Generalized sampling in U-invariant subspaces

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***Abstract*—In this work we carry out some results in sampling theory for U-invariant subspaces of a separable Hilbert space $\mathcal{H}$, also called atomic subspaces:**

$$\mathcal{A}_a = \big\{ \sum_{n\in\mathbb{Z}} a_n U^n a : \{a_n\} \in \ell^2(\mathbb{Z}) \big\},$$

where U is an unitary operator on $\mathcal{H}$ and a is a fixed vector in $\mathcal{H}$. These spaces are a generalization of the well-known shift-invariant subspaces in $L^2(\mathbb{R})$; here the space $L^2(\mathbb{R})$ is replaced by $\mathcal{H}$, and the shift operator by U. Having as data the samples of some related operators, we derive frame expansions allowing the recovery of the elements in $\mathcal{A}_a$. Moreover, we include a frame perturbation-type result whenever the samples are affected with a jitter error.

I. Introduction

Our work is motivated by the generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$. Namely, assume that our functions (signals) belong to some shift-invariant space of the form:

$$V_\varphi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R})}\big\{\varphi(t-n),\ n\in\mathbb{Z}\big\},$$

where the generator function φ belongs to $L^2(\mathbb{R})$ and the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$. Thus, the shift-invariant space V_φ^2 can be described as

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\ \varphi(t-n)\ :\ \{\alpha_n\}\in\ell^2(\mathbb{Z}) \Big\}. \tag{1}$$

On the other hand, in many common situations the available data are samples of some filtered versions $f*\mathsf{h}_j$ of the signal f itself, where the average function h_j reflects the characteristics of the adquisition device.

Suppose that s convolution systems (linear time-invariant systems or filters in engineering jargon) $\mathcal{L}_j f = f*\mathsf{h}_j$, $j=1,2,\dots,s$, are defined on V_φ^2. Assume also that the sequence of samples $\{(\mathcal{L}_j f)(kr)\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$, where $r\in\mathbb{N}$, is available for any f in V_φ^2.

Mathematically, the generalized sampling problem consists of the stable recovery of any $f\in V_\varphi^2$ from the above sequence of samples, i.e., to obtain sampling formulas in V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(kr)\, S_j(t-kr)\,, \quad t\in\mathbb{R}\,, \tag{2}$$

such that the sequence of reconstruction functions $\{S_j(\cdot - kr)\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for the shift-invariant space V_φ^2 (see, for instance, [3], [5], [6], [7], [9], [10], [15], [16], [17]).

In the present work we provide a generalization of the above problem in the following sense: Let $\{U^t\}_{t\in\mathbb{R}}$ denote a continuous *group of unitary operators* in $\mathcal{H}$ containing our unitary operator U (see Section C) below). For a fixed $a\in\mathcal{H}$, we consider the subspace of $\mathcal{H}$ given by

$$\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n\in\mathbb{Z}\big\}\,.$$

In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ (see, for instance, a necessary and sufficient condition in [13]) we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}\in\ell^2(\mathbb{Z}) \Big\}\,.$$

On the other hand, for $b_j\in\mathcal{H},\ j=1,2\dots,s$ we consider the linear operators $x\in\mathcal{H}\mapsto\mathcal{L}_j x\in C(\mathbb{R})$ defined on $\mathbb{R}$ as

$$\big(\mathcal{L}_j x\big)(t) := \langle x, U^t b_j\rangle_{\mathcal{H}}, \quad t\in\mathbb{R}\,. \tag{3}$$

These operators $\mathcal{L}_j$ can be seen as a generalization of the previous convolution systems.

II. Goals and procedure

Given $b_j\in\mathcal{A}_a,\ j=1,2\dots,s$, our aim is to recover any $x\in\mathcal{A}_a$, in a stable way, by means of the sequence of generalized samples

$$\big\{\big(\mathcal{L}_j x\big)(kr)\big\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

obtained from (3) (here r denotes a fixed number in $\mathbb{N}$). In order to do this we only deal with the discrete group $\{U^n\}_{n\in\mathbb{Z}}$ completely determined by U, but we might be in presence of a time jitter error, and then, the study of the continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ becomes essential. Having as data a perturbed sequence of samples

$$\big\{\big(\mathcal{L}_j x\big)(kr+\epsilon_{kj})\big\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

with errors $\epsilon_{kj}\in\mathbb{R}$, again we want to recover $x\in\mathcal{A}_a$.

In order to attack these problems we have proceeded in the following steps:

(a) The study of when the sequence $\big\{U^{kr} b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a complete system, a Bessel sequence, a frame or a Riesz basis for $\mathcal{A}_a$.

(b) In the frame case, search for a family of dual frames of the form $\big\{U^{kr} c_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$, where $c_j\in\mathcal{A}_a,\ j=$

$1, 2 \ldots, s$, allowing to recover any $x \in \mathcal{A}_a$ by means of the sampling formula

$$x = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j x\big)(kr)\, U^{kr} c_j \quad \text{in } \mathcal{H}\,. \tag{4}$$

(c) Using the standard perturbation theory of frames (see Ref. [4]) and the group of unitary operators theory [2], [18], to find a condition on the error sequence $\{\epsilon_{kj}\}$ allowing the recovery of any $x \in \mathcal{A}_a$ by means of a sampling expansion as

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \big(\mathcal{L}_j x\big)(kr + \epsilon_{kj})\, C^{\epsilon}_{k,j} \quad \text{in } \mathcal{H}\,, \tag{5}$$

where the sequence $\{C^{\epsilon}_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_a$.

At stages (a) and (b) we have used some borrowed ideas from [13]; mainly related to the stationary properties of a sequence of the form $\{U^n b\}_{n\in\mathbb{Z}}$, $b \in \mathcal{H}$, and the spectral measure associated with the (auto)-covariance function of b.

III. Main results

A. The study of the sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$

If for every $j = 1, 2, \ldots s$ the spectral measure in the integral representation of the (cross)-covariance function of the sequences $\{U^k a\}_{k\in\mathbb{Z}}$, $\{U^k b_j\}_{k\in\mathbb{Z}}$ has no singular part, we have the following representation

$$\langle U^k a, U^{nr} b_j\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta})\, d\theta.$$

where ϕ_{a,b_j} stands for the cross spectral density of the stationary correlated sequences $\{U^k a\}_{k\in\mathbb{Z}}$ and $\{U^k b_j\}_{k\in\mathbb{Z}}$. Consider the $s \times 1$ matrices of functions defined on the torus $\mathbb{T} := \{e^{i\theta} \,:\, \theta \in [-\pi, \pi)\}$

$$\Phi_{a,b}(e^{i\theta}) := \begin{pmatrix} \phi_{a,b_1}(e^{i\theta}) \\ \phi_{a,b_2}(e^{i\theta}) \\ \vdots \\ \phi_{a,b_s}(e^{i\theta}) \end{pmatrix},$$

and

$$\Psi^l_{a,b}(e^{i\theta}) := (D_r S^{-l} \Phi_{a,b})(e^{i\theta})\,, \quad l = 0, 1, \ldots, r-1\,,$$

where $D_r : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the decimation operator

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \overset{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta}$$

and $S : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the (left) shift operator

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \overset{S}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{k+1} e^{ik\theta}\,.$$

Finally, defining the $s \times r$ matrix of functions on the torus $\mathbb{T}$

$$\Psi_{a,b}(e^{i\theta}) := \big(\Psi^0_{a,b}(e^{i\theta})\ \Psi^1_{a,b}(e^{i\theta})\ \ldots \Psi^{r-1}_{a,b}(e^{i\theta})\big)\,, \tag{6}$$

and its related constants,

$$\begin{aligned} A_\Psi &:= \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}}\ \lambda_{\min}\big[\Psi^*_{a,b}(\zeta)\Psi_{a,b}(\zeta)\big]; \\ B_\Psi &:= \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}}\ \lambda_{\max}\big[\Psi^*_{a,b}(\zeta)\Psi_{a,b}(\zeta)\big] \end{aligned} \tag{7}$$

we have the following result:

Theorem 3.1: Let b_j be in $\mathcal{A}_a$ for $j = 1, 2, \ldots, s$ and let $\Psi_{a,b}$ be the associated matrix given in (6) and its related constants (7). Then, the following results hold:

i) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a complete system in $\mathcal{A}_a$ if and only the rank of the matrix $\Psi_{a,b}(\zeta)$ is r a.e. ζ in $\mathbb{T}$.

ii) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Bessel sequence for $\mathcal{A}_a$ if and only the constant $B_\Psi < \infty$.

iii) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$ if and only if constants A_Ψ and B_Ψ satisfy $0 < A_\Psi \leq B_\Psi < \infty$. In this case, A_Ψ and B_Ψ are the optimal frame bounds for $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$.

iv) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Riesz basis for $\mathcal{A}_a$ if and only if it is a frame and $s = r$.

B. The frame expansion

Define the $r \times s$ matrix Γ of functions on $\mathbb{T}$ as

$$\Gamma(e^{i\theta}) := \sum_{k\in\mathbb{Z}} \Gamma_k e^{ik\theta} = [\Psi^*_{a,b}(e^{i\theta})\Psi_{a,b}(e^{i\theta})]^{-1}\Psi^*_{a,b}(e^{i\theta}). \tag{8}$$

Note that $\Psi^\dagger_{a,b}(e^{i\theta}) := [\Psi^*_{a,b}(e^{i\theta})\Psi_{a,b}(e^{i\theta})]^{-1}\Psi^*_{a,b}(e^{i\theta})$ stands for the Moore-Penrose left-inverse. In case that condition iii) in Theorem 3.1 is satisfied, we can define,

$$\widetilde{a}_n := \begin{pmatrix} U^{nr} a \\ U^{nr+1} a \\ \vdots \\ U^{nr+r-1} a \end{pmatrix}$$

and

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} := \sum_{k\in\mathbb{Z}} \Gamma^\top_k \widetilde{a}_k\,.$$

Note that, under condition iii) in Theorem 3.1, the matrix $\Gamma(e^{i\theta})$ has entries in $L^\infty(\mathbb{T})$.

Then, the sequences $\{U^{kr} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ and $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ are a pair of dual frames for $\mathcal{A}_a$. Hence we obtain the following recovery formula in $\mathcal{A}_a$: For any $x \in \mathcal{A}_a$, the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{kr} b_j\rangle U^{kr} c_j \quad \text{in } \mathcal{H}$$

holds.

The analysis done provides a whole family of dual frames; in fact, everything works if we choose in (8) a matrix of the form

$$\Gamma_{\mathbb{U}}(e^{i\theta}) := \Psi^\dagger_{a,b}(e^{i\theta}) + \mathbb{U}(e^{i\theta})\big[\mathbb{I}_s - \Psi_{a,b}(e^{i\theta})\Psi^\dagger_{a,b}(e^{i\theta})\big]\,,$$

where $\mathbb{U}(e^{i\theta})$ denotes any $r\times s$ matrix with entries in $L^\infty(\mathbb{T})$, and $\Psi^\dagger_{a,b}$ the Moore-Penrose left pseudo-inverse.

Notice that if $s=r$, $\Psi^\dagger_{a,b}=\Psi^{-1}_{a,b}$ which implies that Γ is unique and we are in presence of a pair of dual Riesz basis.

Remark: In Theorem 3.1 we have assumed that b_j belongs to $\mathcal{A}_a$ for each $j=1,2,\dots,s$ since we want the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ to be contained in $\mathcal{A}_a$. In case that some $b_j\notin\mathcal{A}_a$, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is not necessarily contained in $\mathcal{A}_a$. However, whenever $0<A_\Psi\le B_\Psi<\infty$, the inequalities

$$A_\Psi\|x\|^2\le\sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}|\langle x,U^{rk}b_j\rangle|^2\le B_\Psi\|x\|^2\quad\text{for all } x\in\mathcal{A}_a$$

hold, and conversely. Hence, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a pseudo-frame for $\mathcal{A}_a$ (see Refs. [11], [12]).

Denoting by $P_{\mathcal{A}_a}$ the orthogonal projection onto $\mathcal{A}_a$, since for each $x\in\mathcal{A}_a$ we have

$$\langle x,U^{rk}b_j\rangle=\langle x,P_{\mathcal{A}_a}\big(U^{rk}b_j\big)\rangle\,,\ k\in\mathbb{Z}\ \text{and}\ j=1,2,\dots,s\,,$$

and, as a consequence, Theorem 3.1 can be reformulated in terms $\{P_{\mathcal{A}_a}\big(U^{rk}b_j\big)\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$, a sequence in $\mathcal{A}_a$.

C. The study of the time jitter error

In Sections A) and B) it is not strictly necessary to have a group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ to obtain the announced results. However, in order to deal with the time-jitter error this formalism becomes essential in our approach.

Let $\{U^t\}_{t\in\mathbb{R}}$ denote a continuous *group of unitary operators* in $\mathcal{H}$ containing our unitary operator U, i.e., say for instance $U:=U^1$. Recall that $\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying (see Ref. [2, vol. 2; p. 29]):

1) $U^t\,U^{t'}=U^{t+t'}$,
2) $U^0=I_\mathcal{H}$,
3) $\langle U^tx,y\rangle_\mathcal{H}$ is a continuous function of t for any $x,y\in\mathcal{H}$.

Note that $(U^t)^{-1}=U^{-t}$, and since $(U^t)^*=(U^t)^{-1}$, we have $(U^t)^*=U^{-t}$.

Classical Stone's theorem [14] assures us the existence of a self-adjoint operator T (possibly unbounded) such that $U^t\equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T, defined on the dense domain of $\mathcal{H}$

$$D_T:=\Big\{x\in\mathcal{H}\ \text{such that}\ \int_{-\infty}^{\infty}w^2\,d\|E_wx\|^2<\infty\Big\}\,,$$

admits the *spectral representation* $T=\int_{-\infty}^{\infty}w\,dE_w$ which means:

$$\langle Tx,y\rangle=\int_{-\infty}^{\infty}w\,d\langle E_wx,y\rangle\qquad\text{for any } x\in D_T \text{ and } y\in\mathcal{H}\,,$$

where $\{E_w\}_{w\in\mathbb{R}}$ is the corresponding *resolution of the identity*, i.e., a one-parameter family of projection operators E_w in $\mathcal{H}$ such that

1) $E_{-\infty}:=\lim\limits_{w\to-\infty}E_w=O_\mathcal{H}$, $E_\infty:=\lim\limits_{w\to\infty}E_w=I_\mathcal{H}$,
2) $E_{w^-}=E_w$ for every $-\infty<w<\infty$,
3) $E_u\,E_v=E_w$ where $w=\min\{u,v\}$.

Recall that $\|E_wx\|^2$ and $\langle E_wx,y\rangle$, as functions of w, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Furthermore, for any $x\in D_T$ we have that $\lim\limits_{t\to0}\dfrac{U^tx-x}{t}=\mathrm{i}\,Tx$ and the operator T is said to be the *infinitesimal generator* of the group $\{U^t\}_{t\in\mathbb{R}}$. For each $x\in D_T$, U^tx is a continuous differentiable function of t. Notice that, whenever the self-adjoint operator T is bounded, $D_T=\mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t\equiv\mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^tx,y\rangle=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}wt}d\langle E_wx,y\rangle\,,\quad t\in\mathbb{R}\,,$$

where $x\in D_T$ and $y\in\mathcal{H}$.

The following result on frame perturbation, which proof can be found in [4, p. 354] has been used:

Lemma 3.2: Let $\{x_n\}_{n=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{y_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R<A$ such that

$$\sum_{n=1}^{\infty}|\langle x_n-y_n,x\rangle|^2\le R\|x\|^2\ \ \text{for each } x\in\mathcal{H}\,,$$

then the sequence $\{y_n\}_{n=1}^\infty$ is also a frame for $\mathcal{H}$ with bounds $A\big(1-\sqrt{R/A}\big)^2$ and $B\big(1+\sqrt{R/B}\big)^2$. If $\{x_n\}_{n=1}^\infty$ is a Riesz basis, then $\{y_n\}_{n=1}^\infty$ is a Riesz basis.

Thus, we have the following result:

Theorem 3.3: Assume that for some $b_j\in D_T$, i.e., $\int_{-\infty}^{\infty}w^2d\|E_wb_j\|^2<\infty$ for each $1\le j\le r$, the sequence $\{U^{kr}b_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$ with Riesz bounds $0<A_\Psi\le B_\Psi<\infty$. For a sequence $\boldsymbol{\epsilon}:=\{\epsilon_{kj}\}_{k\in\mathbb{Z},\,j=1,2,\dots,r}$ of errors, let R be the constant given by

$$R:=\|\boldsymbol{\epsilon}\|^2\max_{j=1,2,\dots,r}\Big\{\int_{-\infty}^{\infty}w^2d\|E_wb_j\|^2\Big\}\,,$$

where $\|\boldsymbol{\epsilon}\|$ denotes the ℓ^2_s-norm of the sequence $\boldsymbol{\epsilon}$.

If $R<A_\Psi$, then the sequence $\{U^{kr+\epsilon_{kj}}b_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,r}$ is a Riesz sequence in $\mathcal{H}$ with Riesz bounds $A_\Psi\big(1-\sqrt{R/A_\Psi}\big)^2$ and $B_\Psi\big(1+\sqrt{R/B_\Psi}\big)^2$.

Next, we deal with the problem of the recovery of any $x\in\mathcal{A}_a$ in a stable way from the perturbed sequence

$$\{(\mathcal{L}_jx)(kr+\epsilon_{kj})\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}\,,$$

where $\boldsymbol{\epsilon}:=\{\epsilon_{kj}\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ denotes a sequence of real errors.

Taking into account the $L^2(0,1)$ functions

$$g_j(w):=\sum_{k\in\mathbb{Z}}\langle a,U^kb_j\rangle_\mathcal{H}\,\mathrm{e}^{2\pi\mathrm{i}kw}\,,\ j=1,2,\dots,s\,,\qquad(9)$$

we can define the $s\times r$ matrix

$$\mathbb{G}(w):=\Big[g_j\Big(w+\frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s\\k=1,2,\dots,r}}$$

and its related the constants $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$ are given by

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$
$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,.$$

It is worth to mention that in [9] was proved that the sequence $\{\overline{g_j(w)}\, e^{2\pi i rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$. The idea is to consider the sequence $\{\overline{g_{m,j}(w)}\, e^{2\pi i rmw}\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ as a perturbation of the above frame in $L^2(0,1)$, where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, e^{2\pi i kw}\,,\ j=1,2,\dots,s\,.$$

For $|\gamma| < 1/2$, define the functions,

$$M_{a,b_j}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\langle a, U^{k+t} b_j\rangle - \langle a, U^k b_j\rangle|\,,$$

and

$$N_{a,b_j}(\gamma) := \max_{k=0,1,\dots,r-1} \sum_{m\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\langle a, U^{rm+k+t} b_j\rangle - \langle a, U^{rm+k} b_j\rangle|\,.$$

Notice that $N_{a,b_j}(\gamma) \le M_{a,b_j}(\gamma)$ and for $r=1$ the equality holds. Moreover, assuming that the continuous functions $\varphi_j(t) := \langle a, U^t b_j\rangle$, $j=1,2,\dots,s$, satisfy a decay condition as $\varphi_j(t) = O\big(|t|^{-(1+\eta_j)}\big)$ when $|t|\to\infty$ for some $\eta_j > 0$, we deduce that the functions $N_{a,b_j}(\gamma)$ and $M_{a,b_j}(\gamma)$ are continuous near to 0.

Theorem 3.4: Assume that for the functions g_j, $j = 1,2,\dots,s$, given in (9) we have $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$. For an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$, define the constant $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$ for each $j=1,2,\dots,s$. Then the condition $\sum_{j=1}^s M_{a,b_j}(\gamma_j) N_{a,b_j}(\gamma_j) < \alpha_{\mathbb{G}}/r$ implies that there exists a frame $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x\in\mathcal{A}_a$, the sampling expansion

$$x = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, C^{\boldsymbol{\epsilon}}_{m,j} \quad \text{in } \mathcal{H}\,, \tag{10}$$

holds. Moreover, when $r = s$ the sequence $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$, and the interpolation property $\big\langle C^{\boldsymbol{\epsilon}}_{n,j}, U^{rm+\epsilon_{ml}} b_l\big\rangle_{\mathcal{H}} = \delta_{j,l}\,\delta_{n,m}$ holds.

Sampling formula (10) is useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$. As a consequence, in order to recover $x\in\mathcal{A}_a$ from the sequence of inner products $\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ we could implement a frame algorithm in $\ell^2(\mathbb{Z})$. Another possibility is given in the recent Ref. [1].

IV. Conclusion

By way of conclusion we may say that we have obtained a complete characterization of the sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ in $\mathcal{A}_a$, where $b_j\in\mathcal{A}_a$, $1\le j\le s$. We have found a necessary and sufficient condition ensuring that it is a complete system, a Bessel sequence, a frame or a Riesz basis for $\mathcal{A}_a$.

In the case that this sequence is a frame for $\mathcal{A}_a$ we can give an explicit family of dual frames allowing to recover any $x\in\mathcal{A}_a$ by means of a sampling formula like (4).

Concerning the perturbation framework, we have found a condition related to the ℓ^2-norm of $\boldsymbol{\epsilon} = \{\epsilon_{kj}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ and the $\max_{j=1,2,\dots,s}\Big\{\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2\Big\}$ such that the sequence $\{U^{kr+\epsilon_{kj}} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz sequence in $\mathcal{H}$ and we have obtained a sampling expansion allowing us to recover any $x\in\mathcal{A}_a$ in a stable way from the perturbed sequence of samples $\{(\mathcal{L}_j x)(kr+\epsilon_{kj})\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

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