Sparse Signal Reconstruction from Phase-only Measurements

Petros T. Boufounos
Mitsubishi Electric Research Laboratories,
Cambridge, MA, 02139
petrosb@merl.com

Abstract—**We demonstrate that the phase of complex linear measurements of signals preserves significant information about the angles between those signals. We provide stable angle embedding guarantees, akin to the restricted isometry property in classical compressive sensing, that characterize how well the angle information is preserved. They also suggest that a number of measurements linear in the sparsity and logarithmic in the dimensionality of the signal contains sufficient information to acquire and reconstruct a sparse signal within a positive scalar factor. We further show that the reconstruction can be formulated and solved using standard convex and greedy algorithms taken directly from the CS literature. Even though the theoretical results only provide approximate reconstruction guarantees, our experiments suggest that exact reconstruction is possible.**

I. Introduction

The advent of compressive sensing (CS) has significantly improved our ability to sense a variety of signals. Classical CS theory reveals that it is possible to acquire signals at a rate dictated by the complexity of the signal model, rather than the signal dimensionality [1]–[3]. The acquisition is performed using incoherent measurements that preserve all the information in the signal. The signal is recovered from those measurements by exploiting a signal model such as sparsity. Computation—increasingly available thanks to Moore's law—plays an important role in this recovery. Thus it is possible to simplify sensing systems in a number of applications and substitute inexpensive computational complexity in place of frequently expensive sampling complexity.

In this paper we explore how compressive sensing can be used to reconstruct signals from phase-only measurements. Specifically, we demonstrate that the phase of linear complex measurements preserves information about angles of signals. This information can be sufficient to reconstruct the signal within a positive scaling factor. We further show that the measurements contain sufficient information to formulate a convex program or a greedy algorithm to recover the signal.

In many ways, this paper extends earlier work on 1-bit CS, in which a signal is acquired by quantizing the measurements to 1-bit per measurement, i.e. only preserving their signs [4]–[6]. Similar to phase measurements, this operation preserves the angles of signals but not amplitude information. Thus, the signal can only be reconstructed within a scaling factor and only approximated since the measurements are quantized. This paper extends 1-bit CS in the same way that phase/magnitude representations of complex numbers extend sign/magnitude representations of a real numbers.

This work also extends earlier results on the importance of phase information in recovering signals, with a number of practical applications [7]–[10]. In summary, the phase of a fully sampled Fourier transform of a signal contains, under a variety of conditions, sufficient information to uniquely specify the signal and enable its reconstruction within a scaling factor. Our results exploit sparse signal models to reduce the number of phase measurements required. In that sense they transfer classical CS results to phase measurements. While we establish the results using random matrices with i.i.d. normal entries, we conjecture that a large variety of distributions could be used, including subsampled Fourier transforms. Note that quantizing the phase, explored in [11], provides an alternative quantized representation to quantizing the linear measurements.

In the next section we provide a brief background on CS and 1-bit CS, which also partly serves to establish notation. Section III describes the problem, discusses the embedding properties of phase-only measurements and explores how to reconstruct the measured signal. Section IV provides experimental results, validating our approach. Finally, Section V provides some discussion and concludes.

II. Background

A. Compressive Sensing

Classical, by now, results in CS have established that it is possible to measure and successfully reconstruct a signal sparse in some basis using a number of linear measurements which is approximately proportional to the small number of non-zero components of the signal in that basis [1]–[3]. This acquisition can be expressed as the linear system

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^N$ denotes the sparse signal, $\mathbf{y} \in \mathbb{R}^M$ denotes the measured data, $\mathbf{A} \in \mathbb{R}^{M \times N}$ denotes the measurement matrix representing the linear system, and M and N denote the dimensionality of the data and the acquired signal, respectively. The sparsity of $\mathbf{x}$, i.e., the number of non-zero coefficients, is denoted using K. We assume, without loss of generality, that the signal is sparse in the canonical basis.

A sufficient condition to recover the signal from the measurements, is the Restricted Isometry Property (RIP). The matrix $\mathbf{A}$ satisfies the RIP of order K, with RIP constant δ_K if for all K-sparse vectors $\mathbf{x}$:

$$(1-\delta_K)\|\mathbf{x}\|_2 \leq \|\mathbf{A}\mathbf{x}\|_2 \leq (1+\delta_K)\|\mathbf{x}\|_2, \quad (2)$$

i.e., approximately preserves the norm of all K-sparse vectors. Thus, a matrix satisfying the RIP of order $2K$ describes an embedding of K-sparse vectors in N dimensions into an M-dimensional space. This embedding preserves the ℓ_2 distance.

If the RIP of order $2K$ holds with a small RIP constant, the signal can be exactly recovered using the convex program

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}, \quad (3)$$

or one of many available greedy algorithms [1], [12]–[16]. Variations of this program, as well as the recovery guarantees have also been developed for a variety of measurement noise conditions and relaxations of the strict sparsity requirement.

The RIP has been established for a variety of matrix classes. With high probability, a properly scaled random matrix with entries generated from an i.i.d. normal or subgaussian distribution satisfies the RIP as long as $M = O(K \log N)$. Similar results have been shown for other matrices, such as ones generated by randomly selecting rows of a DFT matrix.

B. 1-bit Compressive Sensing

Practical acquisition systems quantize their measurements. 1-bit CS examines extreme quantization to one bit per measurement, i.e., preserving only the sign of each measurement:

$$\mathbf{y} = \operatorname{sign}(\mathbf{A}\mathbf{x}), \quad (4)$$

where $\operatorname{sign}(\cdot)$ is applied element-wise to its argument. Since $\operatorname{sign}(\mathbf{A}\mathbf{x}) = \operatorname{sign}(\mathbf{A}c\mathbf{x})$ for all $c > 0$, 1-bit CS acquisition eliminates amplitude information about the signal. Thus, we can only hope to recover the signal within a scaling factor. Furthermore, the solution of an ℓ_1 minimization program similar to (3) degenerates to a zero $\mathbf{x}$. Some way to enforce a norm constrain is necessary [4].

The constraint proposed originally, $\|\mathbf{x}\|_2 = 1$, leads to non-convex program, difficult to analyze and provide guarantees for. More recently, [17] showed that a convex program can be formulated if we exploit the fact that the sign measurements of the signal reveal the hyperoctant in which the measurements lie. Thus a linear constraint can be used to enforce a non-trivial solution, resulting to the convex program

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} = \operatorname{sign}(\mathbf{A}\mathbf{x}) \text{ and } \mathbf{y}^T(\mathbf{A}\mathbf{x}) = 1. \quad (5)$$

This program enforces an ℓ_1 norm constraint by exploiting the fact that $\mathbf{y}^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|_1$ at the correct solution.

In the context of 1-bit CS, a condition similar to the RIP can be established, the Binary ϵ-Stable Embedding (BeSE) [6]. The BeSE guarantees the correctness of a sign-consistent reconstruction and characterizes the reconstruction error. The BeSE is in fact an angle embedding, which preserves the angles between signals, defined as

$$d_\angle(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi} \arccos \frac{\langle \mathbf{x}, \mathbf{x}' \rangle}{\|\mathbf{x}\|_2 \|\mathbf{x}'\|_2} \quad (6)$$

for two signals $\mathbf{x}$ and $\mathbf{x}'$. The angle is preserved in the normalized Hamming distance between the measurements, defined as $d_H(\mathbf{y}, \mathbf{y}') = (\sum_i y_i \oplus y_i')/M$, according to

$$d_\angle(\mathbf{x}, \mathbf{x}') - \epsilon \leq d_H(\mathbf{y}, \mathbf{y}') \leq d_\angle(\mathbf{x}, \mathbf{x}') + \epsilon. \quad (7)$$

Thus, if a signal with consistent measurements is found, i.e., $d_H = 0$, it will be within angle ϵ of the measured signal. Similar to the RIP, the BeSE holds for measurement matrices with i.i.d. normal entries, although not in more general ensembles. Furthermore, successful signal recovery from 1-bit measurements with more general ensembles and without requiring the BeSE has also been shown in [18].

III. Phase-Only Compressive Sensing

A. Phase-Only Signal Acquisition

In this paper we consider the following acquisition model

$$\mathbf{z} = \mathbf{A}\mathbf{x}, \ \mathbf{y} = \angle(\mathbf{z}), \quad (8)$$

where $\mathbf{x} \in \mathbb{R}^N$ is a real signal, $\mathbf{A} \in \mathbb{C}^{M \times N}$, $\mathbf{z}$ represents the linear measurement, $\angle(\cdot)$ denotes the principal angle of a complex number, applied element-wise to each vector coefficient, and $\mathbf{y}$ represents the final phase measurements. We also use $\mathbf{a}_m$ to denote the m^{th} row of $\mathbf{A}$.

Obviously, $\angle(\mathbf{A}\mathbf{x}) = \angle(\mathbf{A}c\mathbf{x})$ for any $c > 0$. Thus, angle measurements are similar to sign measurements in 1-bit CS and eliminate any norm information on $\mathbf{x}$. Furthermore, if the acquisition matrix $\mathbf{A}$ only contains real elements, the information in $\mathbf{y}$ is essentially the sign of the measurement—0 and π for positive and negative measurements, respectively. In that case, the problem reverts to 1-bit CS. While complex signals $\mathbf{x}$ can also be considered in this formulation, we defer development of the theory to subsequent work.

B. Stable Angle Embedding

Similar to sign measurements, phase measurements also provide stable embeddings. If two signals $\mathbf{x}, \mathbf{x}'$ in a finite set $\mathcal{W}$ of size L are measured with a random Gaussian vector, the expected value of the measured phase difference is equal to

$$E\left\{\left|\angle\left(\frac{z_m}{z_m'}\right)\right|\right\} = E\left\{\left|\angle\left(e^{i(y_m - y_m')}\right)\right|\right\} = \pi d_\angle(\mathbf{x}, \mathbf{x}'). \quad (9)$$

Hoeffding's inequality bounds the probability that the average of M random variables $|\angle(e^{i(y_m - y_m')})|$ deviates from (9). Using the union bound on L^2 point pairs, a property reminiscent of Johnson-Lindenstrauss (JL) embeddings [19] follows.

Theorem 3.1: Consider a finite set $\mathcal{W} \subset \mathbb{R}^N$ of L points measured using (8), with $\mathbf{A} \in \mathbb{C}^{M \times N}$ consisting of i.i.d elements drawn from the standard complex normal distribution. With probability greater than $1 - 2e^{2\log L - 2\epsilon^2 M}$ the following holds for all $\mathbf{x}, \mathbf{x}' \in \mathcal{S}$ and corresponding measurements $\mathbf{y}, \mathbf{y}' \in \mathbb{R}^M$.

$$\left|\frac{1}{M}\sum_m \left|\frac{1}{\pi}\angle\left(e^{i(y_m - y_m')}\right)\right| - d_\angle(\mathbf{x}, \mathbf{x}')\right| \leq \epsilon \quad (10)$$

Furthermore, the absolute value of the phase difference $\left|\angle\left(e^{i(y_m-y'_m)}\right)\right|$ is Lipschitz continuous with Lipschitz constant equal to 1. Thus, an argument similar to [12] provides a continuous version of the embedding guarantees, similar to the BeSE and the RIP, which is appropriate for sparse signals.

Theorem 3.2: Consider the set $\mathcal{S}_K \subset \mathbb{R}^N$ of all K-sparse signals in $\mathbb{R}^N$, measured as in Thm. 3.1. Eq. (10) holds with probability greater than $1 - 2e^{2K\log\left(\frac{12e}{\epsilon}\frac{N}{K}\right)-\frac{\epsilon^2 M}{2}}$, for all $\mathbf{x}, \mathbf{x}' \in \mathcal{S}_K$ and corresponding measurements $\mathbf{y}, \mathbf{y}' \in \mathbb{R}^M$

These theorems demonstrate that if the mean phase difference between the embedding of two signals is small, then the angle between these signals is also very small. Their nature is similar to the JL lemma, the RIP and the BeSE. They suggest that, similar to classical CS, $M = O(K\log(N/K))$ measurements are sufficient to acquire and reconstruct a signal. The embedding guarantees can be extended to other structured signal sets, such as unions of subspaces or manifolds, using the Kolmogorov complexity of the set in a manner similar to [20].

Unfortunately, the additive form of (10) does not guarantee exact reconstruction. Even if we manage to determine a sparse signal estimate $\widehat{\mathbf{x}}$ that has the same embedding as the measured signal $\mathbf{x}$, Thm. 3.2 can only guarantee that we have identified the signal within an angle ϵ from $\mathbf{x}$, i.e., $|d_\angle(\mathbf{x}, \widehat{\mathbf{x}})| \leq \epsilon$. This behavior is similar to quantized embeddings, such as the BeSE, rather than continuous embeddings such as the RIP. Our experimental results suggest that exact reconstruction guarantees should be possible to derive—not necessarily provided in the form of a stable embedding. However, we do not attempt a proof in this paper.

C. Reconstruction

As discussed above, acquiring a signal using (8) eliminates all information on the total magnitude of the signal. Thus, a reconstruction algorithm, especially one based on ℓ_1-norm minimization, should use a norm constraint to avoid trivial solutions. The original 1-bit CS formulation uses $\|\mathbf{x}\|_2 = 1$, which seems like a natural constraint but leads to a non-convex problem [4]. Instead, we use an approach inspired by the convex formulation in [17].

Specifically, we use the phase of each measurement to rotate that measurement to a positive real number. To do so, we define a vector of unit-magnitude complex coefficients whose phase is equal to the phase of the measurements. Abusing notation, we denote it using $e^{i\mathbf{y}}$, i.e., $(e^{i\mathbf{y}})_m = e^{iy_m}$. Since $e^{-iy_m}z_m = |z_m|$, it follows that $(e^{i\mathbf{y}})^H\mathbf{z} = \|\mathbf{z}\|_1$, where $(\cdot)^H$ denotes the Hermitian (conjugate) transpose. Thus, the convex constraint $(e^{i\mathbf{y}})^H\mathbf{A}\mathbf{x} = 1$ can be used as a norm constraint to prevent degenerate solutions.

In addition to the norm constraint, the phase measurements of a solution should be the same as the original phase measurements. This means that when the linear measurements are properly rotated they should produce positive real numbers: $\Re\{e^{-iy_m}z_m\} \geq 0$ and $\Im\{e^{-iy_m}z_m\} = 0$, where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denotes the real and the imaginary part, respectively.

Combining all constraints we obtain the following program:

$$\begin{aligned} \widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} &\|\mathbf{x}\|_0 \\ \text{s.t. } &(e^{i\mathbf{y}})^H\mathbf{A}\mathbf{x} = 1, \\ &\Re\left\{e^{-iy_m}\langle\mathbf{a}_m, \mathbf{x}\rangle\right\} \geq 0 \\ \text{and } &\Im\left\{e^{-iy_m}\langle\mathbf{a}_m, \mathbf{x}\rangle\right\} = 0. \end{aligned} \tag{11}$$

Of course, this ℓ_0 minimization can exhibit combinatorial complexity. Thus, (11) can be relaxed to the convex program:

$$\begin{aligned} \widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} &\|\mathbf{x}\|_1 \\ \text{s.t. } &(e^{i\mathbf{y}})^H\mathbf{A}\mathbf{x} = 1, \\ &\Re\left\{e^{-iy_m}\langle\mathbf{a}_m, \mathbf{x}\rangle\right\} \geq 0 \\ \text{and } &\Im\left\{e^{-iy_m}\langle\mathbf{a}_m, \mathbf{x}\rangle\right\} = 0. \end{aligned} \tag{12}$$

Alternatively we can use a greedy algorithm that attempts to find a sparse vector satisfying the constraints. This is the approach we follow in this work. We first define a rotated matrix $\widetilde{\mathbf{A}}$ such that $\widetilde{\mathbf{a}}_m = e^{-iy_m}\mathbf{a}_m$, i.e., such that if the original signal was measured it would produce positive real measurements. This means that the signal should be in the nullspace of the imaginary part of $\widetilde{\mathbf{A}}$. Thus we can attempt to use a greedy algorithm to solve the following optimization:

$$\begin{aligned} \widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} &\left\| \begin{bmatrix} \left(e^{-i\mathbf{y}}\right)^H \mathbf{A} \\ \Im\left\{\widetilde{\mathbf{A}}\right\} \end{bmatrix} \mathbf{x} - \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \right\|_2^2 \\ \text{s.t. } &\|\mathbf{x}\|_0 \leq K \\ \text{and } &\Re\left\{e^{-iy_m}\langle\mathbf{a}_m, \mathbf{x}\rangle\right\} \geq 0. \end{aligned} \tag{13}$$

This can be solved with straightforward modifications to standard CS algorithms,such as CoSaMP [14], IHT [15], or ALPS [16], to incorporate the positivity constraint on the real part, in a manner similar to the constraints enforcing quantization consistency in [4]–[6]. However, our experimental results showed that the positivity constraint does not contribute significantly to the performance of the system and can be ignored. In this case, the program can be solved using the existing algorithms without any modification. Since a number of implementations of those algorithms expect real matrices as inputs, the complex constraint $(e^{i\mathbf{y}})^H\mathbf{A}\mathbf{x} = 1$ can be implemented as two real constraints $\Re\{(e^{i\mathbf{y}})^H\mathbf{A}\}\mathbf{x} = 1$ and $\Im\{(e^{i\mathbf{y}})^H\mathbf{A}\}\mathbf{x} = 0$. Similarly for the part of the cost function enforcing that constraint in (13).

IV. Experimental Results

To validate the theory we performed experiments in a range of conditions. The results presented are for $N = 1000$ and a variety of K and M, although different values of N exhibited similar behavior. The experiments examined the correlation of the recovered and the measured signals as well as the correct support recovery. Using $\mathbf{x}$ and $\widehat{\mathbf{x}}$ to denote the measured and recovered signals, respectively, the correlation coefficient is

$$\rho = \frac{\langle\mathbf{x}, \widehat{\mathbf{x}}\rangle}{\|\mathbf{x}\|_2\|\widehat{\mathbf{x}}\|_2} \tag{14}$$

and is equal to 1 if and only if the signal is perfectly recovered. Similarly, using $\mathcal{T}(\cdot)$ to denote the support set, the support recovery can be measured using the ratio

$$P_s = \frac{|\mathcal{T}(\mathbf{x}) \cap \mathcal{T}(\widehat{\mathbf{x}})|}{|\mathcal{T}(\mathbf{x})|}. \tag{15}$$

Note that although perfect signal recovery implies perfect support recovery, the opposite is not true. The support could be perfectly recovered without perfect signal recovery.

For reconstruction we used the very efficient ALPS algorithm [16] to solve (13) without enforcing the positivity constraint $\Re\left\{e^{-iy_m}\langle\mathbf{a}_m, \mathbf{x}\rangle\right\} \geq 0$. The acquisition matrix $\mathbf{A}$ was generated randomly with coefficients drawn from a standard complex normal distribution. The signal $\mathbf{x}$ was generated by first selecting its support set uniformly from the $\binom{N}{K}$ possible sets and then drawing coefficients from a standard normal distribution. The results are averaged over 1500 trials, each with different draw of matrix and signal.

The results are illustrated in Fig. 1. The left plot shows the average correlation as a function of the number of measurements for different values of K. Similarly, the right plot shows the fraction of support recovered as a function of the number of measurements. As evident from the results, the recovery performance exhibits similar behavior to classical compressive sensing. The recovery fails if there is an insufficient number of measurements and the performance exhibits a rapid phase transition as the number of measurements increase. Once a sufficient number of measurements is obtained the signal is perfectly recovered.

V. Discussion and Conclusion

In summary, we demonstrated that the phase of complex measurements contains sufficient information to fully reconstruct a sparse signal within a scaling factor. The theory we present demonstrates that two sparse signals with similar measurements also have very high correlation. Unfortunately, the stable angle embeddings we establish do not guarantee exact reconstruction, even if the phase measurements of the reconstructed signal are identical to those of the measured signal. The small error ϵ characterizes the worst-case reconstruction ambiguity. However, the experimental results suggest that Thm. 3.2 can be tightened to guarantee exact reconstruction

We should also note that the theorem does not guarantee that reconstruction is computationally tractable. The program in (11) will recover the signal if $\mathbf{A}$ provides a stable angle embedding. However, a stable angle embedding does not guarantee that the relaxations in (12) and (13) also recover the correct signal. In that sense, a stable angle embedding is not equivalent to the RIP. The latter has a dual role: In addition to its function as an embedding, the RIP also guarantees that ℓ_1 relaxation and greedy algorithms do provide an exact solution, robust to noise and sparsity level. Whether stable angle embeddings can provide such guarantees is still open.

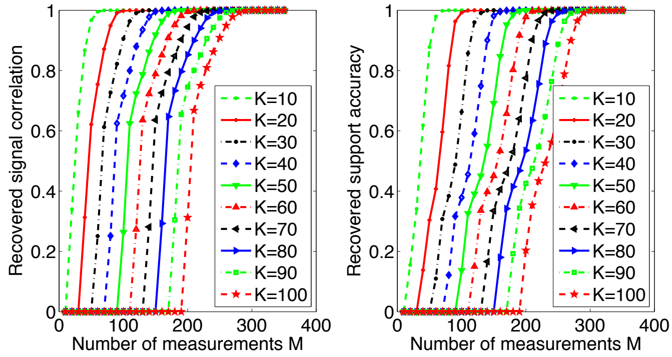

Fig. 1. Reconstruction from phase measurements. Left: average correlation of reconstructed signal with measured signal. Right: probability of correct support recovery in the reconstruction.

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