(Non-)Density Properties of Discrete Gabor Multipliers

Dominik Bayer
Acoustics Research Institute
Austrian Academy of Sciences
Wien, Austria
Email: bayerd@kfs.oeaw.ac.at

Peter Balazs
Acoustics Research Institute
Austrian Academy of Sciences
Wien, Austria
Email: peter.balazs@oeaw.ac.at

***Abstract*—This paper is concerned with the possibility of approximating arbitrary operators by multipliers for Gabor frames or more general Bessel sequences. It addresses the question of whether sets of multipliers (whose symbols come from prescribed function classes such as ℓ^2) constitute dense subsets of various spaces of operators (such as Hilbert-Schmidt class). We prove a number of negative results that show that in the discrete setting subspaces of multipliers are usually not dense and thus too small to guarantee arbitrary good approximation. This is in contrast to the continuous case.**

I. Preliminaries

All Hilbert spaces are assumed to be separable and infinite-dimensional.

A. Bessel sequences

Let H be a Hilbert space with inner product, linear in the first argument, denoted by $\langle \cdot, \cdot \rangle$. A sequence (f_n), $n \in \mathbb{N}$, of elements of H is called a *Bessel sequence* if there exists a constant $B > 0$ such that $\sum_{n\in\mathbb{N}} |\langle h, f_n \rangle|^2 \leq B\|h\|^2$ for all $h \in H$. Any such number B is called a *Bessel bound* of the Bessel sequence, the smallest such constant the *optimal Bessel bound*. If a Bessel sequence satisfies additionally the analogous inequality from below, i.e. there exists a constant $A > 0$ such that $\sum_{n\in\mathbb{N}} |\langle h, f_n \rangle|^2 \geq A\|h\|^2$ for all $h \in H$, then the sequence is called a *frame* for H. Prominent examples of Bessel sequences are orthonormal systems, which are Bessel sequences with Bessel bound 1. For a Bessel sequence (f_n), the *analysis operator* $C : H \to \ell^2$, $h \mapsto Ch := (\langle h, f_n \rangle)_{n\in\mathbb{N}}$, and the *synthesis operator* $D : \ell^2 \to H$, $c = (c_n) \mapsto Dc := \sum_{n\in\mathbb{N}} c_n f_n$ (the series converges in the norm topology of H), are well-defined and adjoint to each other: $C = D^*$.

A useful characterization of Bessel sequences is the following (cf. [4]):

Lemma I.1. *Let (f_n) be a sequence in H and (e_n) be an arbitrary orthonormal basis. Then (f_n) is a Bessel sequence if and only if there exists a bounded operator $T \in B(H)$ with $f_n = Te_n$ for all $n \in \mathbb{N}$. The optimal Bessel bound B is given by $B = \|T\|^2_{B(H)}$.*

We will often use the following basic fact about Bessel sequences (see e.g. [4]):

Lemma I.2. *Let (f_n) be a Bessel sequence with Bessel bound B. Then, for all $n \in \mathbb{N}$,*

$$\|f_n\| \leq \sqrt{B}.$$

B. Time-frequency analysis

In the Hilbert space $L^2(\mathbb{R})$, define the *translation operator* $T_x f(t) = f(t - x)$ and the *modulation operator* $M_\omega f(t) = e^{-2\pi i\omega t} f(t)$ (for $f \in L^2$ and $x, \omega \in \mathbb{R}$). These are unitary operators on L^2. They combine to form the *time-frequency shift* $\pi(x, \omega) = M_\omega T_x$. The *short-time Fourier transform (STFT)* of f with window g is defined as the bilinear time-frequency distribution

$$V_g f(x, \omega) = \int_{\mathbb{R}} f(t)\overline{g(t - x)} e^{-2\pi i\omega t}\, dt = \langle f, \pi(x, \omega)g \rangle.$$

If (x_n, ω_n), $n \in \mathbb{N}$, is a discrete subset of $\mathbb{R}^2$ and $h \in L^2$, then the family of functions $(\pi(x_n, \omega_n)h)$ is called a *Gabor system*. If a Gabor system constitutes a Bessel sequence or a frame for L^2, we speak of a *Bessel Gabor system* or *Gabor frame*, respectively. Another important time-frequency distribution is the *(cross) Wigner distribution* of f and g:

$$W(f, g)(x, \omega) = \int_{\mathbb{R}} f(x + \frac{t}{2})\overline{g(x - \frac{t}{2})} e^{-2\pi i\omega t}\, dt.$$

It is related to the STFT via the formula $W(f, g)(x, \omega) = 2e^{4\pi i\omega x} V_{\tilde{g}} f(2x, 2\omega)$ with $\tilde{g}(t) = g(-t)$. Both STFT and Wigner distribution are in $L^2(\mathbb{R}^2)$ if f and g are in $L^2(\mathbb{R})$. Both can be defined for larger classes of functions or even distributions for f and g. The Wigner distribution is associated to the *Weyl calculus*: every continuous operator $T : \mathcal{S} \to \mathcal{S}'$ from *Schwartz class* $\mathcal{S}$ to the *tempered distributions* $\mathcal{S}'$ can be described in the form $\langle Tf, g \rangle = \langle \sigma, W(g, f) \rangle$ for $f, g \in \mathcal{S}$, with a suitable unique (distributional) *Weyl symbol* $\sigma \in \mathcal{S}'(\mathbb{R}^2)$. If T is a Hilbert-Schmidt operator, then one has $\sigma \in L^2(\mathbb{R}^2)$. For all of these facts and many more we refer to [6].

C. Compact and Schatten class operators

A bounded operator $T : H \to H$ is *compact* if the image of any bounded sequence under T contains a convergent subsequence. A compact operator always has a *spectral* *representation* $T(\cdot) = \sum_k s_k(T) \langle \cdot, \phi_k \rangle \psi_k$ with suitably chosen orthonormal systems (ϕ_k), (ψ_k) and a unique sequence $(s_k(T))$ with $s_1(T) \geq s_2(T) \geq \ldots \geq 0$, $k \in \mathbb{N}$. The sequence $(s_k(T))$ is the sequence of *singular values* of T. The operator belongs to *Schatten p-class* $\mathcal{S}^p(H)$, $1 \leq p < \infty$, if $\sum_k |s_k(T)|^p < \infty$. These are Banach spaces with norm $\|T\|_{\mathcal{S}^p} = \|(s_k(T))\|_p = \left(\sum_k |s_k(T)|^p\right)^{1/p}$. $\mathcal{S}^2(H)$ is also called *Hilbert-Schmidt class*, $\mathcal{S}^1(H)$ *trace class*. We use the notation $\mathcal{S}^\infty(H)$ to denote the set $B(H)$ of all bounded operators on H, and $\mathcal{S}^0(H) = K(H)$ to denote the set of all compact operators on H. For more information, refer to e.g. [5] or [7].

II. Bessel Multipliers

Definition II.1. *Let (f_n) and (g_n) be Bessel sequences in H and $m = (m_n) \in \ell^\infty$. The* Bessel multiplier *with* symbol m *(associated to the sequences (f_n) and (g_n)) is defined as the linear operator on H given by*

$$\mathcal{A}^{(f_n),(g_n)}(m)(h) := \sum_n m_n \langle h, f_n \rangle g_n, \qquad h \in H.$$

In order to simplify notation, we will usually suppress the dependence on the Bessel sequences (f_n) and (g_n) and simply write $\mathcal{A}(m)$ instead of $\mathcal{A}^{(f_n),(g_n)}(m)$.

We cite without proof several results from [1].

Lemma II.2. *Let (f_n) and (g_n) be Bessel sequences with Bessel bounds B_F and B_G, respectively. If $(m_n) \in \ell^\infty$, then $\mathcal{A}(m)$ is a well-defined bounded operator on H with norm $\|\mathcal{A}(m)\|_{B(H)} \leq \sqrt{B_F B_G}\, \|m\|_\infty$.*

Lemma II.3. *Let (f_n) and (g_n) be Bessel sequences with Bessel bounds B_F and B_G, respectively. If $(m_n) \in \ell^1$, then $\mathcal{A}(m)$ is a trace class operator on H with norm $\|\mathcal{A}(m)\|_{\mathcal{S}^1} \leq \sqrt{B_F B_G}\, \|m\|_1$.*

Lemma II.4. *Let (f_n) and (g_n) be Bessel sequences with Bessel bounds B_F and B_G, respectively. If $\lim_{n\to\infty} m_n = 0$, i.e. $m \in c_0(\mathbb{N})$, then $\mathcal{A}(m)$ is a compact operator.*

From Lemma II.2 and Lemma II.3, the following is easily proved by interpolation:

Lemma II.5. *Let (f_n) and (g_n) be Bessel sequences with Bessel bounds B_F and B_G, repsectively. If $m \in \ell^p(\mathbb{N})$, $1 \leq p < \infty$, then $\mathcal{A}(m)$ is a Schatten p-class operator, and $\|\mathcal{A}(m)\|_{\mathcal{S}^p} \leq \sqrt{B_F B_G}\, \|m\|_p$.*

Table I summarizes these results.

Symbol	Bessel Multiplier
$\ell^\infty(\mathbb{N})$	$B(H) = \mathcal{S}^\infty(H)$
$c_0(\mathbb{N}) = \ell^0(\mathbb{N})$	$K(H) = \mathcal{S}^0(H)$, compact operator
$\ell^p(\mathbb{N})$, $1 \leq p < \infty$	$\mathcal{S}^p(H)$, Schatten class operator

TABLE I
Bessel multipliers with different symbols

See also the paper [3], which contains somewhat related results for Gabor multipliers.

III. Berezin Transform

Definition III.1. *Let (f_n) and (g_n) be Bessel sequences in H and $T \in B(H)$. The* Berezin transform *of T (associated to the sequences (f_n) and (g_n)) is defined as the function on $\mathbb{N}$ given by*

$$\mathcal{B}^{(f_n),(g_n)}(T)(n) := \langle T f_n, g_n \rangle, \qquad n \in \mathbb{N}.$$

In order to simplify notation we will usually suppress the dependence on the Bessel sequences (f_n) and (g_n) and simply write $\mathcal{B}(T)$ instead of $\mathcal{B}^{(f_n),(g_n)}(T)$.

Lemma III.2. *Let (f_n) and (g_n) be Bessel sequences with Bessel bounds B_F and B_G, respectively. Then the Berezin transform $\mathcal{B}(T)$ is bounded, hence in $\ell^\infty(\mathbb{N})$, and*

$$\|\mathcal{B}(T)\|_\infty \leq \sqrt{B_F B_G}\, \|T\|_{B(H)}.$$

Proof: We have

$$|\mathcal{B}(T)(n)| \leq \|T\|_{B(H)} \|f_n\| \|g_n\| \leq \|T\|_{B(H)} \sqrt{B_F} \sqrt{B_G}$$

by Lemma I.2, for all $n \in \mathbb{N}$. ∎

For later use, we calculate the Berezin transform of a rank-one operator.

Corollary III.3. *Let $\phi, \psi \in H$ and $T : H \to H$, $h \mapsto \langle h, \phi \rangle \psi$ a rank-one operator. Then*

$$\mathcal{B}(T)(n) = \langle f_n, \phi \rangle \langle \psi, g_n \rangle.$$

We collect further mapping properties of the Berezin transform.

Lemma III.4. *Suppose $T \in B(H)$ is a compact operator and (f_n) and (g_n) are Bessel sequences. Then $\lim_{n\to\infty} |\mathcal{B}(T)(n)| = 0$, i.e. $\mathcal{B}(T) \in c_0(\mathbb{N})$.*

Proof: Since $f_n \xrightarrow{w} 0$ for $n \to \infty$, we have $\|T f_n\| \to 0$ for $n \to \infty$. Together with Lemma I.2 this yields

$$|\langle T f_n, g_n \rangle| \leq \|T f_n\| \|g_n\| \to 0, \quad \text{for } n \to \infty.$$

∎

Lemma III.5. *Let (f_n) and (g_n) be Bessel sequences with Bessel bounds B_F and B_G, respectively. Let T be a Schatten class operator, with $1 \leq p < \infty$. Then the Berezin transform $\mathcal{B}(T)$ is in $\ell^p(\mathbb{N})$, and*

$$\|\mathcal{B}(T)\|_p \leq \sqrt{B_F B_G}\, \|T\|_{\mathcal{S}^p}.$$

Proof: Let (e_n) be an arbitrary orthonormal basis for H. By Lemma I.1, there are bounded operators R and S in $B(H)$ such that $f_n = R e_n$ and $g_n = S e_n$ for all n and $\|R\|_{B(H)} \leq \sqrt{B_F}$ and $\|S\|_{B(H)} \leq \sqrt{B_G}$. Hence

$$\langle T f_n, g_n \rangle = \langle T R e_n, S e_n \rangle = \langle S^* T R e_n, e_n \rangle$$

for all n. The operator $\tilde{T} = S^* T R$ is again in $\mathcal{S}^p$, so

$$\left(\sum_n |\mathcal{B}(T)(n)|^p \right)^{\frac{1}{p}} = \left(\sum_n |\langle \tilde{T} e_n, e_n \rangle|^p \right)^{\frac{1}{p}} \leq \|\tilde{T}\|_{\mathcal{S}^p}.$$

Since

$$\|\tilde{T}\|_{\mathcal{S}^p} \leq \|S^*\|_{B(H)}\|T\|_{\mathcal{S}^p}\|R\|_{B(H)} \leq \|T\|_{\mathcal{S}^p}\sqrt{B_F B_G},$$

the proof is finished. ■

Table II summarizes these results.

Operator	Berezin transform
$B(H) = \mathcal{S}^\infty(H)$	$\ell^\infty(\mathbb{N})$
$K(H) = \mathcal{S}^0(H)$, compact operator	$c_0(\mathbb{N}) = \ell^0(\mathbb{N})$
$\mathcal{S}^p(H)$, $1 \leq p < \infty$, Schatten class	$\ell^p(\mathbb{N})$

TABLE II
BEREZIN TRANSFORM OF DIFFERENT OPERATORS

Suggested by the results given above, it becomes obvious that the concept of Bessel multiplier and the Berezin transform are dual to each other.

The following theorem gives the connection between the Berezin transform and Bessel multipliers.

Theorem III.6. *Let $m = (m_n) \in \ell^p(\mathbb{N})$, $1 \leq p \leq \infty$, and $T \in \mathcal{S}^q$, with q the conjugate exponent to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\langle \mathcal{A}(m), T\rangle_{\mathcal{S}^p,\mathcal{S}^q} = \langle m, \mathcal{B}(T)\rangle_{\ell^p,\ell^q}.$$

Proof: For the moment, let (e_k) be an arbitrary orthonormal basis of H. Then the left hand side can be written as

$$\langle \mathcal{A}(m), T\rangle = \sum_k \langle \mathcal{A}(m)(e_k), Te_k\rangle.$$

Inserting $\mathcal{A}(m) = \sum_n m_n\langle \cdot, f_n\rangle g_n$ yields

$$\langle \mathcal{A}(m), T\rangle = \sum_k \sum_n m_n \langle e_k, f_n\rangle\langle g_n, Te_k\rangle. \qquad (*)$$

The right hand side gives

$$\langle m, \mathcal{B}(T)\rangle = \sum_n m_n\langle g_n, Tf_n\rangle = \sum_n m_n \sum_k \langle T^* g_n, e_k\rangle\langle e_k, f_n\rangle$$

by Parseval's equality. Thus

$$\langle m, \mathcal{B}(T)\rangle = \sum_n \sum_k m_n \langle e_k, f_n\rangle\langle g_n, Te_k\rangle. \qquad (**)$$

Comparing $(*)$ and $(**)$, we see that the claimed equality is proved, if we can justify the change of order of summation in the double sum. In order to do so, we examine the corresponding double sum of the absolute values

$$S := \sum_n \sum_k |m_n||\langle e_k, f_n\rangle||\langle g_n, Te_k\rangle|.$$

Consider the case $p = 1$ (i.e. $m \in \ell^1$ and $T \in \mathcal{S}^\infty = B(H)$). Then

$$\begin{aligned} S &\leq \sum_n |m_n| \Big(\sum_k |\langle e_k, f_n\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_k |\langle T^* g_n, e_k\rangle|^2\Big)^{\frac{1}{2}} \\ &\leq \sum_n |m_n|\|f_n\|\|T^* g_n\| \\ &\leq \sqrt{B_F B_G}\ \|m\|_1 \|T\|_{B(H)} < \infty. \end{aligned}$$

If $1 < p \leq \infty$, then $1 \leq q < \infty$ and T is a compact operator in $\mathcal{S}^q$. As such, it has a spectral representation

$$T = \sum_k \lambda_k \langle \cdot, \sigma_k\rangle \tau_k$$

with orthonormal bases (σ_k) and (τ_k), and $\lambda_k \geq 0$ with $\sum_k \lambda_k^q = \|T\|_{\mathcal{S}^q}^q$. Choose the particular orthonormal basis $(e_k) = (\sigma_k)$. Then $Te_k = T\sigma_k = \lambda_k \tau_k$ for all k, and thus

$$\begin{aligned} S &= \sum_{n,k} |m_n||\lambda_k||\langle \sigma_k, f_n\rangle||\langle g_n, \tau_k\rangle| \\ &\leq \Big(\sum_{n,k} |m_n|^p |\langle \sigma_k, f_n\rangle||\langle g_n, \tau_k\rangle|\Big)^{\frac{1}{p}} \times \\ &\qquad \Big(\sum_{n,k} |\lambda_k|^q |\langle \sigma_k, f_n\rangle||\langle g_n, \tau_k\rangle|\Big)^{\frac{1}{q}}. \end{aligned}$$

These two sums can be estimated, the first as

$$\begin{aligned} &\sum_{n,k} |m_n|^p |\langle \sigma_k, f_n\rangle||\langle g_n, \tau_k\rangle| \\ &\quad \leq \sum_n |m_n|^p \Big(\sum_k |\langle \sigma_k, f_n\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_k |\langle g_n, \tau_k\rangle|^2\Big)^{\frac{1}{2}} \\ &\quad \leq \sqrt{B_F B_G}\ \|m\|_p^p \end{aligned}$$

and the second similarly as

$$\begin{aligned} &\sum_{n,k} |\lambda_k|^q |\langle \sigma_k, f_n\rangle||\langle g_n, \tau_k\rangle| \\ &\quad \leq \Big(\sum_k \lambda_k^q\Big) \sqrt{B_F}\|\sigma_k\|\sqrt{B_G}\|\tau_k\| \\ &\quad = \sqrt{B_F B_G}\ \|T\|_{\mathcal{S}^q}^q. \end{aligned}$$

So, finally, we have for $1 < p \leq \infty$

$$S \leq \sqrt{B_F B_G}\ \|m\|_p \|T\|_{\mathcal{S}^q} < \infty.$$

Since in every case $S < \infty$, Fubini's theorem yields the desired conclusion, the equality of $(*)$ and $(**)$. ■

Corollary III.7. *(1) Let $\mathcal{A} : \ell^\infty \to B(H)$ and $\mathcal{B} : \mathcal{S}^1 \to \ell^1$. Then $\mathcal{A} = \mathcal{B}^*$ is the Banach space adjoint.*
(2) Let $\mathcal{A} : c_0 \to K(H)$ and $\mathcal{B} : \mathcal{S}^1 \to \ell^1$. Then $\mathcal{B} = \mathcal{A}^*$.
(3) Let $\mathcal{A} : \ell^p \to \mathcal{S}^p$, $1 \leq p < \infty$, and $\mathcal{B} : \mathcal{S}^q \to \ell^q$, with $1 < q \leq \infty$ the conjugate exponent. Then $\mathcal{B} = \mathcal{A}^*$.

Proof: Observe that $B(H) = (\mathcal{S}^1)^*$ and $\ell^\infty = (\ell^1)^*$ in case (1), $\mathcal{S}^1 = (K(H))^*$ and $c_0 \subseteq \ell^\infty$ with $\ell^1 = (c_0)^*$ in case (2), and $\mathcal{S}^q = (\mathcal{S}^p)^*$ and $\ell^q = (\ell^p)^*$ in case (3). The statements then follow immediately from Theorem III.6. ■

IV. (NON-)DENSITY RESULTS

In this section we investigate whether a given operator on H can be approximated by a Gabor multiplier with respect to various norms. In particular, we would like to understand when the set of Gabor multipliers (associated to a fixed pair of Gabor systems) is dense in $B(H)$ or in $\mathcal{S}^p(H)$ (if ever).

In order to examine such density properties, we employ some well known results from functional analysis. Precisely,

we use the following facts ([5]):
Let X, Y be Banach spaces, $T : X \to Y$ be a bounded operator and let $T^* : Y^* \to X^*$ be the (Banach space) adjoint operator.

- T^* is one-to-one on Y^*, if and only if the range of T is dense in Y with respect to the norm topology on Y.
- T is one-to-one on X, if and only if the range of T^* is dense in X^* with respect to the weak* topology on X^*.

To understand when the mapping $a \to \mathcal{A}(a)$ has dense range, it suffices, in view of Theorem III.6 and its corollary, to check when the Berezin transform $\mathcal{B}$ is one-to-one.

Lemma IV.1. *Let $a, b > 0$ and assume that $(f_{n,m}) = (\pi(an, bm)f)$ and $(g_{n,m}) = (\pi(an, bm)g)$ are Gabor systems, $f, g \in L^2(\mathbb{R})$. Let $(z, \nu) \in \mathbb{R}^2$ and $T = \pi(z, \nu)$ be the corresponding time-frequency shift. Then*

$$\mathcal{B}(T)(n, m) = e^{2\pi i(an\nu - bmz)}\overline{V(g, f)(z, \nu)}$$

for all $n, m \in \mathbb{Z}$.

Corollary IV.2. *Let $(f_{n,m}) = (\pi(an, bm)f)$ and $(g_{n,m}) = (\pi(an, bm)g)$ be Bessel Gabor systems. If there exists a point $(z, \nu) \in \mathbb{R}^2$ such that $V(g, f)(z, \nu) = 0$, then the Berezin transform $\mathcal{B} : B(L^2) \to \ell^\infty$ is not one-to-one.*

Proof: We have $T = \pi(z, \nu) \neq 0$ in $B(L^2)$, but $\mathcal{B}(T) = 0$ by the preceding lemma. ■

For the particular case of Hilbert-Schmidt operators, we have the following negative result:

Theorem IV.3. *Let $(f_{n,m})$ and $(g_{n,m})$ be Bessel Gabor systems. Then the range of $\mathcal{A} : \ell^2 \to \mathcal{S}^2$ is not a norm-dense subspace of Hilbert-Schmidt class. There are thus Hilbert-Schmidt operators on $L^2(\mathbb{R})$ that cannot be approximated in Hilbert-Schmidt norm by Gabor multipliers (with a given fixed pair of Gabor systems).*

Proof: In view of Corollary III.7, it suffices to show that $\mathcal{B} : \mathcal{S}^2 \to \ell^2$ is not one-to-one. Let $T \in \mathcal{S}^2$. Now note that there is a bijective correspondence between Hilbert-Schmidt operators and Weyl symbols in $L^2(\mathbb{R}^2)$. Thus there exists a unique Weyl symbol $\sigma \in L^2(\mathbb{R}^2)$ such that

$$\langle T\phi, \psi\rangle = \langle \sigma, W(\psi, \phi)\rangle$$

for all $\phi, \psi \in L^2(\mathbb{R})$. Thus

$$\begin{aligned}\mathcal{B}(T)(n, m) &= \langle \sigma, W(g_{n,m}, f_{n,m})\rangle \\ &= \langle \sigma, W(\pi(an, bm)g, \pi(an, bm)f)\rangle \\ &= \langle \sigma, T_{(an,bm)}W(g, f)\rangle.\end{aligned}$$

Observe that $W(g, f) \in L^2(\mathbb{R}^2)$. As is well-known, a discrete countable family of translates of a function $F \in L^2(\mathbb{R}^2)$ is never complete, thus

$$U := \overline{\text{span}\{T_{(an,bm)}W(g, f) \mid n, m \in \mathbb{Z}\}}$$

is a proper closed subspace of $L^2(\mathbb{R}^2)$. Choose $0 \neq \sigma \in U^\perp$. Then the corresponding Hilbert-Schmidt operator T satisfies $\mathcal{B}(T)(n, m) = \langle \sigma, W(g_{n,m}, f_{n,m})\rangle = 0$ for all $n, m \in \mathbb{Z}$, thus $\mathcal{B}(T) = 0$, but $T \neq 0$. Hence $\mathcal{B} : \mathcal{S}^2 \to \ell^2$ is not one-to-one. ■

We can extend this result to the cases $1 \leq p < 2$.

Theorem IV.4. *Let $(f_{n,m})$ and $(g_{n,m})$ be Bessel Gabor systems and $1 \leq p < 2$. Then the range of $\mathcal{A} : \ell^p \to \mathcal{S}^p$ is not a norm-dense subspace of the Schatten class $\mathcal{S}^p$.*

Proof: Let $2 < q \leq \infty$ be the conjugate exponent to p. Observe that $\mathcal{S}^2 \subseteq \mathcal{S}^q \subseteq \mathcal{S}^\infty$. By Theorem IV.3, the Berezin transform $\mathcal{B} : \mathcal{S}^2 \to \ell^2$ is not one-to-one, hence, a fortiori, the Berezin transform $\mathcal{B} : \mathcal{S}^q \to \ell^q$ is not one-to-one, either. By Corollary III.7, this is equivalent to the range of $\mathcal{A} : \ell^p \to \mathcal{S}^p$ not being norm-dense. ■

For the cases $2 < p < \infty$, we conjecture analogous results.

For the case $p = \infty$, we have the following result (whose proof we omit for lack of space):

Theorem IV.5. *Let $(f_{n,m})$ and $(g_{n,m})$ be Bessel Gabor systems. Then there exists an operator $R \in B(L^2)$ and a constant $\delta > 0$ such that*

$$\|R - \mathcal{A}(m)\|_{B(L^2)} \geq \delta$$

for all $m \in \ell^\infty$. In particular, the range of $\mathcal{A} : \ell^\infty \to B(L^2)$ is not a norm-dense subspace of $B(L^2)$.

One can take for R the Fourier transform, fractional Fourier transforms or any other operator that incorporates time-frequency shifts of arbitrarily large size.

V. Conclusion

Our results show that subsets of Gabor multipliers with symbols in ℓ^p-spaces are not dense in the respective Schatten classes, but span proper subspaces. There exist thus operators in these Schatten classes that cannot be approximated arbitrarily well by multipliers in the respective Schatten norm. This is in sharp contrast to the case of continuous (STFT) multipliers, as shown in [2]. For approximation of bounded operators in operator norm, however, the negative result shown in this paper also holds analogously in the continuous case.

VI. Acknowledgement

The first author would like to thank K. Gröchenig for his substantial help (in particular with [2]) and continual encouragement.

References

[1] P. Balazs. Basic definition and properties of Bessel multipliers. *J. Math. Anal. Appl.*, 325(1):571–585, 2007.

[2] D. Bayer. *Bilinear Time-Frequency Distributions and Pseudodifferential Operators*. PhD thesis, University of Vienna, 2010.

[3] J. Benedetto and G. Pfander. Frame expansions for Gabor multipliers. *Appl. Comp. Harm. Anal.*, 20(1):26–40, 2006.

[4] O. Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, 2003.

[5] J.B. Conway. *A Course in Functional Analysis*. Springer, New York, 2nd edition, 1990.

[6] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

[7] K. Zhu. *Operator Theory in Function Spaces*. Mathematical Surveys and Monographs. American Mathematical Society, 2nd edition, 2007.