# A Lie group approach to diffusive wavelets

Swanhild Bernstein
Institute of Applied Analysis
TU Bergakademie Freiberg
D-09596 Freiberg, Germany
Email: swanhild.bernstein@math.tu-freiberg.de

***Abstract*—The aim of this paper is to give an overview of diffusive wavelets on compact Lie groups, homogenous spaces and the Heisenberg group. This approach is based on Lie groups and representation theory and generalizes well-known constructions of wavelets on the sphere. We give also examples for the construction of diffusive wavelets.**

## I. Introduction

The task of analyzing data, reconstructing functions from measurements or to save data in an handable way occurs in a lot of applications. Problems in geophysics, astronomy and in material sciences involve groups and homogeneous spaces such as the group SO(3) or the spheres $S^2$ or $S^3$. The group theoretic approach to wavelets as coherent states fails and it has been an open problem for along time to construct wavelets on the sphere. The breakthrough was the construction of spherical wavelets on $S^2$ based on convolution-type integrals by W. Freeden and co-workers [14]. An alternate successful approach was made by J.-P. Antoine and P. Vandergheynst [1], [2] by lifting up rotations and dilations on the sphere into the Lorentz group. The aim of this paper is to demonstrate that both approaches can be generalized to continuous diffusive wavelets. Diffusive wavelets can be build not only on compact groups and homogeneous spaces but also on stratified groups. The most well-known example here is the Heisenberg group.

Classical wavelet theory is based on the group generated by translations and dilations. The key idea of diffusive wavelets is to generate a dilation from a diffusive semigroup and to substitute translation by action of a compact group. A related approach based on spectral calculus of the Laplacian on closed manifolds was proposed by D. Geller and A. Mayeli [15] and related work by I. Pesenson and D. Geller [17], [18].

The construction of diffusive wavelets is based on convolution-type operators. These ideas were used in [6] to construct wavelets on the sphere $S^3$. Discrete wavelet transforms of that type were used by R. R. Coifman, M. Maggioni and others [11], [10], where the heat evolution was combined with an orthogonalization procedure to model a multi-resolution analysis in $L^2(S^3)$.

These ideas can be combined with other group structures to get wavelets invariant under finite reflection groups [3]. A similar construction is possible for the torus in [5].

An application material sciences and specifically to the crystallographic Radon-transform [8], [7], [9] we need wavelets on $S^3$, $SO(3)$ and $S^2 \times S^2$ [4].

This approach was generalized by a representation theory based approach where the heat flow was replaced by a more general approximate convolution identity in [13] and [12].

The aim of this paper is to give a general approach to diffusive wavelets on compact groups, homogeneous spaces and stratified groups which is based on the theory of Lie groups. Several examples explain the construction of diffusive wavelets for specific situations.

## II. Diffusive wavelets

### A. Preliminaries on compact Lie groups

Let $\mathscr{G}$ be a compact Lie group. A unitary representation of $\mathscr{G}$ is a continuous group homomorphism $\pi\colon \mathscr{G} \to U(d_\pi)$ of $\mathscr{G}$ into the group of unitary matrices of a certain dimension $d_\pi$. Such a representation is irreducible if $\pi(g)M = M\pi(g)$ for all $g \in \mathscr{G}$ and some $M \in \mathbb{C}^{d_\pi \times d_\pi}$ implies $M = cId$, where $Id$ is the identity matrix.

**Theorem 1** (Peter-Weyl). *Let $\widehat{\mathscr{G}}$ be the set of all equivalence classes of irreducible representations of the compact Lie group $\mathscr{G}$, choose one unitary representation $\pi_\alpha(g)$ from each class, and let the dimension of the representation $\pi_\alpha(g)$ be $d_\alpha$, and its matrix elements be $\pi^\alpha_{ij}$, $1 \le i,j \le d_\alpha$, and $\mathcal{H}_\alpha = \operatorname{span}(\pi^\alpha_{ij})^{d_\alpha}_{i,j}$. Then*

$$L^2(\mathscr{G}) = \oplus_\alpha \mathcal{H}_{\pi_\alpha} = \oplus_{\pi\in\widehat{\mathscr{G}}} \mathcal{H}_\pi$$

*and any function $f \in L^2(\mathscr{G})$ has a unique decomposition into*

$$f(g) = \sum_\alpha \sum_{i,j}^{d_\alpha} c^\alpha_{ij} \pi^\alpha_{ij},$$

*with Fourier coefficents $c^\alpha_{ij}$.*
*The orthogonal projection $L^2(\mathscr{G}) \to \mathcal{H}_\alpha$ is given by*

$$f_\alpha(g) = \sum_{i,j}^{d_\alpha} c^\alpha_{ij}\pi^\alpha_{ij} = d_\alpha(\phi * \chi_{pi_\alpha}),$$

*where $\chi_{\pi_\alpha}(g) = \operatorname{trace}\pi_\alpha(g)$ is the character of the representation.*

The Fourier coefficient $\hat{f}(\pi)$ can be calculated as

$$\hat{f}(\pi) = \int_{\mathscr{G}} f(g)\pi^*(g)\,dg.$$

and the inversion formula (the Fourier expansion) is then given by

$$f(g) = \sum_{\pi \in \hat{\mathscr{G}}} d_\pi \operatorname{trace}(\pi(g)\hat{f}(\pi)).$$

The Laplace-Beltrami operator $\Delta_{\mathscr{G}}$ on $\mathscr{G}$ is bi-invariant. Therefore, all of its eigenspaces are also bi-invariant subspaces of $L^2(\mathscr{G})$. As $\mathcal{H}_\pi$ are minimal bi-invariant subspaces, each of them has to be an eigenspace of $\Delta_G$ with respect to an eigenvalue $-\lambda_\pi^2$. Hence,

$$\Delta_{\mathscr{G}}\phi = -\sum_{\pi \in \hat{\mathscr{G}}} d_\pi \lambda_\pi^2 \operatorname{trace}\left(\pi(g)\hat{\phi}(\pi)\right)$$

and the solution to the heat equation

$$(\partial_t - \Delta_{\mathscr{G}})u = 0,\ u(0,\cdot) = \phi,$$

is given as convolution with the heat kernel $p_t(g)$ as $u(t,\cdot) = \phi * p_t$, where

$$\hat{p}_t(\pi) = e^{-t\lambda_\pi^2} I \text{ and } p_t(g) = \sum_{\hat{\mathscr{G}}} d_\pi e^{-t\lambda_\pi^2}\chi_\pi(g).$$

In particular $\phi * p_t \to \phi$ for all $\phi \in L^p(\mathscr{G})$, $1 \le p < \infty$.

### B. Wavelets on compact groups

**Definition 1** (Diffusive approximate identity). *Let $\hat{\mathscr{G}}_+ \subset \hat{\mathscr{G}}$ be cofinite. A family $t \to p_t$ from $C^1(\mathbb{R}_+; L^1(\mathscr{G}))$ will be called diffusive approximate identity with respect to $\hat{\mathscr{G}}_+$ if it satisfies*

- *$||\hat{p}_t(\pi)|| \le C$ uniform in $\pi \in \hat{\mathscr{G}}_+$ and $t \in \mathbb{R}_+$;*
- *$\lim_{t\to 0} \hat{p}_t(\pi) = I$ for all $\pi \in \hat{\mathscr{G}}_+$;*
- *$\lim_{t\to\infty} \hat{p}_t(\pi) = 0$ for all $\pi \in \hat{\mathscr{G}}_+$;*
- *$-\partial_t\hat{p}_t(\pi)$ is a positive matrix for all $t \in \mathbb{R}_+$ and $\lim_{t\to 0} \hat{p}_t(\pi) = I$ for all $\pi \in \hat{\mathscr{G}}_+$.*

For $f \in L^2(\mathscr{G})$ the projection onto $L_0^2(\mathscr{G})$ is

$$f\,|_{\widehat{\mathscr{G}}_+} = \sum_{\pi \in \widehat{\mathscr{G}}_+} f * \chi_\pi.$$

**Definition 2** (Diffusive wavelets on a compact Lie group). *Let $p_t$ be a diffusive approximate identity and $\alpha(\rho) > 0$ a given weight function.*
*A family $\psi_\rho \in L_0^2(\mathscr{G}) = \bigoplus_{\pi\in\hat{\mathscr{G}}_+} H_\pi$ is called* diffusive wavelet family*, if it satisfies the admissibility condition*

$$p_t|_{\hat{\mathscr{G}}_+} = \int_t^\infty \check{\psi}_\rho * \psi_\rho\, \alpha(\rho)\, d\rho,$$

*where $\check{\psi}_\rho(g) = \overline{\psi_\rho(g^{-1})}$.*

Applying Fourier transform to the admissibility condition yields:

$$\hat{p}_t(\pi) = \int_t^\infty \hat{\psi}_\rho(\pi)\hat{\psi}_\rho^*(\pi)\alpha(\rho).$$

Differentiation with respect to $t$ results in

$$-\partial_t\hat{p}_t(\pi) = \hat{\psi}_\rho(\pi)\hat{\psi}_\rho^*(\pi)\alpha(\rho).$$

If $\widehat{\psi}_\pi(\pi)$ are the Fourier coefficients than a multiplication with a unitary matrix $\eta_\rho(\pi)$ does not change the last equality.

### C. Wavelets based on the heat kernel

Let $p_t$ be the heat kernel $e_t^{heat}$ on the group $\mathscr{G}$. We know that

$$\lim_{t\to\infty} \hat{e}_t^{heat}(\pi) = 0$$

for all nontrivial representations of $\mathscr{G}$. Since the character of the trivial representation $\pi_0$ is $\chi_{\pi_0} \equiv 1$ the corresponding invariant subspace in $L^2(\mathscr{G})$ is the space of constant functions and hence the eigenvalue vanishes, which implies $\hat{e}_t^{heat}(\pi) = Id$ and contradicts the definition of the diffusive approximate identity. Therefore we choose

$$\widehat{\mathscr{G}}_+ = \widehat{\mathscr{G}}\backslash\{\pi_0\}.$$

That means $L_0^2(\mathscr{G})$ contains all square integrable functions with vanishing mean. The admissibility condition reads now as

$$\partial_\rho\hat{e}_\rho^{heat}(\pi) = \lambda_\pi^2 e^{-\rho\lambda_\pi^2} Id = \hat{\psi}_\rho(\pi)\hat{\psi}_\rho^*(\pi)\alpha(\rho).$$

Due to the freedom in choosing a unitary matrix $\eta_\rho(\pi)$ we get

$$\hat{\psi}_\rho(\pi) = \tfrac{1}{\sqrt{\alpha(\rho)}}\lambda_\pi e^{-\lambda_\pi\frac{\rho}{2}}\, Id$$

and the wavelet has the form

$$\psi_\rho = \tfrac{1}{\sqrt{\alpha(\rho)}} \sum_{\pi\in\widehat{G}_+} d_\pi\lambda_\pi e^{-\lambda_\pi\frac{\rho}{2}} \operatorname{trace}\left(\eta_\rho(\pi)\pi(g)\right).$$

**Definition 3** (Wavelet transform). *Let $\mathscr{G}$ be a compact group, $\alpha(\rho) > 0$ a weight function on $\mathscr{G}$ and $\psi_\rho \in L_0^2(\mathscr{G})$ a diffusive wavelet family. The wavelet transform $\mathscr{W} : L_0^2(\mathscr{G}) \to L^2(\mathbb{R}_+ \times \mathscr{G},\, \alpha(\rho)\, d\rho \otimes dg)$ is defined as*

$$(\mathscr{W} f)(\rho,\, g) := (f * \check{\psi}_\rho)(g)$$

**Theorem 2.** *The wavelet transform $\mathscr{W} : L_0^2(\mathscr{G}) \to L^2(\mathbb{R}_+\times\mathscr{G}, \alpha(\rho)\, d\rho\otimes dg)$ is a unitary operator and the wavelet transform is invertible on its range by*

$$\int_{\mathbb{R}_+}\int_{\mathscr{G}} (\mathscr{W} f)(\rho,\, h)\, \psi_\rho(h^{-1}g)\, dh\, \alpha(\rho)\, d\rho$$
$$= \int_{\to 0}^\infty (\mathscr{W} f)(\rho,\cdot) * \psi_\rho\, \alpha(\rho)\, d\rho = f(g), \quad \forall f \in L_0^2(\mathscr{G}).$$

### D. Wavelets on homogeneous spaces

We have two options to construct wavelets on homogeneous spaces:
The naive way: We apply the wavelet transform to the lifted function $\tilde{f}(g) = f(g\cdot x_0)$ with base-point $x_0 \in \mathscr{X} = \mathscr{G}/\mathscr{H}$ for some $f \in L^2(\mathscr{X})$. This defines a function on $\mathbb{R}_+\times\mathscr{G}$ via

$$(\mathscr{W}\tilde{f})(\rho, g) = \int_{\mathscr{G}} \tilde{f}(h)\check{\psi}_\rho(h^{-1}g)\, dh$$
$$= \int_{\mathscr{G}} f(h\cdot x_0)\check{\psi}_\rho(h^{-1}g)\, dh$$

But we would prefer to have a transform living on $\mathbb{R}_+ \times \mathscr{X}$ instead of $\mathbb{R}_+ \times \mathscr{G}$.

For that we introduce the following zonal product

$$f\hat{\bullet}\psi(x) = \int_{\mathscr{G}} \overline{f(g\cdot x_0)}\psi(g\cdot x)\, dg \in L^1(\mathscr{G}).$$

**Definition 4.** *Let $\mathscr{X} = \mathscr{G}/\mathscr{H}$ be a homogeneous space and $p_t$ be a diffusive approximate identity and $\alpha(\rho) > 0$ be a given weight function. A family $\psi_\rho \in L^2(\mathscr{X})$ is called a* diffusive wavelet family *if the admissibility condition*

$$p_t^{\mathscr{X}}(x)\big|_{\hat{\mathscr{G}}_+} = \int_t^\infty \psi_\rho \hat{\bullet} \psi_\rho(x)\, \alpha(\rho)\, d\rho$$

*is satisfied.*
*We associate to this family the wavelet transform*

$$(\mathscr{W}_{\mathscr{X}} f)(\rho, g) = f \bullet \psi_\rho(g) = \int_{\mathscr{X}} f(x)\overline{\psi(g^{-1}\cdot x)}\, dx.$$

**Theorem 3.** *The wavelet transform $\mathscr{W}_{\mathscr{X}} : L_0^2(\mathscr{X}) \to L^2(\mathbb{R}_+ \times \mathscr{G},\, \alpha(\rho)\, d\rho \otimes dg)$ is invertible on its range by*

$$\tilde{f} = \int_{\to 0}^\infty (\mathscr{W}_{\mathscr{X}} f)(\rho, \cdot) * \tilde{\psi}_\rho\, \alpha(\rho)\, d\rho \quad \textit{for all} \quad f \in L_0^2(X).$$

*E. The non-compact case*

We only mention the key points for this case. The spectrum of the Laplacian of non-compact groups becomes continuous. Consequently, the expansion in eigenfunctions of the Laplacian becomes a direct integral

$$f(g) = \int_{\mathbb{R}}^{\oplus} \hat{f}(\lambda)\pi_\lambda(g)\, d\mu(\lambda).$$

The critical question here is to have an appropriate Fourier transform. That means, does there exist a measure $d\mu$ on $\widehat{\mathscr{G}}$, such that the integral

$$\int_{\widehat{G}} \hat{f}(\lambda)\pi_\lambda\, d\mu(\lambda), \quad \text{where } \hat{f}(\lambda) := \int_{\mathscr{G}} \pi_\lambda^*(g)\, f(g)\, dg$$

is well-defined for some function space on $\mathscr{G}$. If such measure exists it is called Plancherel measure. In this case the construction of diffusive wavelets works similar to the compact case. In general a Plancherel measure does not exist for locally compact groups. But since the Plancherel measure exists for nilpotent Lie groups, one can extend diffusive wavelets to nilpotent Lie groups.

## III. Wavelet packets

**Definition 5.** *Let $\{\rho_j,\, j \in \mathbb{Z}\}$ be a strictly decreasing sequence of real numbers such that*

$$\lim_{j\to\infty} \rho_j = 0 \textit{ and } \lim_{j\to-\infty} \rho_j = \infty.$$

*Let $\{\Psi_\rho,\, \rho > 0\}$ be a family of diffusive wavelets. A* wavelet packet *is defined by*

$$\hat{\Psi}_j^P(\pi) = \left( \int_{\rho_{j+1}}^{\rho_j} (\hat{\Psi}_\rho)^2 \alpha(\rho)\, d\rho \right)^{\frac{1}{2}},$$

*and in spatial domain*

$$\Psi_j^P = \sum_{\pi \in \hat{G}} d_\pi \lambda_\pi \left( \int_{\rho_{j+1}}^{\rho_j} e^{-\rho \lambda_\pi^2}\, d\rho \right)^{\frac{1}{2}} \mathrm{trace}\,(\eta(\pi)\pi(g)).$$

The wavelet transform is now given by

$$(\mathscr{W}^P f)(j, g) := (f * \check{\Psi}_j^P)(g).$$

**Theorem 4.** *The wavelet transform $\mathscr{W}^P$ is an isometry $L^2(\mathscr{G}) \to L^2(\mathbb{Z} \times \mathscr{G})^2$.*

**Theorem 5.** *The wavelet transform $\mathscr{W}^P$ is invertible on its range by*

$$f(g) = \sum_{j\in\mathbb{Z}} (\mathscr{W}^P f)(j, \cdot) * \Psi_j^P(\cdot))(g).$$

A common strategy is to build up a multiresolution analysis corresponding to $\Psi^P$.

## IV. Examples

*1) The torus $\mathbb{T}^n$:* Let $\mathbb{T}^n$ denote the $n$-dimensional torus which can be identified with

$$\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n.$$

We will identify $n$-fold periodic functions on $\mathbb{R}^n$ with their projection on $\mathbb{T}^n$. The corresponding projection will be called periodization and is defined by

$$\mathbb{P}f(x) = \sum_{\omega \in 2\pi\mathbb{Z}^n} f(x + \omega).$$

In particular, the periodization of the heat kernel on $\mathbb{R}^n$ give the heat kernel on $\mathbb{T}^n$. We have

$$e_t^{heat,\mathbb{R}^n}(x) = \frac{1}{2(\pi t)^n} e^{\frac{||x||^2}{4t}}$$

Let $m \in \mathbb{Z}^n$. For $f \in L^2(\mathbb{T}^n)$ we have

$$f(x) = \sum_{m\in\mathbb{Z}^n} \hat{f}(m)\, e^{i\sum_{j=1}^n m_j x_j},$$

$$\hat{f}(m) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} f(x) e^{-i\sum_{j=1}^n m_j x_j}\, dx.$$

The Fourier coefficients of the heat kernel $e_t^{heat,\mathbb{T}^n}$ can be given explicitly

$$\hat{e}_t^{heat,\mathbb{T}^n} = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \sum_{\omega\in 2\pi\mathbb{Z}^n} e_t^{heat,\mathbb{R}^n}(x+\omega) e^{-i\sum_{j=1}^n m_j x_j} dx$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{2(\pi t)^n} e^{\frac{||x||^2}{4t}} e^{-i\sum_{j=1}^n m_j x_j}\, dx = \frac{1}{2\pi^n} e^{-\sum_{j=1}^n m_j^2 t}.$$

Let $\{\psi_\rho\}$ be a subfamily of $L^2(\mathbb{T}^n)$. the wavelet we are looking for has the Fourier series expansion

$$\hat{\psi}_\rho(x) = \sum_{m\in\mathbb{Z}^n} \frac{1}{\sqrt{2\pi^n}} \sum_{j=1}^n m_j^2 e^{-\sum_{j=1}^n m_j^2 \rho} e^{i\sum_{j=1}^n m_j x_j}.$$

*2) The unit sphere $S^n$:* The unit sphere is a homogeneous space $S^n = SO(n+1)/SO(n)$. An orthonormal system in $L^2(S^n)$ is given by the *spherical harmonics* $\{Y_k^i, k = 0,1,\ldots, i = 1,\ldots d_k(n)\}$, where $d_k(n) = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$. We denote by $C_k^\lambda$ the Gegenbauer polynomials of order $\lambda = \frac{n-1}{2}$. The eigenvalues of the Laplace-Beltrami operator on $S^n$ are $-\lambda_k^2 = -k(k+n-2)$ and the heat kernel is given by

$$e_t^{heat,S^n}(x) = \sum_{k=0}^{\infty} d_k(n) e^{-\lambda_k^2 t} \frac{C_k^\lambda(x_0\cdot x)}{C_k^\lambda(1)} = \sum_{k=0}^{\infty} \frac{2k+n-1}{n-1} e^{-k(k+n-2)t} C_k^\lambda(x_0\cdot x),$$

where $x_0$ is base point. Let $\alpha(\rho) > 0$ be a weight function on $S^n$. Then (radial) diffusive wavelets are given by

$$\Psi_\rho(x) = \frac{1}{\sqrt{\alpha(\rho)}} \sum_{k=0}^{\infty} \frac{(2k+n-1)\lambda_k}{n-1} e^{-\lambda_k^2\rho/2} C_k^\lambda(x_0\cdot x),$$

where $\lambda_k^2 = k(k+n-2)$. This construction is based on the Gauß-Weierstraß kernel. A similar construction can be done with the Abel-Poisson kernel, where $\lambda_k^2 = k$.

*3) The compact group SO(3):* For SO(3) all irreducible representations are unitary equivalent to one of the irreducible components of the quasi-regular representation in $L^2(S^2)$. In $L^2(S^2)$ the translation invariant subspaces are spanned by the spherical harmonics of the same degree of homogenity. We have $d(2) = 2k+1$ and the eigenvalues of the Laplace-Beltrami operator are $-\lambda_k^2 = -k(k+1)$. The eigenfunctions are the so-called Wigner polynomials. Hence the heat kernel on $SO(3)$ is

$$e_t^{SO(3)}(g) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1) e^{-k(k+1)} C_{2k}\left(\sin\left(\tfrac{\gamma(g)}{2}\right)\right),$$

where $\gamma(g)$ denotes the angle of $g$ [16]

$$\gamma(g) = \arccos\left(\tfrac{\mathrm{trace}\,(g)-1}{2}\right).$$

By our construction a family of wavelets on $SO(3)$ corresponding to the heat kernel is given by

$$\Psi_\rho(g) = \frac{1}{\sqrt{\alpha(g)}} \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)\sqrt{k(k+1)} e^{\frac{k(k+1)}{2}\rho} C_{2k}^1\left(\sin\left(\tfrac{\gamma(g)}{2}\right)\right).$$

*4) The Heisenberg group:* The construction of diffusive wavelets is not restricted to compact groups and homogeneous spaces. As long as we have some Plancharel formula we can construct diffusive wavelets. Therefore we can construct diffusive wavelets on the Heisenberg group. Since the Heisenberg group is noncompact we cannot use the Peter-Weyl theorem. But fortunately similar results can be obtained from the Stone-von-Neumann theorem. Due to the existence of a Plancherel measure the Fourier transform can be developed in a similar way, where the sum over irreducible representations becomes an integral since the spectrum of the Laplacian is continuous. While the Laplacian involves a complete basis of the Lie algebra, the sub-Laplacian involves only those operators which corresponds to vector fields belonging to the sub-Riemannian structure. Therefore we consider the heat equation

$$(\Delta_{sub} - \partial_r)u((x,y,t),r) = 0$$

with fundamental solution

$$p_r(x,y,t) = \int (2\pi)^{n/2} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} e^{-((2|\alpha|+n)|\lambda|)r} \phi_k^\lambda(x,y,t)\, d\mu(\lambda),$$

where $\phi_k^\lambda(x,y,t)$ are the radial-symmetric eigenfunctions of $\Delta_{sub}$. For the three dimensional Heisenberg group $H^1$ we obtain the diffusive wavelets

$$\Psi_\rho(x,y,t) = -\sum_{k=0}^{\infty} \left( \frac{1}{k!} \frac{1}{(it-(2k+1)\frac{\rho}{2})} \left(1 + \frac{\frac{1}{2}|x+iy|^2}{it-(2k+1)\frac{\rho}{2}}\right)^k + \frac{(-1)^k}{k!} \frac{1}{(-it-(2k+1)\frac{\rho}{2})} \left(1 + \frac{\frac{1}{2}|x+iy|^2}{it-(2k+1)\frac{\rho}{2}}\right)^k \right) e^{-\frac{1}{4}|x+iy|^2}.$$

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