Analogies and differences in optical and mathematical systems and approaches

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***Abstract*—We review traditions and trends in optics and imaging recently arising by applying programmable optical devices or by sophisticated approaches for data evaluation and image reconstruction. Furthermore, a short overview is given about modeling of well-known classical optical elements, and vice versa, about optical realizations of classical mathematical transforms, as in particular Fourier, Hilbert, and Riesz transforms.**

I. Introduction

In the 18th/19th century the work of physicists and mathematicians was often closely connected. Scientists in that age were often acting concurrently in both fields: if we think about e.g. Augustin Fresnel explaining experimentally and theoretically the phenomena of light propagation and diffraction, or about Joseph Fourier, experimentally discovering mode decomposition of (mechanical) wave fields and delivering the basis for later theory about transforming signals and fields into the (temporal or spatial) frequency domain. In the nearer past both disciplines were developing rather independently in their own directions. In the field of optics important discoveries as the laser, wave-guides, novel microscopic or holographic techniques should be named as examples among others. In the field of mathematics the huge field of harmonic analysis, bringing up wavelets, frames etc., the several numerical approaches for solving differential equations and also the development of the functional theoretic background in analysis should be quoted as representatives here.

II. Optical devices and mathematical descriptions; Mathematical approaches and optical realizations

A. Analogies between optical and mathematical approaches

Due to the contemporary possibilities given on one hand by advanced digital optical devices, as spatial light modulators (SLM) or micro mirror arrays (MMA), deformable mirrors or phased arrays in combination with traditional optical elements, and on the other hand by the computational power of modern hard- and software architecture allowing sophisticated mathematical reconstruction algorithms, new fields of research and perspectives are opened. Computational or programmable optics are examples for this modern development and interdisciplinary entanglement of the different disciplines. They open a new branch of methods as for digital holography, lensless microscopy, or adaptive optics [1]–[3]; they comprehend several phase retrieval and reconstruction techniques [4], [5], adaptive wave front correction methods up to compressive sensing for optical applications [6], [7]. Whereas in the past optical imaging performance has often been hampered by scattering within materials, by turbulences within fluids, or speckles at rough surfaces, nowadays computational techniques and programmable optics deliver novel approaches as focusing through or within scattering materials, turbulence corrections or contrast enhancement by SLM-based techniques, [8]–[10].

Bringing now together optics and mathematics in such a way, touching points are noticed and furthermore, awareness is arising that in both fields similar approaches exist, only realization techniques or names may differ. This concerns for instance classical optical devices (as lenses, prism, cones,...) or classical imaging techniques (bright field, dark field, Schlieren or knife edge imaging technique, spiral phase quadrature imaging, or differential interference contrast (DIC) imaging), [11]–[14]. Primarily, these techniques are modifying contrast of the visualized specimen, but to a certain amount they are also quantitative with respect to phase or optical path length, which can be expressed and reconstructed mathematically under knowledge of their (complex-valued) point spread function (PSF) in the spatial domain or of their optical transfer function in the Fourier domain.

Beyond the well-known Fourier transform (FT) other classical mathematical transforms as the two-dimensional (2D) Hilbert transform (HT), also denoted as directional HT [15], with a kernel function H_{HT} defined in the Fourier do-

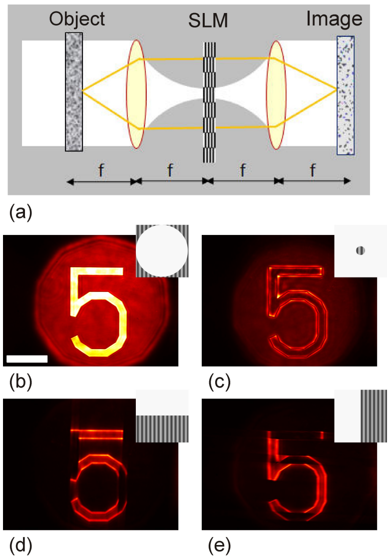

Fig. 1. Different contrast modifications emulated by means of the SLM device being addressed with different filter functions in a Fourier plane filtering unit (a). The emulated imaging type shows as representative contrast: (b) bright field (original scan), (c) dark field, (d) and (e) horizontal and vertical Schlieren/knife edge imaging contrast. (Striped regions encode a zero magnitude, continuous regions encode a constant unit magnitude, and a phase between [0, 2 π] according to the gray level. The white scale bar yields 100 μm.)

main (u, v) by

$$H_{HT_1}(u, v) = -\mathrm{sgn(u)} \exp(il\pi/2), \tag{1}$$
$$H_{HT_2}(u, v) = -\mathrm{sgn(v)} \exp(il\pi/2), \tag{2}$$

where $l = 1$ is chosen for the conventional HT, or as the 2D Riesz transform (RT), also denoted as complexified-valued Riesz transform [15] or radial Hilbert transform [16], with a kernel function H_{RT} defined in the Fourier domain by

$$H_{RT}(\hat{r}, \hat{\varphi}) = \exp(il\hat{\varphi}), \tag{3}$$

with $(\hat{r}, \hat{\varphi})$ denoting polar coordinates in the Fourier domain, and $l = 1$ is chosen for the conventional RT, find entrance in optical modeling, emulation, and settings. Optically these transforms can be realized by classical elements (lenses, apertures, spiral phase plates) or nowadays more and more by programmable SLM devices allowing flexible realizations. Vice versa, in the mathematical modeling of optical imaging techniques these transforms build the base for an (approximated) description of the PSF e.g. for Schlieren and DIC imaging, for pyramid and roof sensors (all with a PSF model based on the directional HT), [17], [18]. Also spiral phase/vortex filtering (with a PSF model based on the RT), and their fractional expressions as fractional half-plane and spiral phase filters (corresponding to a fractional HT resp. RT with $0 < l < 1$ in eq. (2) and (3)) can be modeled in such a way, as shown in Fig.1 and Fig.2, [19]–[23].

Here we can connect now optics with classical functional analysis. The PSF of a pyramid sensor [17] given in the spatial

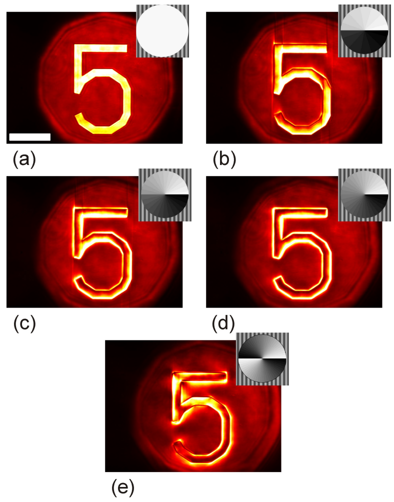

Fig. 2. Contrast modifications emulated by means of the SLM device being addressed with fractional spiral phase filter functions (fractional Riesz transform) of fractional coefficient (a) l=0, (b) l=0.4, (c) l=0.8, (d) l=1.0, (e) l=2.

domain (x, y) by

$$\begin{aligned} h_{PY}((-1)^n x, (-1)^m y) &= \frac{1}{4}\delta(x, y) \\ + \ &\frac{(-1)^{m+n}}{4\pi^2} p.v. \frac{1}{xy} \\ + \ &\frac{i}{4\pi}\left[(-1)^n p.v.(\frac{1}{x}\delta(y)) + (-1)^m p.v.(\delta(x)\frac{1}{y})\right], \end{aligned} \tag{4}$$

with p.v. denoting a principal value and (n, m) are enumerators $(0, 1)$, resembles in its structure the 2D analytic signal, as introduced by Hahn [24]. Whereas the PSF of a spiral phase filter or so called vortex filter [22] can be described by the (2D) Riesz kernel

$$h_{SP}(r, \varphi) = \frac{i}{2\pi r^2} \exp(il\varphi), \tag{5}$$

with (r, φ) denoting polar coordinates in spatial domain, and $l = 1$ is chosen in the conventional case. Furthermore, it should be noted that Riesz transform has been introduced by [25] in the field of optics under the name spiral phase quadrature transform. This filter tends rather to a monogenic signal approach, as introduced by Felsberg, [26]. Knowing now in principle the PSF of these imaging modalities, we can emulate the special imaging types by addressing SLMs in a corresponding way with amplitude or phase transfer functions in optical Fourier domain. So we can flexibly change the contrast corresponding to the envisaged imaging technique [27] and can go towards a quantitative reconstruction based on the emulated PSF in future.

B. Optical Fourier plane filtering and wavelet-like filters

In applied mathematics and signal analysis orthogonal, isotropic or anisotropic wavelet-based decomposition approaches play an important role for image processing, naming applications as image denoising, edge enhancement, or

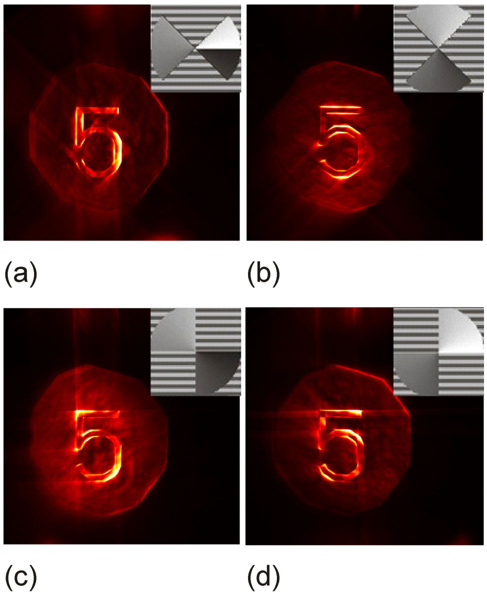

Fig. 3. The original image has been filtered optically in Fourier domain by different monogenic wavelet-like filters, the applied filter kernels are sketched in the inset. (Striped regions encode a zero magnitude, continuous regions encode a constant unit magnitude and a phase between [0, 2 π] according to the gray level.)

compression methods among other. In particular, analytic or monogenic wavelet approaches have found entrance in image processing delivering additional phase and orientation information or may be used for scale-based demodulation [28]–[32].

On the other hand we can also ask whether and to which extent a wavelet-like filtering can be performed analogously in an optical way. Here the classical principle for optical Fourier plane filtering finds its modern application anew. In combination with programmable optics as SLMs or MMAs we also can emulate to a certain amount the (compact and positive) support and transfer function of suitable wavelets (curvelets, shearlets) in Fourier domain. And for their analytic and monogenic wavelet complements, also in this case, optical realizations of Hilbert and Riesz transform build up the basis for the filtering approaches. Here, the methods are usable for isotropic or anisotropic contrast improvement in imaging [33], as shown in Fig.3, for orientation emphasizing, or for salient point detection.

C. Restrictions and differences between optical and mathematical approaches

However, we also should keep in mind the restrictions and differences between optical realizations and mathematical approaches. For instance SLMs or MMAs as pixelated and discrete arrays exhibit only a finite resolution; therefore, the available spatial frequency range is restricted for filtering. Furthermore, in optics without introducing additional sensors or working with interferometric imaging setups, we can only measure intensities at a conventional camera applied as detector. So the separated information given in the amplitude and phase spectrum - as easily obtained by Fourier transform in mathematics - is lost in their optical counterparts. At least for phase reconstruction an additional phase retrieval step using a multiple recording of the modified image would be required.

Coherence aspects in optical Fourier plane filtering provides an additional discussion point. Coherence may be regarded as an imaging feature closely related to the considered scale. Furthermore, it must be distinguished between temporal coherence and spatial coherence. Operating with broadband light sources for illumination, these sources exhibit a smaller temporal coherence length than conventional narrow band laser sources used in coherent imaging. Therefore, the phase filter applied on the SLM mask is exactly matching only for the central wavelength. This mismatch may result in a slight blurring of the image features such as edges. Spatial coherence can be maintained by coupling the illumination beam into a single mode fiber.

However, scattering within turbid materials severly restricts the fixed phase relationship within the electro-magnetic wave field required e.g. for the Fourier plane (phase-only) filtering (or correspondingly within a convolution kernel of a defined support). This demands again methods for wavefront correction to cope with scattering materials for future successful implementations.

III. Conclusion

In summary, the close connection between the modeling of well-known optical devices or elements and classical mathematical approaches or transforms has been demonstrated. Furthermore, by linear filtering in optics we can realize similar effects as with classical filtering in signal or image processing. The explanation of the obtained effects in optics and in mathematics is partly similar, but due to the complex-valued nature of the light also different mechanism, as e.g. interference or diffraction has to be considered.

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