manifold.

One can think of a Riemannian manifold as of a surface in a Euclidean space. A homogeneous manifold is a surface with "many" symmetries like the sphere $x_1^2 + \dots + x_d^2 = 1$ in Euclidean space \mathbf{R}^d .

Our construction of frames in a function space $L_2(\mathbf{M})$ heavily depends on proper notions of bandlimitedness and Shannon-type sampling on a manifold \mathbf{M} . The crucial role in this development is played by positive cubature formulas (Theorem 1.3) and by the product property (Theorem 1.2), which were proved in [1] and [10].

The notion of bandlimitedness on a compact manifold \mathbf{M} is introduced in terms of eigenfunctions of a certain second-order differential elliptic operator on \mathbf{M} . The most important fact for our construction of frames is that in a space of ω -bandlimited functions the regular $L_2(\mathbf{M})$ norm can be discretized. This result in the case of compact manifolds (and even non-compact manifolds of bounded geometry) was first discovered and explored in many ways in our papers [?]-[?]. In the classical cases of straight line \mathbf{R} and circle \mathbf{S} the corresponding results are known as Plancherel-Polya and Marcinkiewicz-Zygmund inequalities. Our generalization of Plancherel-Polya and Marcinkiewicz-Zygmund inequalities implies that ω -bandlimited functions on manifolds are completely determined by their values on discrete sets of points "uniformly" distributed over \mathbf{M} with a spacing comparable to $1/\sqrt{\omega}$ and can be completely reconstructed in a stable way from their values on such sets. The last statement is an extension of the Shannon sampling theorem to the case of

Riemannian manifolds.

Our article is a summary of some results for Riemannian manifolds that were obtained in [1]-[12]. To the best of our knowledge these are the pioneering papers which contain the most general results about frames, Shannon sampling, and cubature formulas on compact and non-compact Riemannian manifolds. In particular, the paper [1] gives an "end point" construction of tight localized frames on homogeneous compact manifolds. The paper [11] is the first systematic development of localized frames on compact domains in Euclidean spaces.

A. Compact homogeneous manifolds

A homogeneous compact manifold \mathbf{M} is a C^∞ -compact manifold on which a compact Lie group G acts transitively. In this case \mathbf{M} is necessary of the form G/H , where H is a closed subgroup of G . The notation $L_2(\mathbf{M})$, is used for the usual Hilbert spaces, where dx is an invariant measure.

If \mathfrak{g} is the Lie algebra of a compact Lie group G then it is a direct sum $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g}, \mathfrak{g}]$, where \mathfrak{a} is the center of \mathfrak{g} , and $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple algebra. Let Q be a positive-definite quadratic form on \mathfrak{g} which, on $[\mathfrak{g}, \mathfrak{g}]$, is opposite to the Killing form. Let X_1, \dots, X_d be a basis of \mathfrak{g} , which is orthonormal with respect to Q . Since the form Q is $Ad(G)$ -invariant, the operator

$$-X_1^2 - X_2^2 - \dots - X_d^2, \quad d = \dim G$$

is a bi-invariant operator on G , which is known as the Casimir operator. This implies in particular that the corresponding operator on $L_2(\mathbf{M})$,

$$L = -D_1^2 - D_2^2 - \dots - D_d^2, \quad D_j = D_{X_j}, \quad d = \dim G, \quad (1)$$

commutes with all operators $D_j = D_{X_j}$. Operator L , which is usually called the Laplace operator, is the image of the Casimir operator under differential of quazi-regular representation in $L_2(\mathbf{M})$. Note that if $\mathbf{M} = G/H$ is a compact symmetric space then the number $d = \dim G$ of operators in the formula (1) can be strictly bigger than the dimension $n = \dim \mathbf{M}$. For example on a two-dimensional sphere \mathbf{S}^2 the Laplace-Beltrami operator $L_{\mathbf{S}^2}$ is written as $L_{\mathbf{S}^2} = D_1^2 + D_2^2 + D_3^2$, where $D_i, i = 1, 2, 3$, generates a rotation in \mathbf{R}^3 around coordinate axis x_i : $D_i = x_j \partial_k - x_k \partial_j$, where $j, k \neq i$.

It is important to realize that in general, the operator L is not necessarily the Laplace-Beltrami operator of the natural

invariant metric on \mathbf{M} . But it coincides with such operator at least in the following cases:

- 1) If the manifold \mathbf{M} is itself a compact Lie group G then L is exactly the Laplace-Beltrami operator of an invariant metric on G . In particular it happens if \mathbf{M} is an n -dimensional torus, and L is the sum of squares of partial derivatives;
- 2) If $\mathbf{M} = G/H$ is a compact symmetric space of rank one, then the operator L is proportional to the Laplace-Beltrami operator of an invariant metric on G/H . This follows from the fact that, in the rank one case, every second-order operator which commutes with all isometries $x \rightarrow g \cdot x$, $x \in \mathbf{M}$, $g \in G$, is proportional to the Laplace-Beltrami operator. The important examples of such manifolds are spheres and projective spaces.

Since manifold \mathbf{M} is compact and L is a second-order differential elliptic self-adjoint positive definite operator $L_2(\mathbf{M})$ it has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ which goes to infinity and there exists a complete family $\{u_j\}$ of orthonormal eigenfunctions which form a basis in $L_2(\mathbf{M})$.

Definition 1.1: The span of eigenfunctions u_j

$$Lu_j = \lambda_j u_j$$

with $\lambda_j \leq \omega$, $\omega > 0$, is denoted as $\mathbf{E}_\omega(L)$ and is called the space of bandlimited functions on \mathbf{M} of bandwidth ω .

According to the Weyl's asymptotic formula one has

$$\dim \mathbf{E}_\omega(L) \sim C \text{Vol}(\mathbf{M}) \omega^{n/2}, \quad (2)$$

where $n = \dim \mathbf{M}$ and C is an absolute constant.

Let $B(x, r)$ be a metric ball on a compact Riemannian manifold \mathbf{M} whose center is x and radius is r . The following lemma can be found in [2], [5].

Lemma 1.1: There exists a natural number $N_{\mathbf{M}}$, such that for any sufficiently small $\rho > 0$, there exists a set of points $\{x_k\}$ such that:

- 1) the balls $B(x_k, \rho/4)$ are disjoint,
- 2) the balls $B(x_k, \rho/2)$ form a cover of \mathbf{M} ,
- 3) the multiplicity of the cover by balls $B(x_k, \rho)$ is not greater than $N_{\mathbf{M}}$.

Definition 1.2: Any set of points $\mathbf{M}_\rho = \{x_k\}$ which is described in Lemma 1.1 will be called a metric ρ -lattice.

The following theorems are of primary importance.

Theorem 1.2: (Product property [1], [10]) If $\mathbf{M} = G/H$ is a compact homogeneous manifold and L is the same as above, then for any f and g belonging to $\mathbf{E}_\omega(L)$, their product fg belongs to $\mathbf{E}_{4d\omega}(L)$, where d is the dimension of the group G .

Remark 1: At this moment it is not known if the constant $4d$ can be lowered in general situation. However, it is easy to verify that in the case of two-point homogeneous manifolds (which include spheres and projective spaces) a stronger result holds: if $f, g \in \mathbf{E}_\omega(L)$ then $fg \in \mathbf{E}_{2\omega}(L)$.

Theorem 1.3: (Cubature formula [1], [10]) There exists a positive constant $c = c(\mathbf{M})$, such that if $\rho = c\omega^{-1/2}$, then for any ρ -lattice \mathbf{M}_ρ , there exist strictly positive coefficients

$\alpha_{x_k} > 0$, $x_k \in \mathbf{M}_\rho$, for which the following equality holds for all functions in $\mathbf{E}_\omega(\mathbf{M})$:

$$\int_{\mathbf{M}} f dx = \sum_{x_k \in \mathbf{M}_\rho} \alpha_{x_k} f(x_k). \quad (3)$$

Moreover, there exists constants c_1, c_2 , such that the following inequalities hold:

$$c_1 \rho^n \leq \alpha_{x_k} \leq c_2 \rho^n, \quad n = \dim \mathbf{M}. \quad (4)$$

II. HILBERT FRAMES

Since eigenfunctions have perfect localization properties in the spectral domain they cannot be localized on the manifold.

It is the goal of our development to construct "better bases" in corresponding $L_2(\mathbf{M})$ spaces which will have rather strong localization on a manifold and in the spectral domain.

In fact, the "kind of basis" which we are going to construct is known today as a frame.

A set of vectors $\{\psi_v\}$ in a Hilbert space \mathcal{H} is called a frame if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$

$$A \|f\|_2^2 \leq \sum_v |\langle f, \psi_v \rangle|^2 \leq B \|f\|_2^2. \quad (5)$$

The largest A and smallest B are called lower and upper frame bounds.

The set of scalars $\{\langle f, \psi_v \rangle\}$ represents a set of measurements of a signal f . To synthesize signal f from this set of measurements one has to find another (dual) frame $\{\Psi_v\}$ and then a reconstruction formula is

$$f = \sum_v \langle f, \psi_v \rangle \Psi_v. \quad (6)$$

Dual frame is not unique in general. Moreover it is difficult to find a dual frame. If in particular $A = B = 1$ the frame is said to be tight or Parseval.

The main feature of Parseval frames is that decomposing and synthesizing a signal or image from known data are tasks carried out with the same set of functions. In other words in (6) one can have $\Psi_v = \psi_v$.

Parseval frames are similar in many respects to orthonormal wavelet bases. For example, if in addition all vectors ψ_v are unit vectors, then the frame is an orthonormal basis. However, the important differences between frames and, say, orthonormal bases is their redundancy that helps reduce the effect of noise in data.

Frames in Hilbert spaces of functions whose members have simultaneous localization in space and frequency arise naturally in wavelet analysis on Euclidean spaces when continuous wavelet transforms are discretized. Such frames have been constructed, studied, and employed extensively in both theoretical and applied problems.

III. BANDLIMITED LOCALIZED PARSEVAL FRAMES ON COMPACT HOMOGENEOUS MANIFOLDS

According to spectral theorem if F is a Schwartz function on the line, then there is a well defined operator $F(L)$ in the

space $L_2(\mathbf{M})$ such that for any $f \in L_2(\mathbf{M})$ one has

$$(F(L)f)(x) = \int_{\mathbf{M}} \mathcal{K}^F(x, y) f(y) dy, \quad (7)$$

where dy is the invariant normalized measure on \mathbf{M} and

$$\mathcal{K}^F(x, y) = \sum_{j=0}^{\infty} F(\lambda_j) u_j(x) \overline{u_j(y)}. \quad (8)$$

We will be especially interested in operators of the form $F(t^2L)$, where F is a Schwartz function and $t > 0$. The corresponding kernel will be denoted as $\mathcal{K}_t^F(x, y)$ and

$$\mathcal{K}_t^F(x, y) = \sum_{j=0}^{\infty} F(t^2\lambda_j) u_j(x) \overline{u_j(y)}. \quad (9)$$

Note, that variable t here is a kind of scaling parameter.

Localization properties of the kernel $\mathcal{K}_t^F(x, y)$ are given in the following statement.

Lemma 3.1: If L is an elliptic self-adjoint second order differential operators on compact manifolds, then the following holds

1) If F is any Schwartz function on \mathbf{R} , then

$$\mathcal{K}_t^F(x, x) \sim c t^{-d}, \quad t \rightarrow 0. \quad (10)$$

2) If, in addition, $F \in C_c^\infty(\mathbf{R})$ is even, then on $\mathbf{M} \times \mathbf{M} \setminus \Delta$, where $\Delta = \{(x, x)\}$, $x \in \mathbf{M}$, $\mathcal{K}_t^F(x, y)$ vanishes to infinite order as t goes to zero.

Let $g \in C^\infty(\mathbf{R}_+)$ be a monotonic function such that $\text{supp } g \subset [0, 2^2]$, and $g(s) = 1$ for $s \in [0, 1]$, $0 \leq g(s) \leq 1$, $s > 0$. Setting $G(s) = g(s) - g(2^2s)$ implies that $0 \leq G(s) \leq 1$, $s \in \text{supp } G \subset [2^{-2}, 2^2]$. Clearly, $\text{supp } G(2^{-2j}s) \subset [2^{2j-2}, 2^{2j+2}]$, $j \geq 1$. For the functions $F_0(s) = \sqrt{g(s)}$, $F_j(s) = \sqrt{G(2^{-2j}s)}$, $j \geq 1$, one has $\sum_{j \geq 0} F_j^2(s) = 1$, $s \geq 0$. Using the spectral theorem for L one can define bounded self-adjoint operators $F_j(L)$ as

$$F_j(L)f(x) = \int_{\mathbf{M}} \mathcal{K}_{2^{-j}}^F(x, y) f(y) dy,$$

where

$$\mathcal{K}_{2^{-j}}^F(x, y) = \sum_{\lambda_m \in [2^{2j-2}, 2^{2j+2}]} F(2^{-2j}\lambda_m) u_m(x) \overline{u_m(y)}. \quad (11)$$

The same spectral theorem implies $\sum_{j \geq 0} F_j^2(L)f = f$, $f \in L_2(\mathbf{M})$, and taking inner product with \bar{f} gives

$$\|f\|^2 = \sum_{j \geq 0} \langle F_j^2(L)f, f \rangle = \sum_{j \geq 0} \|F_j(L)f\|^2. \quad (12)$$

Moreover, since the function $F_j(s)$ has its support in $[2^{2j-2}, 2^{2j+2}]$ the functions $F_j(L)f$ are bandlimited to $[2^{2j-2}, 2^{2j+2}]$.

Consider the sequence $\omega_j = 2^{2j+2}$, $j = 0, 1, \dots$. By (12) the equality $\|f\|^2 = \sum_{j \geq 0} \|F_j(L)f\|^2$ holds, were every function $F_j(L)f$ is bandlimited to $[2^{2j-2}, 2^{2j+2}]$. Since for every $F_j(L)f \in \mathbf{E}_{2^{2j+2}}(L)$ one can use Theorem 1.2 to conclude that

$$|F_j(L)f|^2 \in \mathbf{E}_{4d2^{2j+2}}(L),$$

where $d = \dim G$, $\mathbf{M} = G/H$. This shows that for every $f \in L_2(\mathbf{M})$ we have the following decomposition

$$\sum_{j \geq 0} \|F_j(L)f\|_2^2 = \|f\|_2^2, \quad |F_j(L)f|^2 \in \mathbf{E}_{4d2^{2j+2}}(L). \quad (13)$$

According to Theorem 1.3 there exists a constant $a > 0$ such that for all integers j if

$$\rho_j = ad^{-1/2}2^{-j} \sim 2^{-j}, \quad d = \dim G, \quad \mathbf{M} = G/H, \quad (14)$$

then for any ρ_j -lattice \mathbf{M}_{ρ_j} one can find coefficients $\mu_{j,k}$ with $\mu_{j,k} \sim \rho_j^n$, $n = \dim \mathbf{M}$, for which the following exact cubature formula holds

$$\|F_j(L)f\|_2^2 = \sum_{k=1}^{K_j} \mu_{j,k} |F_j(L)f(x_{j,k})|^2, \quad (15)$$

where $x_{j,k} \in \mathbf{M}_{\rho_j}$, $k = 1, \dots, K_j = \text{card}(\mathbf{M}_{\rho_j})$. Using the kernel $\mathcal{K}_{2^{-j}}^F$ of the operator $F_j(L)$ we define the functions

$$\Theta_{j,k}(y) = \sqrt{\mu_{j,k}} \overline{\mathcal{K}_{2^{-j}}^F}(x_{j,k}, y) = \sqrt{\mu_{j,k}} \sum_{\lambda_m \in [2^{2j-2}, 2^{2j+2}]} \overline{F}(2^{-2j}\lambda_m) \overline{u_m(x_{j,k})} u_m(y). \quad (16)$$

We find that for every $f \in L_2(\mathbf{M})$ the following equality holds $\|f\|_2^2 = \sum_{j,k} |\langle f, \Theta_{j,k} \rangle|^2$.

Theorem 3.2: (Kernel localization [1]) If \mathbf{M} is compact then the functions $\Theta_{j,k}$ are localized around the points $x_{j,k}$ in the sense that for any $N > 0$ there exists a $C(N) > 0$ such that

$$|\Theta_{j,k}(x)| \leq C(N) \frac{2^{dj}}{\max(1, 2^j d(x, x_{j,k}))^N}, \quad (17)$$

for all natural j .

Theorem 3.3: (Bandlimited localized Parseval localized frames on homogeneous manifolds) For any compact homogeneous manifold \mathbf{M} the set of functions $\{\Theta_{j,k}\}$, constructed in (16) forms a Parseval frame in the Hilbert space $L_2(\mathbf{M})$. In particular the following reconstruction formula holds true

$$f = \sum_{j \geq 0} \sum_{k=1}^{K_j} \langle f, \Theta_{j,k} \rangle \Theta_{j,k}, \quad (18)$$

with convergence in $L_2(\mathbf{M})$. Every $\Theta_{j,k}$ is bandlimited to $[2^{2j-2}, 2^{2j+2}]$ and its localization on manifold is given by (17).

The condition (14) imposes a specific rate of sampling in (15). It is interesting to note that this rate is essentially optimal. Indeed, on one hand the Weyl's asymptotic formula (2) gives the dimension of the space $\mathbf{E}_\omega(L)$. On the other hand, the condition (14) and the definition of a ρ -lattice imply that the number of points in an "optimal" lattice \mathbf{M}_{ρ_j} for $\rho_j \sim 2^{-j}$ can be approximately estimated as

$$\text{card } \mathbf{M}_{\rho_j} \sim c \frac{\text{Vol}(\mathbf{M})}{2^{-jn/2}} = c \text{Vol}(\mathbf{M}) 2^{jn/2}, \quad n = \dim \mathbf{M},$$

which is in agreement with the Weyl's formula (2) with $\omega \sim 2^j$.

IV. SHANNON SAMPLING OF BANDLIMITED FUNCTIONS

We consider an even $F \in C_c^\infty(\mathbf{R})$ which equals 1 on $[-1, 1]$, and which is supported in $[-\Omega, \Omega]$, $\Omega > 1$. Let $\mathcal{K}_{\Omega^{-1/2}}^F(x, y)$ be the kernel of $F(\Omega^{-1}L)$ defined by (9). If $0 < \omega \leq \Omega$ then since $F(\Omega^{-1}\lambda_k) = 1$ whenever $\lambda_k \leq \omega$, we have that according to (7) - (9) for every $f \in \mathbf{E}_\omega(L)$ the following reproducing formula holds

$$f(x) = [F(\Omega^{-1}L)f](x) = \int_{\mathbf{M}} \mathcal{K}_{\Omega^{-1/2}}^F(x, y)f(y)dy \quad (19)$$

where dy is the normalized invariant measure. Clearly, for a fixed $x \in \mathbf{M}$ the kernel $\mathcal{K}_{\Omega^{-1/2}}^F(x, y)$ as a function in y belongs to $\mathbf{E}_\Omega(L)$. Thus, for $f \in \mathbf{E}_\omega(L)$, $\omega < \Omega$, the Product property (Theorem 1.2) implies that the product $\mathcal{K}_{\Omega^{-1/2}}^F(x, y)f(y)$ belongs to $\mathbf{E}_{4d\Omega}(L)$, where $d = \dim G$. Now an application of the Cubature formula (Theorem 1.3) implies the following theorem.

Theorem 4.1: For every compact homogeneous manifold $\mathbf{M} = G/H$ there exists a constant $c = c(\mathbf{M})$ such that for any $\Omega > 0$ and any lattice $\mathbf{M}_\rho = \{x_k\}_{k=1}^{m_\Omega}$ with $\rho = c\Omega^{-1/2}$ one can find positive weights μ_k

$$\mu_k \asymp \Omega^{-n/2}, \quad n = \dim \mathbf{M},$$

such that for any $f \in \mathbf{E}_\omega(L)$ with $\omega \leq \Omega$ the following analog of the Shannon formula holds

$$f(x) = \sum_{k=1}^{m_\Omega} \mu_k f(x_k) \mathcal{K}_{\Omega^{-1/2}}^F(x, x_k), \quad f \in \mathbf{E}_\omega(L). \quad (20)$$

Remark 2: Note that our definition of a ρ -lattice and the Weyl's asymptotic formula (2) for eigenvalues of L imply that m_Ω is "essentially" the dimension of the space $\mathbf{E}_{4d\Omega}(L)$ with $d = \dim G$. In other words there exists a constants $C_1(\mathbf{M}) > 0$, $C_2(\mathbf{M}) > 0$ (which are independent on Ω) such that the number m_Ω of sampling points satisfies the following inequalities

$$C_1(\mathbf{M})\Omega^{n/2} \leq m_\Omega \leq C_2(\mathbf{M})\Omega^{n/2} \\ C_1(\mathbf{M})\mathbf{E}_{4d\Omega}(L) \leq m_\Omega \leq C_2(\mathbf{M})\mathbf{E}_{4d\Omega}(L). \quad (21)$$

Remark 3: Lemma 3.1 shows that for large Ω functions $\mathcal{K}_{\Omega^{-1/2}}^F(x, x_k)$ in (20) are essentially localized around sampling points x_k .

V. A DISCRETE FORMULA FOR EVALUATING FOURIER COEFFICIENTS ON MANIFOLDS.

As another application of the Product Property and the Cubature Formula, we prove an analog of the Shannon Sampling Theorem on compact homogeneous manifolds.

Theorems 1.2 and 1.3 imply the following theorem which shows that on a compact homogeneous manifold \mathbf{M} there are finite sets of points which yield exact discrete formulas for computing Fourier coefficients of bandlimited functions.

Theorem 5.1: For every compact homogeneous manifold $\mathbf{M} = G/H$ there exists a constant $c = c(\mathbf{M})$ such that

for any $\omega > 0$ and any lattice $\mathbf{M}_\rho = \{x_k\}_{k=1}^{r_\omega}$ with $\rho = c\omega^{-1/2}$ one can find positive weights μ_k comparable to $\omega^{-n/2}$, $n = \dim \mathbf{M}$, such that Fourier coefficients $c_i(f)$ of any f in $\mathbf{E}_\omega(L)$ with respect to the basis $\{u_i\}_{i=1}^\infty$ can be computed by the following *exact* formula

$$c_i(f) = \int_{\mathbf{M}} f(x)\overline{u_i}(x)dx = \sum_{k=1}^{r_\omega} \mu_k f(x_k)\overline{u_i}(x_k),$$

with r_ω satisfying relations

$$C_1(\mathbf{M})\omega^{n/2} \leq r_\omega \leq C_2(\mathbf{M})\omega^{n/2}$$

$$C_1(\mathbf{M})\mathbf{E}_{4d\omega}(L) \leq r_\omega \leq C_2(\mathbf{M})\mathbf{E}_{4d\omega}(L), \quad (22)$$

where $C_1(\mathbf{M})$ and $C_2(\mathbf{M})$ are the same as in (21).

We obviously have the following "discrete" representation formula of f in $\mathbf{E}_\omega(L)$ in terms of eigenfunctions u_i

$$f = \sum_i \sum_{k=1}^{r_\omega} \mu_k f(x_k)\overline{u_i}(x_k)u_i. \quad (23)$$

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