Variation and approximation for Mellin-type operators

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***Abstract*—Mellin analysis is of extreme importance in approximation theory, also for its wide applications: among them, for example, it is connected with problems of Signal Analysis, such as the Exponential Sampling. Here we study a family of Mellin-type integral operators defined as**

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}) f(\mathtt{st}) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}, \ \mathtt{s} \in \mathbb{R}^N_+, \ w > 0, \qquad \text{(I)}$$

where $\{K_w\}_{w>0}$ are (essentially) bounded approximate identities, $\langle \mathtt{t} \rangle := \prod_{i=1}^N t_i$, $\mathtt{t} = (t_1, \dots, t_N) \in \mathbb{R}^N_+$, and $f : \mathbb{R}^N_+ \to \mathbb{R}$ is a function of bounded $\varphi-$variation. We use a new concept of multidimensional $\varphi-$variation inspired by the Tonelli approach, which preserves some of the main properties of the classical variation. For the family of operators (I), besides several estimates and a result of approximation for the $\varphi-$modulus of smoothness, the main convergence result that we obtain proves that

$$\lim_{w \to +\infty} V^{\varphi}[\mu(T_w f - f)] = 0,$$

for some $\mu > 0$, provided that f is $\varphi-$absolutely continuous. Moreover, the problem of the rate of approximation is studied, taking also into consideration the particular case of Fejér-type kernels.

I. Introduction

An important topic in approximation theory is the study of convergence of classes of integral operators in the frame of $BV-$spaces, namely spaces of functions of bounded variation. This problem was faced in the literature from several points of view, using different families of operators and different notions of variation, such as the classical variation ([4]), the distributional variation ([7]), the Cesari variation ([16]) or the Musielak-Orlicz $\varphi-$variation ([26], [15], [24], [28], [13], [17], [5]). An important direction of this research is the multidimensional case, in particular in view of the application of such results in several fields, such as image reconstruction. Results in this sense can be found, for example, in [10], [4] in the case of Tonelli variation and in [6], where the authors introduce a new multidimensional concept of $\varphi-$variation and give approximation results for functions of bounded $\varphi-$variation by means of the classical convolution integral operators. The nonlinear case was explored in [3].

An interesting development of the theory is the case of Mellin-type integral operators. Mellin operators are well known and widely used in approximation theory (see, e.g., [23], [19]), also because of their important applications in various fields, for example in Signal Processing. Indeed, Mellin analysis is strictly connected to Signal Analysis, in particular to the Exponential Sampling. A seminal paper in this sense is [20], where the authors establish a Sampling Theorem in which the samples are not equally spaced, as in the classical Shannon Sampling Theorem, but exponentially spaced, by means of Mellin transform methods. This theory has important applications, for example in optical physics and engineering (see, e.g., [22], [18]), in problems in which information accumulates near time $t = 0$. With this respect, to develop a theory about Mellin-type operators becomes useful and interesting. Results in this sense can be also found, for example, in [11], [12].

Here we consider a family of Mellin-type integral operators of the form

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}) f(\mathtt{st}) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}, \ \mathtt{s} \in \mathbb{R}^N_+, \ w > 0 \qquad \text{(I)}$$

and we develop an approximation theory in the frame of $BV-$spaces. In particular, $f : \mathbb{R}^N_+ \to \mathbb{R}$ will be a function of bounded $\varphi-$variation on $\mathbb{R}^N_+$ and $\{K_w\}_{w>0}$ will be a family of (essentially) bounded approximate identities (see Section IV). Here φ is a convex $\varphi-$function (see Section II) such that $u^{-1}\varphi(u) \to 0$ as $u \to 0^+$. The above operators (I) allow us to obtain, as particular cases, several classes of integral operators well-known and used in approximation theory, such as, for example, the moment-type or average operators, the Gauss-Weierstrass-type operators and others.

The new multidimensional concept of variation that we will use is inspired to the Tonelli approach ([29]) (see also [27] and [30]). Such concept of variation was introduced in [8], and it was adapted to the setting of $\mathbb{R}^N_+$ from the multidimensional $\varphi-$variation defined in [6] in the case of $\mathbb{R}^N$ endowed with the Lebesgue measure. Indeed, in order to treat the Mellin case, it is natural to frame the theory in $\mathbb{R}^N_+$ endowed with the Haar measure $\mu(A) := \int_A \langle \mathtt{t} \rangle^{-1} \, d\mathtt{t}$, where A is a Borel subset of $\mathbb{R}^N_+$ and $\langle \mathtt{t} \rangle := \prod_{i=1}^N t_i$, $\mathtt{t} = (t_1, \dots, t_N) \in \mathbb{R}^N_+$. We recall that, in the case of the Lebesgue measure, similar approximation results were obtained in [2], [9], while the one-dimensional case was explored in [15] and [14] (nonlinear case).

In order to get convergence of the family $\{T_w f\}_{w>0}$ to f, a crucial tool is to prove that

$$\lim_{\delta\to 0^+} \omega(f,\delta) = 0, \qquad (1)$$

where $\omega(f,\delta)$ denotes the modulus of smoothness of f. It is well known that (1) holds if and only if f is absolutely continuous working with the classical (Jordan or Tonelli) variation (see, e.g., [21], [10], [4]). On the contrary, dealing with the $\varphi-$variation, due to the lack of an integral representation of $\varphi-$variation in terms of $\varphi-$absolute continuity, the result is no more trivial. In particular, working with the Musielak-Orlicz $\varphi-$variation, the result can be obtaind by means of a direct construction (see, e.g., [26], [5]). In the multidimensional setting, the situation becomes more delicate. The result was obtaind in [8] where, through an approximation technique by means of step-type functions, we proved that

$$\lim_{\delta\to 0^+} \omega^\varphi(\lambda f,\delta) = 0, \qquad (2)$$

for some $\lambda > 0$, provided that the function f is $\varphi-$absolutely continuous. Here $\omega^\varphi(\lambda f,\delta) := \sup_{|\mathtt{1}-\mathtt{t}|\le\delta} V^\varphi[\lambda(\tau_{\mathtt{t}} f - f)]$ ($\tau_{\mathtt{t}} f(\mathtt{s}) = f(\mathtt{st})$, $\mathtt{s},\mathtt{t} \in \mathbb{R}^N_+$ is the dilation operator and $\mathtt{1}$ is the unit vector of $\mathbb{R}^N_+$) represents the natural reformulation of the classical modulus of smoothness in terms of $\varphi-$variation (see, e.g., [25], [13]). The above result proves that the situation is analogous to the one-dimensional case (see [15], [14]) and to the case of the Lebesgue measure (see, e.g., [1], [2]).

In this paper we develop a new theory about convergence and rate of approximation for the operators (I). In particular we first obtain several estimates for $\{T_w f\}_{w>0}$. Then, by means of such results and using (2), we are able to prove the main convergence theorem, which states that there exists a constant $\mu > 0$ such that

$$\lim_{w\to+\infty} V^\varphi[\mu(T_w f - f)] = 0,$$

whenever $f \in AC^\varphi(\mathbb{R}^N_+)$ (the space of $\varphi-$absolutely continuous functions). Introducing suitable Lipschitz classes, we also study the problem of the rate of approximation. Moreover, in the particular case of Fejér-type kernels, we obtain that all the assumptions for the rate of approximation are implied by the classical condition that the absolute moments of order α of the kernels are finite.

We finally point out that the case of the classical variation can be also treated, by using a direct approach: indeed, taking the identity function instead of the $\varphi-$function φ, it is possible to obtain a new multidimensional version of the classical Jordan variation in the sense of Tonelli for functions defined on $\mathbb{R}^N_+$ equipped with the logarithmic measure.

II. Notations and definitions

We denote by Φ the class of all the functions φ such that

1) φ is a convex $\varphi-$function, where a $\varphi-$function is a nondecreasing continuous function $\varphi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty}\varphi(u) = +\infty$;
2) $u^{-1}\varphi(u) \to 0$ as $u \to 0^+$.

From now on we will assume that $\varphi \in \Phi$.

Given $f : \mathbb{R}^N_+ \to \mathbb{R}$ and $\mathtt{x} = (x_1,\ldots,x_N) \in \mathbb{R}^N_+$, $N \in \mathbb{N}$, if we are interested in particular in the $j-$th coordinate, $j = 1,\ldots,N$, we will write

$$x'_j = (x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N) \in \mathbb{R}^{N-1}_+,$$

so that $\mathtt{x} = (x'_j, x_j)$ and $f(\mathtt{x}) = f(x'_j, x_j)$. For a fixed interval $I = \prod_{i=1}^N [a_i,b_i]$, we will denote by $[a'_j, b'_j]$ the $(N-1)-$dimensional interval obtained deleting by I the $j-$th coordinate, so that

$$I = [a'_j, b'_j] \times [a_j, b_j].$$

Moreover, given two vectors $\mathtt{s},\mathtt{t} \in \mathbb{R}^N_+$, we put $\mathtt{st} = (s_1t_1,\ldots,s_Nt_N)$.

In order to define the multidimensional $\varphi-$variation, the first step is to compute the Musielak-Orlicz $\varphi-$variation of the $j-$th section of f, i.e., $V^\varphi_{[a_j,b_j]}[f(x'_j,\cdot)]$, and then to consider the $(N-1)-$dimensional integrals

$$\Phi^\varphi_j(f,I) := \int_{a'_j}^{b'_j} V^\varphi_{[a_j,b_j]}[f(x'_j,\cdot)] \frac{dx'_j}{\langle x'_j\rangle},$$

where by $\langle x'_j\rangle$ we denote the product $\prod_{i=1,i\ne j}^N x_i$. We recall that the $\varphi-$variation of a function $g : [a,b] \to \mathbb{R}$ is defined as

$$V^\varphi_{[a,b]}[g] := \sup_D \sum_{i=1}^n \varphi(|g(s_i) - g(s_{i-1})|),$$

where $D = \{s_0 = a, s_1,\ldots,s_n = b\}$ is a partition of $[a,b]$ ([26], [25]), and g is said to be of bounded $\varphi-$variation ($g \in BV^\varphi([a,b])$) if $V^\varphi_{[a,b]}[\lambda g] < +\infty$, for some $\lambda > 0$.

Let now $\Phi^\varphi(f,I)$ be the Euclidean norm of the vector $(\Phi^\varphi_1(f,I),\ldots,\Phi^\varphi_N(f,I))$, namely

$$\Phi^\varphi(f,I) := \left\{\sum_{k=1}^N [\Phi^\varphi_k(f,I)]^2\right\}^{\frac{1}{2}}.$$

We set $\Phi^\varphi(f,I) = +\infty$ if $\Phi^\varphi_k(f,I) = +\infty$ for some $k = 1,\ldots,N$.

We define the multidimensional $\varphi-$variation of f on an interval $I \subset \mathbb{R}^N_+$ as

$$V^\varphi[f,I] := \sup \sum_{i=1}^m \Phi^\varphi(f,J_i),$$

where the supremum is taken over all the finite families of $N-$dimensional intervals $\{J_1,\ldots,J_m\}$ which form partitions of I.

The $\varphi-$variation of f over the whole space $\mathbb{R}^N_+$ is defined as

$$V^\varphi[f] := \sup_{I\subset\mathbb{R}^N_+} V^\varphi[f,I],$$

where the supremum is taken over all the intervals $I \subset \mathbb{R}^N_+$. By $BV^\varphi(\mathbb{R}^N_+)$ we denote the *space of functions of bounded $\varphi-$variation* over $\mathbb{R}^N_+$, i.e.,

$$BV^\varphi(\mathbb{R}^N_+) = \{f \in L^1_\mu(\mathbb{R}^N_+) :\ \exists\lambda > 0 \text{ s.t. } V^\varphi[\lambda f] < +\infty\}.$$

We will say that a function $f : \mathbb{R}^N_+ \to \mathbb{R}$ is *locally $\varphi-$absolutely continuous* ($f \in AC^{\varphi}_{loc}(\mathbb{R}^N_+)$) if f is (uniformly) $\varphi-$absolutely continuous in the Tonelli sense: this means that for every $I = \prod_{i=1}^{N}[a_i, b_i] \subset \mathbb{R}^N_+$ and for every $j = 1, 2, \ldots, N$, the $j-$th sections of f, $f(x'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$, are (uniformly) $\varphi-$absolutely continuous for almost every $x'_j \in [a'_j, b'_j]$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ for which

$$\sum_{i=1}^{n} \varphi(\lambda|f(x'_j, \beta^i) - f(x'_j, \alpha^i)|) < \varepsilon,$$

for a.e. $x'_j \in [a'_j, b'_j]$ and for all finite collections of non-overlapping intervals $[\alpha^i, \beta^i] \subset [a_j, b_j]$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \varphi(\beta^i - \alpha^i) < \delta$.

By $AC^{\varphi}(\mathbb{R}^N_+)$ we will denote the subspace of $BV^{\varphi}(\mathbb{R}^N_+)$ of the *$\varphi-$absolutely continuous functions*, namely all the functions of bounded $\varphi-$variation which are locally $\varphi-$absolutely continuous.

III. Results about the multidimensional $\varphi-$variation

Our multidimensional $\varphi-$variation satisfies similar properties to the Musielak-Orlicz $\varphi-$variation and to the Jordan variation. In particular we prove that $BV^{\varphi}(\mathbb{R}^N_+)$ is a vector space, namely, $\alpha f_1 + \beta f_2 \in BV^{\varphi}(\mathbb{R}^N_+)$ whenever $f_1, f_2 \in BV^{\varphi}(\mathbb{R}^N_+)$, $\alpha, \beta \in \mathbb{R}$. Indeed this is a consequence of the following property ([8])

$$V^{\varphi}[\lambda(f_1 + f_2)] \leq \frac{1}{2}\Big(V^{\varphi}[2\lambda f_1] + V^{\varphi}[2\lambda f_2]\Big), \quad \lambda > 0,$$

and of the trivial consideration that $V^{\varphi}[\lambda f] \leq V^{\varphi}[\mu f]$, if $0 < \lambda \leq \mu$.

Another classical property of variation which is preserved by our definition is the lower semicontinuity with respect to pointwise convergence. Indeed, in this paper we prove that, if $(f_k)_{k\in\mathbb{N}}$ is pointwise convergent to f, then

$$V^{\varphi}[f] \leq \liminf_{k\to+\infty} V^{\varphi}[f_k].$$

Finally it is also possible to prove results about additivity on intervals which are quite similar to the classical ones. Nevertheless we recall that a crucial difference with the Jordan variation is that, in the frame of $\varphi-$variation, even in the one-dimensional case, we don't have at our disposal an integral representation of $\varphi-$variation for absolutely continuous functions.

IV. Mellin operators and convergence results

We will study the following family of Mellin-type integral operators of the form

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}) f(\mathtt{st}) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}, \quad w > 0, \ \mathtt{s} \in \mathbb{R}^N_+, \qquad \text{(I)}$$

for $f \in BV^{\varphi}(\mathbb{R}^N_+)$, where $\{K_w\}_{w>0}$ is a family of bounded approximate identities, i.e.,

$\mathbf{K_w.1}$) $K_w : \mathbb{R}^N_+ \to \mathbb{R}$ is a measurable essentially bounded function such that $K_w \in L^1_{\mu}(\mathbb{R}^N_+)$, $\|K_w\|_{L^1_{\mu}} \leq A$ for an absolute constant $A > 0$ and $\int_{\mathbb{R}^N_+} K_w(\mathtt{t})\langle \mathtt{t} \rangle^{-1} d\mathtt{t} = 1$, for every $w > 0$,

$\mathbf{K_w.2}$) for every fixed $0 < \delta < 1$, $\int_{|\mathbf{1}-\mathtt{t}|>\delta} |K_w(\mathtt{t})|\langle \mathtt{t} \rangle^{-1} d\mathtt{t} \to 0$, as $w \to +\infty$.

We point out that, since K_w is essentially bounded, if $f \in BV^{\varphi}(\mathbb{R}^N_+)$, $(T_w f)(\mathtt{s})$ is well-defined for every $\mathtt{s} \in \mathbb{R}^N_+$ and $w > 0$.

We first obtain two estimates for our integral operators (I). The first one proves that $\{T_w\}_{w>0}$ map $BV^{\varphi}(\mathbb{R}^N_+)$ into itself.

Proposition 1: Let $f \in BV^{\varphi}(\mathbb{R}^N_+)$ and let $\{K_w\}_{w>0}$ be such that $K_w.1$) holds. Then there exists $\lambda > 0$ such that

$$V^{\varphi}[\lambda(T_w f)] \leq V^{\varphi}[\zeta f], \qquad (3)$$

where $\zeta > 0$ is the constant for which $V^{\varphi}[\zeta f] < +\infty$. Therefore, $T_w : BV^{\varphi}(\mathbb{R}^N_+) \to BV^{\varphi}(\mathbb{R}^N_+)$.

The second estimate will be the main tool in order to prove the convergence result. By $\omega^{\varphi}(\lambda f, \delta) := \sup_{|\mathbf{1}-\mathtt{t}|\leq\delta} V^{\varphi}[\lambda(\tau_{\mathtt{t}} f - f)]$, where $\tau_{\mathtt{t}} f(\mathtt{s}) := f(\mathtt{st})$, $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N_+$ is the dilation operator, we denote the $\varphi-$modulus of smoothness of f.

Proposition 2: Let $f \in BV^{\varphi}(\mathbb{R}^N_+)$ and let $\{K_w\}_{w>0}$ be such that $K_w.1$) is satisfied. Then for every $\lambda > 0$, $\delta \in]0, 1[$ and $w > 0$,

$$V^{\varphi}[\lambda(T_w f - f)] \leq \omega^{\varphi}(\lambda A f, \delta) + A^{-1} V^{\varphi}[2\lambda A f] \int_{|\mathbf{1}-\mathtt{t}|>\delta} |K_w(\mathtt{t})|\langle \mathtt{t} \rangle^{-1} \, d\mathtt{t}.$$

This estimate links the $\varphi-$variation of the error of approximation $(T_w f - f)$ to the $\varphi-$modulus of smoothness, hence the convergence result will follow by the assumptions on kernel functions and by the following result of convergence for $\omega^{\varphi}(f, \delta)$ ([8]):

Theorem 1: Let $f \in AC^{\varphi}(\mathbb{R}^N_+)$. Then there exists $\lambda > 0$ such that $\lim_{\delta\to 0^+} \omega^{\varphi}(\lambda f, \delta) = 0$.

By means of Propositions 1 and 2, and using Theorem 1, we can therefore prove the main convergence result:

Theorem 2: Let $f \in AC^{\varphi}(\mathbb{R}^N_+)$ and let $\{K_w\}_{w>0}$ be such that $K_w.1$) and $K_w.2$) are satisfied. Then there exists a constant $\mu > 0$ such that

$$\lim_{w\to+\infty} V^{\varphi}[\mu(T_w f - f)] = 0.$$

We also obtain results about the order of approximation, with suitable singularity assumptions on kernels, for functions which belong to the Lipschitz class $V^{\varphi}Lip_N(\alpha)$, $\alpha > 0$, defined as

$$V^{\varphi}Lip_N(\alpha) := \{f \in BV^{\varphi}(\mathbb{R}^N_+) : \ \exists \mu > 0 \text{ s.t. } V^{\varphi}[\mu \Delta_{\mathtt{t}} f] = O(|\log \mathtt{t}|^{\alpha}), \text{ as } |\mathbf{1} - \mathtt{t}| \to 0\},$$

where $\Delta_{\mathtt{t}} f(\mathtt{x}) := (\tau_{\mathtt{t}} f - f)(\mathtt{x}) = f(\mathtt{xt}) - f(\mathtt{x})$, for $\mathtt{x}, \mathtt{t} \in \mathbb{R}^N_+$, and $\log \mathtt{t} := (\log t_1, \ldots, \log t_N)$.

We point out that, in the particular case of Fejér-type kernels, namely kernels of the form

$$K_w(\mathtt{t}) = w^N K(\mathtt{t}^w), \quad \mathtt{t} \in \mathbb{R}^N_+, \ w > 0,$$

where $K \in L^1_\mu(\mathbb{R}^N_+)$ is essentially bounded and such that $\int_{\mathbb{R}^N_+} K(\mathtt{t})\langle \mathtt{t}\rangle^{-1} d\mathtt{t} = 1$, it is possible to prove that all the assumptions of the results about the rate of approximation are implied by the classical condition that the absolute moments of order α of K are finite.

V. The particular case of $BV(\mathbb{R}^N_+)$

It is immediate to see that assumption 2) on the $\varphi-$functions implies that the identity function does not belong to the class Φ. Nevertheless all the theory can be developed also for the space $BV(\mathbb{R}^N_+)$, i.e., taking $\varphi(u) = u$, $u \in \mathbb{R}^+_0$, in the definition of the variation, and hence replacing everywhere the Musielak-Orlicz $\varphi-$variation with the Jordan variation. In this setting we obtain a new multidimensional concept of variation in the sense of Tonelli in the frame of Mellin theory and approximation results for Mellin-type integral operators in $BV(\mathbb{R}^N_+)$. Indeed assumption 2) on the $\varphi-$function, that now fails, is just used to prove the convergence result for the $\varphi-$modulus of smoothness (Theorem 1) and it replaces the lack of the integral representation of $\varphi-$variation. On the contrary, working with the classical variation, we have at our disposal the integral representation for absolutely continuous functions, and the convergence of the modulus of smoothness can be derived from it: hence, by means of different techniques, we prove the following

Theorem 3: If $f \in AC(\mathbb{R}^N_+)$, then $\lim_{\delta\to 0^+} \omega(f,\delta) = 0$. Here $AC(\mathbb{R}^N_+)$ and $\omega(f,\delta)$ denote the space of the absolutely continuous functions and the modulus of smoothness, respectively, in the case $\varphi(u) = u$, $u \in \mathbb{R}^+_0$.

The other results (estimates, convergence and rate of approximation) can be proved in a similar fashion, and therefore we obtain new results also in the case of the classical multidimensional variation in the present setting.

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