Mellin analysis and exponential sampling. Part II: Mellin differential operators and sampling

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***Abstract*—Here, we introduce a notion of strong fractional derivative and we study the connection with the pointwise fractional derivative, which is defined by means of Hadamard-type integrals. The main result is a fractional version of the fundamental theorem of integral and differential calculus in Mellin frame. Finally there follow the first of several theorems in the sampling area, the highlight being the reproducing kernel theorem as well as its approximate version for non-bandlimited functions in the Mellin sense, both being new.**

I. Introduction

This article is the continuation of the previous one devoted to the study of Mellin fractional integrals. Here we apply the results concerning the Hadamard type integrals in order to define an appropriate notion of the associated pointwise fractional derivative. Moreover we will introduce a notion of a strong fractional derivative in spaces X_c, as an extension to the Mellin setting of the notion of classical strong derivatives in L^p-spaces (see [6]). The pointwise fractional derivative of order $\alpha > 0$, is defined by the Hadamard integrals formally as follows:

$$(D^{\alpha}_{0+,c}f)(x) = x^{-c}\delta^m x^c (J^{m-\alpha}_{0+,c}f)(x),$$

where $m = [\alpha] + 1$ and $\delta := (x\frac{d}{dx})$. The above definition, introduced in [9, Part I], originates from the theory of the classical Mellin differential operator, studied in [6, Part I]. The main result here is an equivalence theorem which strictly connects the two notions of fractional derivatives and the Hadamard integrals. As far as we are aware this kind of equivalence was never stated explicitily in the setting of Fourier transform theory. This is also related to the fundamental theorem of integral and differential calculus in the fractional Mellin setting. For usual Mellin derivatives, this was described in [6, Part I], where, in particular, the representation of the Mellin derivatives in terms of the Stirling numbers of the second kind is discussed in depth. Finally there follow the first of several theorems in the sampling area.

One of the new and important applications regarding the exponential sampling is an error estimate giving the fast rate of approximation depending on the order of the fractional derivative (see Corollary 2 below).

II. The strong and pointwise Mellin fractional differential operators

We recall that X_c denotes the space of all the measurable functions $f : \mathbb{R}^+ \to \mathbb{C}$ such that $f(\cdot)(\cdot)^{c-1} \in L^1(\mathbb{R}^+)$. The Mellin transform of a function $f \in X_c$ is defined by

$$M[f](s) \equiv [f]^{\wedge}_M(s) = \int_0^\infty u^{s-1}f(u)du$$

where $s = c + it, t \in \mathbb{R}$, and the Mellin translation operator τ^c_h, for $h \in \mathbb{R}^+,\ c \in \mathbb{R},\ f : \mathbb{R}^+ \to \mathbb{C}$, by

$$(\tau^c_h f)(x) := h^c f(hx) \quad (x \in \mathbb{R}^+).$$

Setting $\tau_h := \tau^0_h$, then $(\tau^c_h f)(x) = h^c(\tau_h f)(x)$, $\|\tau^c_h f\|_{X_c} = \|f\|_{X_c}$. The Mellin fractional difference of $f \in X_c$ of order $\alpha > 0$, defined by

$$\Delta^{\alpha,c}_h f(x) := (\tau^c_h - I)^\alpha f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^{\alpha-j}\tau^c_{h^j} f(x).$$

for $h > 0$, I being the identity operator over the space of all measurable functions on $\mathbb{R}^+$, and

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!},$$

has the following properties

Proposition 1: For $f \in X_c$ the difference $\Delta^{\alpha,c}_h f(x)$ exists a.e. for $h > 0$, with

i) $\|\Delta^{\alpha,c}_h f\|_{X_c} \leq \|f\|_{X_c} \sum_{j=0}^\infty \left|\binom{\alpha}{j}\right|$
ii) $M[\Delta^{\alpha,c}_h f](c+it) = (h^{-it} - 1)^\alpha M[f](c+it)$.

Proof. As to (ii) it follows by taking the Mellin transforms on the left, thus

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^{\alpha-j} h^{-itj} M[f](c+it).$$

For spaces $X_{[a,b]}$, we have the following Proposition.

Proposition 2: Let $f \in X_{[a,b]}$, and let $c \in]a,b[$.

(i) If $0 < h \leq 1$, then $\Delta^{\alpha,c}_h f \in X_{[a,c]}$, and for every $\nu \in [a,c]$

$$\|\Delta^{\alpha,c}_h f\|_{X\nu} \leq \|f\|_{X_\nu} \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| h^{(c-\nu)j}.$$

Moreover for $t \in I\!R$,

$$M[\Delta_h^{\alpha,c} f](\nu + it) = (h^{c-\nu-it} - 1)^\alpha M[f](\nu + it).$$

(ii) If $h \geq 1$, then $\Delta_h^{\alpha,c} f \in X_{[c,b]}$, and for every $\mu \in [c, b]$

$$\|\Delta_h^{\alpha,c} f\|_{X\mu} \leq \|f\|_{X_\mu} \sum_{j=0}^{\infty} \left| \begin{pmatrix} \alpha \\ j \end{pmatrix} \right| h^{(c-\mu)j}.$$

Moreover for $t \in I\!R$,

$$M[\Delta_h^{\alpha,c} f](\mu + it) = (h^{c-\mu-it} - 1)^\alpha M[f](\mu + it).$$

Definition. If for $f \in X_c$ there exists $g \in X_c$ such that

$$\lim_{h \to 1} \left\| \frac{\Delta_h^{\alpha,c} f(x)}{(h-1)^\alpha} - g(x) \right\|_{X_c} = 0$$

then g is called the strong fractional derivative of f of order α and it is denoted by $g(x) = \text{s-}\Theta_c^\alpha f(x)$, and

$$W_{X_c}^\alpha := \{f \in X_c : \text{s-}\Theta_c^\alpha f \text{ exists and s-}\Theta_c^\alpha f \in X_c\},$$

with $W_{X_c}^0 = X_c$, is the Mellin Sobolev space. Analogously we define the spaces $W_{X[a,b]}^\alpha, W_{X]a,b[}^\alpha$.

Now to our several basic theorems of the two-parts paper.

Theorem 1: (i) If $f \in W_{X_c}^\alpha$, then for $s = c + it, t \in I\!R$,

$$M[\text{s-}\Theta_c^\alpha f](s) = (-it)^\alpha M[f](s).$$

(ii) If $f \in W_{X[a,b]}^\alpha$, then for every $\nu, c \in [a, b]$,

$$M[\text{s-}\Theta_c^\alpha f](\nu+it) = (c-\nu-it)^\alpha M[f](\nu+it), \quad t \in I\!R.$$

Proof. As to (i), it can be shown in view of

$$\lim_{h \to 1} \left(\frac{h^{-it} - 1}{h - 1} \right)^\alpha = (-it)^\alpha,$$

that

$$\lim_{h \to 1} \left| (-it)^\alpha [f]_M^\wedge(s) - [\text{s-}\Theta_c^\alpha f]_M^\wedge(s) \right| = 0.$$

The pointwise fractional derivative of order α, associated with the integral $J_{0+,c}^\alpha f$, $c \in I\!R$, and $f \in Dom J_{0+,c}^{m-\alpha}$, is given by (see e.g. [9, Part I], [5], [15, Part I])

$$(D_{0+,c}^\alpha f)(x) = x^{-c} \delta^m x^c (J_{0+,c}^{m-\alpha} f)(x)$$

where $\alpha > 0, m = [\alpha] + 1$ and $\delta = (x\frac{d}{dx})$. The (classical) pointwise Mellin derivative of integral order, is defined by

$$\begin{aligned} & \lim_{h \to 1} \frac{\tau_h^c f(x) - f(x)}{h - 1} \\ = & \lim_{h \to 1} \left[h^c x \frac{f(hx) - f(x)}{hx - x} + \frac{h^c - 1}{h - 1} f(x) \right] \\ = & \; xf'(x) + cf(x), \end{aligned}$$

provided f' exists a.e. on $I\!R^+$, and the Mellin differential operator of order $r \in I\!N$ iteratively by $\Theta_c^1 := \Theta_c$, $\Theta_c^r := \Theta_c(\Theta_c^{r-1})$.

The following proposition gives the connection between Mellin and ordinary derivatives.

Proposition 3: For the pointwise derivative of order $r \in I\!N$, we have

$$(D_{0+,c}^r f)(x) = (\Theta_c^r f)(x) = \sum_{k=0}^{r} S_c(r, k) x^k f^{(k)}(x),$$

where $S_c(r, k)$, $0 \leq k \leq r$, denote the generalized Stirling numbers of second kind, defined recursively by

$$S_c(r, 0) := c^r, \; S_c(r, r) := 1,$$
$$S_c(r + 1, k) = S_c(r, k - 1) + (c + k)S_c(r, k).$$

In the fractional case, for a given $\alpha > 0$, we define the space $X_{c,loc}^\alpha$ by

$$\{f \in X_{c,loc} : \exists (D_{0+,c}^\alpha f)(x) \text{ a.e}, \; D_{0+,c}^\alpha f \in X_{c,loc}\}.$$

Proposition 4: Let $f \in X_{c,loc}^\alpha$ be such that $\Theta_c^m f \in X_{c,loc}$, where $m = [\alpha] + 1$. Then

$$(D_{0+,c}^\alpha f)(x) = \Theta_c^m (J_{0+,c}^{m-\alpha} f)(x) = J_{0+,c}^{m-\alpha} (\Theta_c^m f)(x).$$

Now to the fundamental theorem of the fractional differential and integral calculus in the Mellin frame.

Theorem 2: Let $\alpha > 0$ be fixed. Let $f \in X_{c,loc}^\alpha$, be such that $D_{0+,c}^\alpha f \in Dom J_{0+,c}^m$ and $\Theta_c^m f \in Dom J_{0+,c}^m$. If $\Theta^{m-1} f \in \widetilde{X}_{c,loc}$, then

$$(J_{0+,c}^\alpha (D_{0+,c}^\alpha f))(x) = f(x), \; a.e. \; x \in I\!R^+.$$

Moreover, let $f \in Dom J_{0+,c}^m$ be such that $J_{0+,c}^\alpha f \in X_{c,loc}$. Then

$$(D_{0+,c}^\alpha (J_{0+,c}^\alpha f))(x) = f(x), \quad a.e. \; x \in I\!R^+.$$

Concerning the connections between the strong and the pointwise Mellin derivatives, we have the following

Theorem 3: Let $\alpha > 0$ and $c \in]a, b]$ be fixed, and $f \in X_{[a,b]}^\alpha$ be such that $\Theta_c^m f \in X_{[a,b]}$. Then $f \in W_{[a,b]}^\alpha$ and

$$(D_{0+,c}^\alpha f)(x) = \text{s-}\Theta_c^\alpha f(x), \quad a.e. \; x \in I\!R^+.$$

Proof. By Proposition 4 we have

$$(D_{0+,c}^\alpha f)(x) = (J_{0+,c}^{m-\alpha} (\Theta_c^m f))(x).$$

Thus passing to Mellin transforms, we have, for $t \in I\!R$,

$$\begin{aligned} & [D_{0+,c}^\alpha f]_M^\wedge(\nu + it) = [(J_{0+,c}^{m-\alpha} (\Theta_c^m f))]_M^\wedge(\nu + it) \\ & = (c - \nu - it)^{\alpha-m} [\Theta_c^m f]_M^\wedge(\nu + it) \\ = & \; (c - \nu - it)^\alpha [f]_M^\wedge(\nu + it). \end{aligned}$$

Hence, $D_{0+,c}^\alpha f$ and $\text{s-}\Theta_c^\alpha f$ have the same Mellin transform along the line $\nu + it$, and so the assertion follows by the identity theorem (see [6, Part I]).

Using the previous results, we give the following equivalence theorem which is the fractional version of Theorem 10 in [6, Part I].

Theorem 4: Let $f \in X_{[a,b]}$, $\alpha > 0$. The following four assertions are equivalent

(i) $f \in W^{\alpha}_{X_{[a,b]}}$.
(ii) There is a function $g_1 \in X_{[a,b]}$ such that, for every $c \in]a,b[$,

$$\lim_{h\to 1}\left\| \frac{\Delta_h^{\alpha,c} f}{(h-1)^{\alpha}} - g_1 \right\|_{X_c} = 0.$$

(iii) There is $g_2 \in X_{[a,b]}$ such that, for every $\nu, c \in]a,b[$,

$$(c-\nu-it)^{\alpha} M[f](\nu+it) = M[g_2](\nu+it).$$

(iv) There is $g_3 \in X_{[a,b]}$ such that for $c \in]a,b[$ and $x \in I\!R^+$,

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^c \left(\log\frac{x}{u}\right)^{\alpha-1} g_3(u)\frac{du}{u} \quad a.e.$$

If one of the above assertions is satisfied, then $D^{\alpha}_{0+,c} f(x) = \text{s-}\Theta^{\alpha}_c f(x) = g_1 = g_2 = g_3$ a.e. $x \in I\!R^+$.

Proof. It is easy to see that (i) implies (ii), and (ii) implies (iii) by Theorem 1. As to (iii) implies (iv), observe

$$M[J^{\alpha}_{0+,c} g_2](\nu+it) = (c-\nu-it)^{-\alpha} M[g_3](\nu+it).$$

As far as we know, a fundamental theorem with four equivalent assertions in the form presented above for the Mellin transform in the fractional case has never been stated for the Fourier transform. As a fundamental theorem in the present sense it was first established for $2\pi-$periodic functions via the finite Fourier transform in [10], and for the Chebyshev transform in [7], [8]. Fractional Chebyshev derivatives were there defined in terms of fractional order differences of the Chebyshev translation operator, the Chebyshev integral by an associate convolution product. The next fundamental theorem, after that for Legendre transforms (see e,g. [2]), was the one concerned with the Jacobi transform, see e.g. [9].

III. The exponential sampling theorem

Let B^T_c denote the class of functions $f \in X_c,\ f \in C(I\!R^+)$, $c \in I\!R$, which are Mellin band-limited in the interval $[-T,T]$, $T \in I\!R^+$, thus for which $[f]^{\wedge}_M(c+it) = 0$ for all $|t| > T$. A mathematician's version of the exponential sampling theorem introduced by the electrical engineers/physicists M.Bertero, E.R. Pike [5, Part I] and F. Gori [14, Part I], reads as follows

Theorem 5: If $f \in B^{\pi T}_c$ for some $c \in I\!R$, and $T > 0$, then the series

$$x^c \sum_{k=-\infty}^{\infty} f(e^{k/T}) \text{lin}_{c/T}(e^{-k}x^T)$$

is uniformly convergent in $I\!R^+$, and one has the representation

$$f(x) = \sum_{k=-\infty}^{\infty} f(e^{k/T}) \text{lin}_{c/T}(e^{-k}x^T) \equiv E^c_T f(x) \quad (x \in I\!R^+).$$

The lin_c-function for $c \in I\!R$, $\text{lin}_c : I\!R^+ \to I\!R$, is defined, for $x \in I\!R^+ \setminus \{1\}$, by

$$\text{lin}_c(x) = \frac{x^{-c}}{2\pi i}\frac{x^{\pi i} - x^{-\pi i}}{\log x} = \frac{x^{-c}}{2\pi}\int_{-\pi}^{\pi} x^{-it}dt,$$

with the continuous extension $\text{lin}_c(1) := 1$, thus $\text{lin}_c(x) = x^{-c}\text{sinc}(\log x)$.

As we all know, bandlimitation in the classical Fourier version of the Whittaker-Kotel'nikov-Shannon sampling theorem is a restriction we try to avoid. Likewise it is so in the Mellin setting. In this respect we have the following approximate version.

Theorem 6: Let $f \in X_c \cap C(I\!R^+)$, $c \in I\!R$, be such that $M[f] \in L^1(\{c\} \times iI\!R)$. Then there holds the error estimate

$$\left| f(x) - \sum_{k=-\infty}^{\infty} f(e^{k/T}) \text{lin}_{c/T}(e^{-k}x^T) \right|$$
$$\leq \frac{x^{-c}}{\pi} \int_{|t|>\pi T} |M[f](c+it)|dt \quad (x \in I\!R^+,\ T > 0).$$

Corollary 1: Let $f \in X_c \cap C(I\!R^+)$, $c \in I\!R$, be such that $M[f] \in L^1(\{c\} \times iI\!R)$. Then

$$\lim_{T\to+\infty} |f(x) - E^c_T f(x)| = 0, \quad x \in I\!R^+.$$

Further, if $f \in B^{\pi\overline{T}}_c$ for some $\overline{T} > 0$, then, for all $T \geq \overline{T}$,

$$f(x) = E^c_T f(x), \quad x \in I\!R^+.$$

The operator $\text{s-}\Theta^{\alpha}_c f$, $\alpha > 0$, plays the basic role in the following corollary, giving the fast rate of approximation of $f(x)$, depending on its order α, by the exponential sampling sum $E^c_T f(x)$.

Corollary 2: If $f \in W^{\alpha}_{X_c}$, $c \in I\!R, \alpha > 0$, is continuous on $I\!R^+$ such that $M[\text{s-}\Theta^{\alpha}_c f] \in L^1(\{c\} \times iI\!R)$, then

$$|f(x) - E^c_T f(x)| = o(T^{-\alpha}), \quad (x \in I\!R^+; T \to +\infty).$$

Proof. According to Theorem 1, $|[f]^{\wedge}_M(c+it)| = |t|^{-\alpha}|[\text{s-}\Theta^{\alpha}_c f]^{\wedge}_M(c+it)|$, $t \in I\!R$, so that:

$$\int_{|t|>\pi T} |[f]^{\wedge}_M(c+it)|dt$$
$$\leq \frac{1}{\pi^{\alpha}T^{\alpha}} \int_{|t|>\pi T} |[\text{s-}\Theta^{\alpha}_c f]^{\wedge}_M(c+it)|dt = o(T^{-\alpha}),$$

so the assertion follows by Theorem 6.

In the previous new corollary, we can consider the pointwise derivative $D^{\alpha}_{0+,c}$ with the assumptions of Theorem 3.
One of the several theorems which are equivalent to the classical Whittaker-Kotel'nikov-Shannon sampling theorem is the well known reproducing kernel formula. In the Mellin setting, it reads as follows, for functions in $B^{T\pi}_c,\ T > 0$

Theorem 7: Let $f \in B^{T\pi}_c$, $c \in I\!R,\ T > 0$, be fixed. Then we have

$$f(x) = T\int_0^{\infty} f(y)\text{lin}_{c/T}\left((\frac{x}{y})^T\right)\frac{dy}{y} \quad (x \in I\!R^+).$$

Proof: Putting $h(y) = f(y^{1/T})$, we have $h \in B^{\pi}_{c/T}$. Then using the reasoning of Lemma 6.3 in [8, Part I] we can write

$$[h(y)\text{lin}_{c/T}(x/y)]^{\wedge}_M(it)$$
$$= \frac{x^{-c/T}}{2\pi}\int_{-\pi}^{\pi}[h]^{\wedge}_M(c/T + i(t+v))x^{-iv}dv \quad (t \in I\!R).$$

Then for $t = 0$ we get

$$\int_0^\infty h(y)\text{lin}_{c/T}(x/y)\frac{dy}{y}$$
$$= \frac{x^{-c/T}}{2\pi}\int_{-\pi}^{\pi}[h]^\wedge_M(c/T+iv)x^{-iv}dv = h(x)$$

by Theorem 2.4 in [8, Part I]. Thus we have

$$f(x^{1/T}) = T\int_0^\infty f(y)\text{lin}_{c/T}(x/y^T)\frac{dy}{y},$$

and putting $x^{1/T} = z$ we have the assertion.

A further new result is the approximate reproducing kernel theorem, namely its version for not necessarily Mellin-bandilimited functions. It states that

Theorem 8: Let $f \in X_c$, $c \in I\!R$, be continuous on $I\!R^+$ such that $M[f] \in L^1(\{c\} \times iI\!R)$. Then there holds, for $y \in I\!R^+$ and $T > 0$, the error estimate

$$\left|f(x) - T\int_0^\infty f(y)\text{lin}_{c/T}\left((\frac{x}{y})^T\right)\frac{dy}{y}\right|$$
$$\leq \frac{x^{-c}}{2\pi}\int_{|v|\geq\pi T}|[f]^\wedge_M(c+iv)|dv.$$

IV. The finite Mellin transform and the Mellin-Poisson summation formula

The Poisson summation formula in the classical frame of Fourier analysis is one the cornestones of all mathematical analysis. To formulate and establish it in the Mellin setting one needs further concepts, namely Mellin Fourier series (introduced in [7, Part I]) and the associated finite Mellin transform, since this Poisson summation connects the Mellin transform with its finite version.

Definitions.

(i) A function $f : I\!R^+ \to \boldsymbol{C}$ will be called *recurrent*, if $f(x) = f(e^{2\pi}x)$ for all $x \in I\!R^+$. The function f is called $c-$*recurrent* for $c \in I\!R$, if $x^c f(x)$ is recurrent, i.e., if $f(x) = e^{2\pi c}f(e^{2\pi}x)$ for all $x \in I\!R^+$.

(ii) The space Y_c of $c-$recurrent functions $f : I\!R^+ \to \boldsymbol{C}$ is defined for $c \in I\!R$, by

$$Y_c := \{f \in L^1_{loc}(I\!R^+) : f \text{ c-recurrent}, \|f\|_{Y_c} < +\infty\},$$

$\|f\|_{Y_c} = \int_{e^{-\pi}}^{e^\pi}|f(u)|u^{s-1}du$, with $s = c + it$.

(iii) The finite Mellin transform of $f \in Y_c$, $c \in I\!R$ is

$$\mathcal{M}_c[f](k) \equiv [f]^\wedge_{\mathcal{M}_c}(k) = \int_{e^{-\pi}}^{e^\pi} f(u)u^{c+it-1}du, \ \ (k \in \boldsymbol{Z}),$$

$\mathcal{M}_c : Y_c \to L^\infty(\boldsymbol{Z})$, $f \mapsto \{[f]^\wedge_{\mathcal{M}_c}(k)\}_{k\in\boldsymbol{Z}}$, with $\|\mathcal{M}_c\|_{[Y_c,L^\infty]} = 1$.

(iv) The associated Mellin-Fourier series of $f \in Y_c$ is

$$f(x) \sim \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}[f]^\wedge_{\mathcal{M}_c}(k)x^{-c-ik}, \ \ (x \in I\!R^+).$$

Theorem 9: Let $f \in X_c$, $c \in I\!R$, be continuous on $I\!R^+$ such that $\sum_{k=-\infty}^{\infty}|M[f](c+ik)| < \infty$. If the series

$$f^c(x) := \sum_{k=-\infty}^{\infty} f(e^{2\pi k}x)e^{2\pi kc}, \ \ (x \in I\!R^+)$$

which is $c-$recurrent and absolutely convergent a.e. on the interval $[e^{-\pi}, e^\pi]$ is also uniformly convergent there, then

$$M[f](c+ik) = \mathcal{M}_c[f^c](k), \ \ (k \in \boldsymbol{Z}),$$

and especially there holds

$$f^c(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}M[f](c+ik)x^{-c-ik}, \ \ (x \in I\!R^+).$$

It is the strong feeling of the authors that not only the Mellin-sampling theorem is equivalent to its approximate version, but also the reproducing kernel theorem and its approximate version are equivalent. Even more so all these theorems are equivalent among themselves, and under suitable conditions, are equivalent to the Mellin-Poisson summation formula. Equivalence is understood in the sense that each is a corollary of the others. This is indeed the situation in the non-fractional version of the Fourier case as recently proved in [3], [4].

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