Generalized and Fractional Prolate Spheroidal Wave Functions

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
Chicago, IL 60614
Email: azayed@depaul.edu

***Abstract*—An important problem in communication engineering is the energy concentration problem, that is the problem of finding a signal bandlimited to $[-\sigma, \sigma]$ with maximum energy concentration in the interval $[-\tau, \tau], 0 < \tau$, in the time domain, or equivalently, finding a signal that is time limited to the interval $[-\tau, \tau]$ with maximum energy concentration in $[-\sigma, \sigma]$ in the frequency domain. This problem was solved by a group of mathematicians at Bell Labs in the early 1960's. The solution involves the prolate spheroidal wave functions which are eigenfunctions of a differential and an integral equations.**

The main goal of this talk is to present a solution to the energy concentration problem in a Hilbert space of functions. This solution will contain as a special case the solution to the energy concentration problem in both the fractional Fourier transform and the linear canonical transform domains. The solution involves a generalization of the prolate spheroidal wave functions, which when restricted to the fractional Fourier transform domain, we may call fractional prolate spheroidal wave functions.

I. Introduction

One of the fundamental problems in communication engineering is the energy concentration problem, that is the problem of finding a signal bandlimited to $[-\sigma, \sigma]$ with maximum energy concentration in the interval $[-\tau, \tau], 0 < \tau$, in the time domain or equivalently, finding a signal that is time limited to the interval $[-\tau, \tau]$ with maximum energy concentration in $[-\sigma, \sigma]$ in the frequency domain. This problem was solved by a group of mathematicians, D. Slepian, H. Landau, and H. Pollak, at Bell Labs [6], [7], [12], [17] in the early 1960's. The solution involves the prolate spheroidal wave functions which are eigenfunctions of a differential and an integral equations.

Because bandlimited functions are entire functions, they cannot vanish outside any interval and as a result the energy concentration in any interval $[-\tau, \tau]$ cannot be 100%. The percentage of the energy concentration depends on σ and τ and involves the eigenvalues of a certain integral equation satisfied by the prolate spheroidal wave functions. The solution of the problem uses properties of the Fourier transform, among them is the fact that the Fourier transform of a prolate spheroidal wave function is a multiple of a scaled version of itself.

Recall that the energy concentration of f in $(-\tau, \tau)$ is given by $\int_{-\tau}^{\tau} |f(t)|^2 \, dt$; therefore, the solution of the concentration problem can be found by finding the function f that maximizes the ratio

$$\alpha^2(\tau) = \frac{\int_{-\tau}^{\tau} |f(t)|^2 \, dt}{\int_{-\infty}^{\infty} |f(t)|^2 \, dt}.$$

A more general problem to consider is the energy concentration problem in the fractional Fourier transform domain. That is to find a signal that is bandlimited to $[-\sigma, \sigma]$ in the fractional Fourier transform domain with maximum energy concentration in the interval $[-\tau, \tau], 0 < \tau$, in the time domain. This problem, in turn, is a special case of the energy concentration problem for the linear canonical transform. The latter problems were solved in [15] and discrete versions of them were solved in [22]. The solutions involved what the authors called the generalized prolate spheroidal wave functions. The generalized prolate spheroidal wave functions associated with the fractional Fourier transform and the linear canonical transform have interesting applications in the analysis of the status of energy preservation of optical systems, self-imaging phenomenon, and the resonance phenomenon of finite-sized one-stage and multiple-stage optical systems [15].

The main goal of this article is to solve the energy concentration problem in a Hilbert space of functions which will contain the fractional Fourier transform and the linear canonical transform as special cases.

A. The Fractional Fourier Transform

The fractional Fourier transform (or FrFT) was first introduced by Namias in 1980 in connection with an application in quantum mechanics [11]. But since its introduction to the signal processing community in the early 1990's, the transform has become an important tool in signal processing applications and signal representation in the fractional Fourier transform domain has been an active area of investigation [1], [3], [4], [5], [8], [9], [10], [13], [14], [19], [20], [21].

The fractional Fourier Transform or FrFT of a signal or a function, say $f(t) \in L^2(I\!R)$, is defined by

$$\widehat{f}_\theta(\omega) = \int_{-\infty}^{\infty} f(t) k_\theta(t, \omega) \, dt \qquad (1)$$

where

$$k_\theta(t, \omega) = \begin{cases} c(\theta) \cdot e^{ja(\theta)(t^2+\omega^2) - jb(\theta)\omega t}, \theta \neq p\pi \\ \delta(t - \omega), \theta = 2p\pi \\ \delta(t + \omega), \theta = (2p-1)\pi \end{cases}$$

is the transformation kernel with

$$c(\theta) = \sqrt{(1 - j\cot\theta)/2\pi}, \ a(\theta) = \cot\theta/2, \ \text{and} \ b(\theta) = \csc\theta.$$

The kernel $k_\theta(t,\omega)$ is parameterized by an angle $\theta \in I\!R$ and p is some integer. For simplicity, we may write a, b, c instead of $a(\theta), b(\theta)$, and $c(\theta)$. The inverse-FrFT with respect to an angle θ is the FrFT with angle $-\theta$, given by

$$f(t) = \int_{-\infty}^{\infty} \widehat{f_\theta}(\omega)\, k_{-\theta}(t,\omega)\, d\omega. \quad (2)$$

When $\theta = \pi/2$, (1) reduces to the classical Fourier transform, which will be denoted by $\hat{f}_{\pi/2} = \hat{f}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{I\!R} f(t) e^{-j\omega t} dt.$$

Let $k_\theta(t,\omega)$ be the kernel of FrFT and define the operator L_θ as

$$L_\theta[f](\omega) = \int_{-\infty}^{\infty} f(t) k_\theta(t,\omega) dt.$$

It is easy to see that

$$L_\theta(L_\phi[f](\omega)) = L_{\theta+\phi}[f](\omega).$$

It can be shown that the solution of the concentration problem for the Fractional Fourier transform is the solution of the integral equation (3)

$$\int_{-\sigma}^{\sigma} F(\omega) K_\tau(\omega,\zeta)\, d\omega = \lambda F(\zeta), \quad (3)$$

that yields the maximum λ, where

$$K_\tau(\omega,\zeta) = \frac{e^{ja\left(t^2-\zeta^2\right)} \sin b\tau\,(t-\zeta)}{\pi(\omega-\zeta)}.$$

The solutions of the integral equation (3) share similar properties with the prolate spheroidal wave functions, but satisfy more general differential and integral equations. For lack of better terminology, we shall call these new functions fractional prolate spheroidal wave functions.

B. *The Linear Canonical Transform*

The linear canonical transform $G_{(a,b,c,d)}(u)$ of a function $f(x)$, which depends on four parameters a, b, c, d, is defined as

$$G_{(a,b,c,d)}(u) = \begin{cases} \int_{-\infty}^{\infty} K_{(a,b,c,d)}(x,u) f(x) dx, & b \neq 0 \\ \sqrt{d} e^{(j/2)cdu^2} f(ud), & b = 0 \end{cases}$$

where

$$K_{(a,b,c,d)} = \frac{1}{\sqrt{2\pi j b}} \exp\left(\frac{j}{2b}\left[du^2 - 2ux + ax^2\right]\right),$$

with $ad - bc = 1$.

For $a = \cos\theta, b = \sin\theta, c = -\sin\theta, d = \cos\theta$, the linear canonical transform reduces to the fractional Fourier transform.

C. *The Prolate Spheroidal Wave Functions*

The prolate spheroidal wave functions (PSWF), were first discovered in [12] as the bounded eigenfunctions of the following differential operator L_c,

$$L_c\varphi(x) = (1-x^2)\frac{d^2}{dx^2}\varphi(x) - 2x\frac{d}{dx}\varphi(x) - c^2x^2\varphi(x), \quad (4)$$

where $c > 0$ is a real number. In the 1960's, the group at Bell Labs discovered that the following integral operator

$$F_c(\varphi_{n,c})(x) = \int_{-1}^{1} \varphi_{n,c}(t) \frac{\sin(c(x-t))}{\pi(x-t)} dt, \quad (5)$$

commutes with L_c, where $\varphi_{n,c}$ are the eigenfunctions of the operator (4). This commutation relation was termed "a lucky accident" by David Slepian. In a series of papers, the group at Bell Labs employed the commutation relation to derive several properties of the prolate spheroidal wave functions, see [6], [7], [17]. For example, they have showed that the PSWFs satisfy the following integral equation

$$\int_{-1}^{1} \varphi_{n,c}(x) e^{icwx} dx = \mu_n(c)\varphi_{n,c}(w). \quad (6)$$

The PSWFs are normalized so that

$$\|\varphi_{n,c}\|_2^2 = \int_{-\infty}^{+\infty} |\varphi_{n,c}(x)|^2\, dx = 1, \quad (7)$$

or equivalently,

$$\|\varphi_{n,c}\chi_{(-1,1)}\|_2^2 = \int_{-1}^{1} |\varphi_{n,c}(x)|^2\, dx = \lambda_n(c), \quad (8)$$

where $\lambda_n(c)$ is the nth eigenvalue of F_c. The most important properties of the PSWFs are:

(P_1) The set of PSWFs $\{\varphi_{n,c}, n \in I\!N\}$ is an orthogonal basis of $L^2([-1,1])$. More precisely, we have

$$\int_{-1}^{1} \varphi_{n,c}(x)\varphi_{m,c}(x)\, dx = \lambda_n(c)\delta_{mn}.$$

(P_2) The Fourier transform of $\varphi_{n,c}$ is given by :

$$\hat{\varphi}_{n,c}(w) = (-i)^n \sqrt{\frac{2\pi}{c\lambda_n(c)}}\, \varphi_{n,c}\left(\frac{w}{c}\right) \chi_{[-c,c]}(w).$$

(P_3) The set of PSWFs $\{\varphi_{n,c}, n \in I\!N\}$ is an orthonormal set of $L^2(I\!R)$ and also an orthonormal basis of B_c, where

$$B_c = \{f \in L^2(I\!R) : \, supp(\widehat{f}) \subset [-c,c]\},$$

is the space of functions bandlimited to $[-c,c]$.

II. Conclusion

The main goal of this talk is to show that the energy concentration problem can be solved in a general Hilbert space of functions using the theory of reproducing-kernel Hilbert spaces. To outline the setting in which the problem will be solved, let us introduce the following notation.

Let $\mathcal{E}$ be an arbitrary set and $\mathcal{F}(\mathcal{E})$ be the linear space of all complex-valued functions defined on $\mathcal{E}$. Let $d\mu$ be a σ-finite positive measure and $\mathcal{T}$ be a $d\mu$-measurable set in $I\!R^N$.

Consider the Hilbert space $\mathcal{H} = L^2(\mathcal{T}, d\mu)$ consisting of all complex-valued functions F such that

$$\|F\|^2_{L^2(\mathcal{T},d\mu)} = \int_{\mathcal{T}} |F(t)|^2 d\mu(t) < \infty.$$

Let $h(t,p)$ denote a complex-valued function on $\mathcal{T} \times \mathcal{E}$, such that

$$h(t,p) \in L^2(\mathcal{T}, d\mu) \text{ for any } p \in \mathcal{E}.$$

Let L be the linear mapping $L : L^2(\mathcal{T}, d\mu) \to \mathcal{F}(\mathcal{E})$ defined by

$$f(p) = (LF)(p) = \int_{\mathcal{T}} F(t)\overline{h}(t,p)d\mu(t), \quad F \in L^2(\mathcal{T}, d\mu). \tag{9}$$

It is not difficult to see that the the function

$$K(p,q) = \int_{\mathcal{T}} h(t,q)\overline{h}(t,p)d\mu(t), \tag{10}$$

is positive definite on $\mathcal{E}$, i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}K(p_i,p_j) \geq 0,$$

for any finite set $\{p_i\}$ of $\mathcal{E}$. Then it follows from [2] that $K(p,q)$ is a reproducing kernel for some Hilbert space of functions defined on $\mathcal{E}$. In fact, the set of all f's given by (9), i.e., the range of the operator L, is a reproducing-kernel Hilbert space $\widetilde{\mathcal{H}}$ whose reproducing kernel is given by (10) so that $f(q) =< f, K(.,q) >_{\widetilde{\mathcal{H}}}$; see [16].

Hereafter, all functions of the form (9) will be called K-bandlimited functions. In this talk we will show that the energy concentration problem can be solved for the class of K-bandlimited functions, but the details will be published somewhere else. The problem will be solved by constructing a sequence of functions that share similar properties to those of the PSWF, in particular Equations (5), (6), and properties P_1 and P_2.

ACKNOWLEDGMENT

The authors would like to thank one of the referees for his constructive comments and bringing to my attention references [15], [22].

REFERENCES

[1] L. B. Almeida, The fractional Fourier transform and time- frequency representations, IEEE Trans. on signal processing, vol.42, no.11 (1994) pp.3084-3091.

[2] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.,* Vol. 68 (1950), pp. 337-404.

[3] L. Auslander and R. Tolimier, Radar amiguity functions and group theory, SIAM J. Math. Anal., Vol. 16, pp. 577-601, 1985.

[4] C. Candan, M. A. Kutay, H. M. Ozakdas, The discrete fractional Fourier transform, IEEE Trans. on signal processing, vol. 48. no. 5, 2000.

[5] M. A. Kutay, H. M. Ozakdas, O. Arikan, and L. Onural , Optimal filtering in fractional Fourier domains, IEEE Trans. Signal Processing, vol. 45. pp. 1129-1143, 1997.

[6] H. J. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty-II,* Bell Syst. Techn. J. (1960). pp.65-84.

[7] H. J. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty-III; The dimension of the space of essentially tine and band-limited signals.*, Bell Syst. Techn. J. (1962). pp.1295-1336.

[8] A. W. Lohmann, Image rotation, Wigner rotation and the fractional Fourier transform, J. Opt. Soc. Amer. A. , vol. 10, pp. 2181-2186, 1993.

[9] D. Mendlovic, Z. Zalevsky, and H. M. Ozakdas, The applications of the fractional Fourier transform to optical pattern recognition, in Optical Pattern Recognition, New York: Academic, 1998, Ch. 3.

[10] D. Mendlovich and H. M. Ozaktas, Fractional Fourier transforms and their optical implementation 1, J. Opt. Soc. Amer. A. , vol.10. pp. 1875-1881, 1993.

[11] V. Namias, The fractional order Fourier transforms and its application to quantum mechanics, J. Inst. Math. Appl. , vol. 25, pp. 241-265, 1980.

[12] C. Niven, *On the conduction of heat in ellipsoids of revolution,* Phylosophical Trans. Roy. Soc. Lond., (1880), 171, pp: 117-151.

[13] H. M. Ozakdas, M. A. Kutay, and D. Mendlovic, Introduction to the fractional Fourier transform and its applications, Advances in Imaging Electronics and Physics. New York: Academic, 1999, ch. 4.

[14] H. M. Ozaktas, B. Barshan, D. Mendlovic, and L. Onural, Convolution filtering, and multiplexing in fractional Fourier domains and their relation to chirp and wavelet transforms, J. Opt. Soc. Amer. A. , vol. 11, pp. 547-559, 1994.

[15] S. Pei, J.J Ding, Generalized prolate spheroidal wave functions for optical finite fractional Fourier and linear canonical transform, J. Opt.Soc. Amer. A, Vol. 22, March 2005.

[16] S. Saitoh, *The Theory of Reproducing Kernels and its Applications,* Pitman Research Notes in Math., Vol. 189 (1988), Longman Scientific & Technical, United Kingdom.

[17] D. Slepian and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty-I,* Bell System Tech. J., (1961), pp.379-393.

[18] D. Slepian, *Prolate spheroidal wave functions, Fourier analysis and uncertainty–IV: Extensions to many dimensions; generalized prolate spheroidal functions,* Bell System Tech. J. **43** (1964), pp. 3009–3057.

[19] A. I. Zayed, Hilbert transform associated with the fractional Fourier transform, IEEE Signal Processing Lett., vol. 5, pp. 206-208, 1998.

[20] A. I. Zayed, Convolution and product theorem for the fractional Fourier transform, IEEE Signal Processing Lett. , vol. 5, pp. 101-103, 1998.

[21] A. I. Zayed, On the relationship between the Fourier and fractional Fourier transforms, IEEE Signal Processing Lett. , vol. 3, pp. 310-311, 1996.

[22] H. Zhao, R. Wang, D. Song, and D. Wu, Maximally concentrated sequences in both time and linear canonical transform domains, Signal Image Video Process, DOI 10.1007/s11760-012-0309-1.