Mellin analysis and exponential sampling. Part I: Mellin fractional integrals

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***Abstract*—The Mellin transform and the associated convolution integrals are intimately connected with the exponential sampling theorem. Thus it is very important to develop the various tools of Mellin analysis. In this part we pave the way to sampling analysis by studying basic theoretical properties, including Mellin-type fractional integrals, and give a new approach and version for these integrals, specifying their basic semigroup property. Especially their domain and range need be studied in detail.**

I. Introduction

The theory of Mellin transforms and Mellin approximation theory was introduced in a systematic form, fully independent of Fourier analysis in [6], papers on the present line of research being [1], [2], [3], [4]. Mellin transform theory is intimately connected with the exponential sampling theorem, stating that

$$f(x) = \sum_{k=-\infty}^{+\infty} f(e^{k/T}) \mathrm{lin}_{c/T}(e^{-k}x^T) \quad (x \in I\!R^+),$$

where f is a function which is Mellin-bandlimited to the interval $[-\pi T, \pi T]$, and

$$\mathrm{lin}_c(x) := x^{-c}\mathrm{sinc}(\log x), \quad \mathrm{lin}_c(1) = 1,$$

(see [8]). This version of the Shannon sampling theorem has many applications in optical physics and engineering ([13], [16], [5], [14]). Here the samples are not equally spaced apart as in the case of the Whittaker-Kotel'nikov-Shannon sampling theorem, but exponentially spaced; such spacing is needed in those applications where independent pieces of information accumulates near time $t = 0$.

The aim of this research is to put into a rigorous framework such applications, making use only of results from the Mellin transform theory. In [6] the following sentence is written: *The proofs of the Mellin results are mostly said to follow by a change of variable and a change of function from the corresponding Fourier or Laplace results. In fact one expresses it as follows: "It is a matter of using the theory of the Fourier or Laplace transform to derive what one needs concerning the Mellin transform". However, the hypotheses upon which the Mellin theory lies are often considered quite uncritically, and certainly by no means in a unified, systematic fashion.*

While the classical proof of the Shannon sampling theorem is based on the Poisson summation formula, the exponential version is established via the Mellin-Poisson summation formula, which connects the classical Mellin transform with the finite Mellin transform. Variuos fundamental facts in exponential sampling theory must be developed and, in this direction, a deep study of Mellin analysis appears necessary. In particular, the properties of Mellin convolution integrals and the Mellin differential operators are fundamental tools. In papers [9], [10], [11] certain Mellin convolution integrals, namely the so-called Hadamard- type fractional integrals, were developed: these integrals represent the appropriate extensions of the classical Riemann-Liouville and Weyl fractional integrals and also lead to definitions of certain Mellin fractional differential operators, (see also the book [15]). The purpose of this article is a continuation of these topics. As remarked in [9] the natural operator of Mellin fractional integration is not the classical Riemann-Liouville fractional integral of order $\alpha > 0$ on $I\!R^+$, (see [17], [12]) but the integral

$$(J^\alpha_{0+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log \frac{x}{u}\right)^{\alpha-1} f(u)\frac{du}{u} \quad (x > 0). \quad (1)$$

Thus the operator of integration (anti-differentiation) is not the integral $\int_0^x f(u)du$, as used throughout the literature in matters Mellin transforms, including its table volumes, but the integral $\int_0^x f(u)du/u$. The use of the latter makes calculations not only much simpler but also more elegant.

For the development of the theory, it is important to consider the following generalization of (1), for $\mu \in I\!R,\ x > 0$ namely (see [9], [10], [11])

$$(J^\alpha_{0+,\mu}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^\mu \left(\log \frac{x}{u}\right)^{\alpha-1} f(u)\frac{du}{u} \quad (2)$$

for functions belonging to the space X_c of all measurable complex-valued functions defined on $I\!R^+$, such that $(\cdot)^{c-1}f(\cdot) \in L^1(I\!R^+)$.

Since the definition of pointwise fractional derivatives, as defined in [10], is based on the Hadamard type integral, it is important to study in depth the domain and the range of these integrals. Here we first introduce the local spaces $X_{c,loc}$ and furnish some results concerning both the domain and the range. In this respect, a fundamental role is played by a new version of the basic semigroup property, which is proved here in a direct way, as an extension of a corresponding property for spaces X_c^p given in [10].

The theory of Hadamard fractional integrals is one of the topics which reveals the importance of a direct approach via Mellin transforms. For example, while the domain of the classical Riemann-Liouville fractional operators of any order α contains all the locally integrable functions over the positive real line, this is no longer true for Hadamard operators. Indeed, for $\alpha > 1$, the domain of $J^\alpha_{0+,c}$ is strictly contained in the space $X_{c,loc}$. This implies that the Hadamard integrals and the corresponding notion of pointwise Mellin fractional derivative, which we develop in the second part of this study, represent new types of integro-differential operators which must be properly treated using Mellin transform theory.

In the second part we apply these results to the exponential sampling.

II. Mellin fractional integrals

Let $L^1 = L^1(\mathbb{R}^+)$ be the space of all the Lebesgue measurable and integrable complex valued functions defined on $\mathbb{R}^+$, endowed with the usual norm.

Let us consider the space, for some $c \in \mathbb{R}$,

$$X_c = \{f : \mathbb{R}^+ \to \mathbb{C} : f(x)x^{c-1} \in L^1(\mathbb{R}^+)\}$$

endowed with the norm

$$\|f\|_{X_c} = \|f(\cdot)(\cdot)^{c-1}\|_{L^1} = \int_0^\infty |f(u)|u^{c-1}du.$$

For $a, b \in \mathbb{R}$ we define the spaces $X_{(a,b)},\ X_{[a,b]}$ by

$$X_{(a,b)} = \bigcap_{c\in]a,b[} X_c, \quad X_{[a,b]} = \bigcap_{c\in[a,b]} X_c$$

and, for every c in the given intervals, $\|f\|_{X_c}$ is a norm on them.

We define for every $f \in X_c$ the Mellin transform, with $s = c + it \in \mathbb{C},\ c, t \in \mathbb{R}$, by

$$M[f](s) \equiv [f]^\wedge_M(s) = \int_0^\infty u^{s-1}f(u)du.$$

Thus $M : X_c \to C(\{c\} \times i\mathbb{R}),\ f \to M[f] = [f]^\wedge_M$, (see [6]). A boundedness property for $J^\alpha_{0+,\mu}$ in the space X_c, is needed when the coefficient μ is greater than c. This is due to the fact that only for $\mu > c$, we can view $J^\alpha_{0+,\mu}f$ as a Mellin convolution between two functions in X_c. However, we are interested here in the domain and the range of these fractional operators when $\mu = c$. We will show that for any non-trivial function f in the domain of $J^\alpha_{0+,c}$ the image $J^\alpha_{0+,c}f$ cannot be in X_c. This implies that we cannot compute its Mellin transform in the space X_c.

We define the domain of $J^\alpha_{0+,c}$, for $\alpha > 0$ and $c \in \mathbb{R}$, as the class of all the functions such that

$$\int_0^x u^c\left(\log\frac{x}{u}\right)^{\alpha-1}|f(u)|\frac{du}{u} < +\infty,$$

for a.e. $x \in \mathbb{R}^+$, denoted by $DomJ^\alpha_{0+,c}$.

Let $X_{c,loc}$ be the space of all the functions such that $(\cdot)^{c-1}f(\cdot) \in L^1(]0, a[)$ for every $a > 0$.

Proposition 1: If $f \in X_{c,loc}$, then the function $(\cdot)^c f(\cdot) \in X_{1,loc}$. Moreover, if $c < c'$, then $X_{c,loc} \subset X_{c',loc}$.

Note that the above inclusion does not hold for spaces X_c.

Concerning the domain of the operator $J^\alpha_{0+,c}$, we begin with

Proposition 2: Let $\alpha > 1,\ c \in \mathbb{R}$ be fixed. Then $DomJ^\alpha_{0+,c} \subset X_{c,loc}$.

For $\alpha = 1$ we have immediately $DomJ^1_{0+,c} = X_{c,loc}$. The case $0 < \alpha < 1$ is more delicate. In this instance $X_{c,loc} \subset DomJ^\alpha_{0+,c}$, due to the following "local" version of the semigroup property of $J^\alpha_{0+,c}$:

Theorem 1: Let $\alpha, \beta > 0,\ c \in \mathbb{R}$ be fixed. Let $f \in DomJ^{\alpha+\beta}_{0+,c}$. Then

(i) $f \in DomJ^\alpha_{0+,c} \cap DomJ^\beta_{0+,c}$
(ii) $J^\alpha_{0+,c}f \in DomJ^\beta_{0+,c}$ and $J^\beta_{0+,c}f \in DomJ^\alpha_{0+,c}$.
(iii) $(J^{\alpha+\beta}_{0+,c}f)(x) = (J^\alpha_{0+,c}(J^\beta_{0+,c}f))(x)$, a.e. $x \in \mathbb{R}^+$.
(iv) If $\alpha < \beta$ then $DomJ^\beta_{0+,c} \subset DomJ^\alpha_{0+,c}$.

Thus if $0 < \alpha \leq 1,\ c \in \mathbb{R}$, then $X_{c,loc} \subset DomJ^\alpha_{0+,c}$.

The inclusion in (iv) of Theorem 1 is strict for any choice of α and β. It is sufficient to consider the function, with $\alpha < \gamma < \beta$,

$$f(x) = \frac{x^{-c}}{|\log x|^\gamma}\chi_{]0,1/2[}(x),$$

$\chi_{]0,1/2[}$ being the characteristic function of interval $]0, 1/2[$.

A sufficient condition in order that a function f belongs to $DomJ^\alpha_{0+,c}$ for $\alpha > 1$, is

Proposition 3: Let $\alpha > 1$. If $f \in X_{c,loc}$ is such that $f(u) = \mathcal{O}(u^{-(r+c-1)})$ for $u \to 0^+$ and $0 < r < 1$, then $f \in DomJ^\alpha_{0+,c}$.

As a consequence, for $c \in \mathbb{R}$ fixed, we have

$$\widetilde{X}_{c,loc} \subset \bigcap_{\alpha>0} DomJ^\alpha_{0+,c}.$$

Concerning the range of the operators $J^\alpha_{0+,c}$ we need the following important propositions.

Proposition 4: Let $\alpha > 0,\ c \in \mathbb{R}$ be fixed. If $f \in DomJ^{\alpha+1}_{0+,c}$, then $J^\alpha_{0+,c}f \in X_{c,loc}$.

As a consequence we can deduce that if $f \in DomJ^\alpha_{0+,c}$, not necessarily does $J^\alpha_{0+,c}f \in X_{c,loc}$.

For spaces X_c we have the following

Proposition 5: Let $\alpha > 0,\ c \in \mathbb{R}$ be fixed. If $f \in DomJ^\alpha_{0+,c}$, then $J^\alpha_{0+,c}f \notin X_c$, unless $f = 0$ a.e. in $\mathbb{R}^+$.

However we have the following property.

Proposition 6: Let $\alpha > 0,\ c, \nu \in I\!R,\ \nu < c$, being fixed. If $f \in Dom J^{\alpha}_{0+,c} \cap X_{[\nu,c]}$, then $J^{\alpha}_{0+,c} f \in X_{\nu}$ and

$$\|J^{\alpha}_{0+,c} f\|_{X_{\nu}} = \frac{\|f\|_{X_{\nu}}}{(c-\nu)^{\alpha}}.$$

Moreover, for any $s = \nu + it$,

$$|M[J^{\alpha}_{0+,c} f](s)| \leq \frac{\|f\|_{X_{\nu}}}{(c-\nu)^{\alpha}}.$$

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