Special Frames

Luis Daniel Abreu
Acoustic Research Institute
Austrian Academy of Sciences
Email: daniel@mat.uc.pt

***Abstract*—Three classes of special frames are presented: special Fourier-type frames, special Gabor frames and special wavelet frames. Known information about density of Fourier-Bessel frames, Gabor (super)frames with Hermite functions and wavelet (super)frames with Laguerre functions will be outlined.**

I. INTRODUCTION

Given a Hilbert space $\mathcal{H}$, a vector $g \in \mathcal{H}$ and a family of operators $\{\pi_\lambda g\}_{\lambda\in\Lambda}$, *the special frame problem* consists of the following question:

- *What conditions one should impose on a discrete set* Λ*, such that* $\{\pi_\lambda g\}_{\lambda\in\Lambda}$ *is a frame for* $\mathcal{H}$*?*

More precisely, we want to find constants $A, B > 0$ such that, for every $f \in \mathcal{H}$,

$$A\left\|f\right\|^2 \leq \sum_{\lambda\in\Lambda} \left|\langle f, \pi_\lambda g\rangle_{\mathcal{H}}\right|^2 \leq B\left\|f\right\|^2. \qquad (1)$$

The term *special* refers to a *viewpoint*: rather than looking at general properties of frames, we want to know detailed information about a *specific example* of a frame with particular interesting structure. We will consider three classes of frames.

1) Fourier Frames: $\mathcal{H} = L^2(-\pi,\pi)$, $g(x) = e^{ix}$ and $\pi_\lambda g(x) = g(\lambda x)$. For g other that e^{ix} we will talk about *Fourier-type frames*.
2) Gabor frames: $\mathcal{H} = L^2(R)$ and $\pi_{\lambda=(\lambda_1,\lambda_2)}g(x) = e^{2\pi i\lambda_2 t}g(t-\lambda_1)$. Several choices of g are possible.
3) Wavelet frames (positive frequencies): $\mathcal{H} = H^2(C^+)$ and $\pi_\lambda g(t) = \lambda_1^{-\frac{1}{2}} g(\lambda_1^{-1}(t-\lambda_2)), \quad t \in R$. Several choices of g are possible.

II. SPECIAL FOURIER-TYPE FRAMES

While the *Fourier orthogonal basis* is of the form $\{e^{ikx}\}_{k\in Z}$, *Fourier frames* are of the form $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$, allowing the set Λ to be nonuniform and redundant. The orthogonal basis case $\Lambda = Z$ works as a threshold for Fourier frames: we know that frames requires Λ to be "denser than Z" [16]. We can think of Fourier frames as being made out of the special function $f(x) = e^x$. Frames of the form $\{f(\lambda x)\}_{\lambda\in\Lambda}$ will be called *Fourier-type frame*s. To keep intact the rich set up of the Fourier frames we want to be able to transfer our Fourier-type frames to a Paley–Wiener-type space using some Fourier-type transform. Moreover, we are interested in cases displaying a second order differential operator commuting with the respective concentration operators. In the case of Fourier frames, the existence of such an operator is regarded as a "fortunate accident", according to Daubechies exposition in [9, page 22]. In the work of Tracy and Widom about the local statistics of the asymptotics of certain random matrices [19], [18], one can find two more instances where this "fortunate accident" occurs. This motivated our investigation of Fourier-Bessel frames [5] and Airy frames [6]. Let us say a bit more about the results in [5].

Let $J_\alpha(x)$ be the Bessel function of order $\alpha > -1/2$ and $j_{n,\alpha}$ its n^{th} zero. While the *Fourier-Bessel orthogonal basis* is of the form $\{x^{\frac{1}{2}}J_\alpha(j_{n,\alpha}x)\}_{n=0}^{\infty}$, *Fourier-Bessel frames* are of the form $\{(\lambda x)^{\frac{1}{2}}J_\alpha(\lambda x)\}_{\lambda\in\Lambda}$, allowing the set Λ to be nonuniform and redundant. To obtain the definition of a Fourier-Bessel frame, choose in (1) $\mathcal{H} = L^2[0,1]$, $g(x) = (x)^{\frac{1}{2}}J_\alpha(x)$ and $(\pi_\lambda g)(x) = g(\lambda x)$. In [5], we have considered a more general situation than frames and obtained analogues of the Landau conditions [16] for interpolation and sampling. As a particular case we obtain precise necessary density conditions for Fourier Bessel frames. Let $n_a(r)$ denote the number of points of $\Lambda \subset (0,\infty)$ to be found in $[a, a+r]$. Then the lower density of Λ is given by $D^-(\Lambda) = \lim_{r\to\infty}\inf\inf_{a\geq 0}\frac{n_a(r)}{r}$. The main result in [5] is the following Landau-type necessary condition for sampling in spaces of functions $\mathcal{B}_\alpha(S)$ whose Hankel transform (the analogue of the Fourier transform in this context) is supported on a set S of bounded measure:

Theorem [5]: Let S be a measurable subset of $(0,\infty)$ and $\alpha > -1/2$. If a separated set Λ is of sampling for $\mathcal{B}_\alpha(S)$, then

$$D^-(\Lambda) \geq \frac{1}{\pi}m(S). \qquad (2)$$

III. SPECIAL GABOR (SUPER)FRAMES

The investigation of special Gabor frames has been a topic of high interest in the last twenty years. See the recent paper [13] and the outline in the Introduction. We can construct Gabor superframes with Hermite functions, which are useful in the multiplexing of non-stationary signals. Consider the Hilbert space $L^2(R, C^n)$ of vector-valued functions $\overrightarrow{\mathbf{f}} = (f_0, ..., f_{n-1})$ together with the inner product $\left\langle \overrightarrow{\mathbf{f}}, \overrightarrow{\mathbf{g}}\right\rangle_{\mathcal{H}} = \sum_{0\leq k\leq n-1}\langle f_k, g_k\rangle_{L^2(R)}$. To obtain the definition of a *Gabor superframe* for the vector valued system $\mathcal{G}(\overrightarrow{\mathbf{g}}, \Lambda) = \{\pi_\lambda \overrightarrow{\mathbf{g}}\}_{\lambda\in\Lambda}$, choose in (1) $\mathcal{H} = L^2(R, C^n)$, $g = \overrightarrow{\mathbf{g}}$ and, given a point $\lambda = (\lambda_1, \lambda_2)$ in R^2, define π_λ as the time-frequency shift $\pi_\lambda g(t) = e^{2\pi i\lambda_2 t}g(t-\lambda_1)$, $t \in R$.

There is a characterization of all lattices generating Gabor superframes with Hermite functions h_n [12], which is equivalent to a sampling problem in a Fock space of polyanalytic functions [1].

Theorem [12] Let $\overrightarrow{\mathbf{h}_n} = (h_0, ..., h_{n-1})$ be the vector of the first n Hermite functions. Then $\mathcal{G}(\overrightarrow{\mathbf{h}_n}, \alpha Z + i\beta Z)$ is a frame for $L^2(R, C^n)$, if and only if $\alpha\beta < \frac{1}{n+1}$.

For a special frame generated by a single Hermite function, the characterization is still an open problem. Nevertheless, some interesting results are known. If $\alpha\beta < \frac{1}{n+1}$ then $\mathcal{G}(h_n, \alpha Z + i\beta Z)$ is a frame [11] but if $\alpha\beta = 1 - \frac{1}{j}$ then $\mathcal{G}(h_1, \alpha Z + i\beta Z)$ is not [14]. Supported by their results and by some numerical evidence, the authors of [14] conjectured that if $\alpha\beta < 1$ and $\alpha\beta \neq 1 - \frac{1}{j}$, then $\mathcal{G}(h_1, \alpha Z + i\beta Z)$ is a frame.

IV. Special Wavelet (super)frames

We can also construct wavelet superframes which are useful in the multiplexing of non-stationary signals of positive frequencies, leading to a sampling problem in certain (Bergman) spaces of polyanalytic functions. We should emphasize again that our viewpoint of wavelet frames is different of those ones documented in [9] and in the more recent monograph [15]. For a vector $\mathbf{g} = (g_1, ..., g_n)$ such that the Fourier transforms of any two functions g_i and g_j are orthogonal in $L^2(R^+, t^{-1})$, define π_z pointwise as $\pi_z \mathbf{g} = (\pi_z g_1, ..., \pi_z g_n)$. To obtain the definition of a *wavelet superframe* for the vector valued system $\mathcal{W}(\overrightarrow{\mathbf{g}}, \Lambda) = \{\pi_\lambda \overrightarrow{\mathbf{g}}\}_{\lambda\in\Lambda}$, let in (1) $\mathcal{H} = H^2(C^+, C^n)$ be the inner product space whose vector components belong to $H^2(C^+)$, the standard Hardy space of the upper half-plane, $g = \overrightarrow{\mathbf{g}}$ and, given a point $\lambda = (\lambda_1, \lambda_2)$ in R^2, define π_λ as the time-scale shift $\pi_\lambda g(t) = \lambda_1^{-\frac{1}{2}} g(\lambda_1^{-1}(t - \lambda_2)), \quad t \in R$.

We consider wavelet superframes with analyzing wavelets $\overrightarrow{\Phi_n^\alpha} = (\frac{\Phi_0^\alpha}{c_{\Phi_0^\alpha}}, ..., \frac{\Phi_n^\alpha}{c_{\Phi_n^\alpha}})$, where $c_{\Phi_n^\alpha}^2 = \frac{\Gamma(n+\alpha+1)}{n!}$ is the admissibility constant of the vector component Φ_n^α defined via its Fourier transform as

$$\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{1}{2}} l_n^\alpha(2t), \quad with \quad l_n^\alpha(t) = t^{\frac{\alpha}{2}} e^{-\frac{t}{2}} L_n^\alpha(t), \qquad (3)$$

where $L_n^\alpha(t)$ is the standard notation for the Laguerre polynomial.

The problem of, given a wavelet g, to characterize the sets of points Λ such that $\mathcal{W}(g, \Lambda)$ is a wavelet frame (and the corresponding problem for the superframes defined above), is more difficult than the corresponding one for Gabor frames. The only characterization known so far concerns the case $n = 0$ in (3). In this case, the problem can be reduced to the density of sampling in the Bergman spaces, which has been completely understood in [17]. An important research problem is to understand how Seip's results extend to the whole family $\{\Phi_n^\alpha\}$. The only thing known to the present date is a necessary condition obtained in [2] in terms of a set of points known as the "hyperbolic lattice" $\Gamma(a, b) = \{a^m bk, a^m\}_{k,m\in Z}$. The quantity $b \log a$ replaces the time-frequency $\alpha\beta$ for purposes of measuring frame density.

Theorem [2]: If $\mathcal{W}(\Phi_n^{2\alpha-1}, \Gamma(a, b))$ is a wavelet frame for $H^2(C^+)$, then $b \log a < 2\pi \frac{n+1}{\alpha}$.

Using the polyanalytic structure of the underlying Bergman spaces [3] one can also prove a result which shows that it is necessary to oversample by a rate of n to obtain superframes. This matches what one would expect from [10].

Theorem [4]: If $\mathcal{W}(\overrightarrow{\Phi_n^{2\alpha-1}}, \Gamma(a, b))$ is a wavelet superframe for $H^2(C^+, C^n)$, then $b \log a < \frac{2\pi}{n+\alpha}$.

Actually in [4] we obtain a much stronger result using Seip´s density [17], as part of our sampling results in polyanalytic Bergman spaces.

Aknowledgements: The author would like to thank to the three referees for their remarks, corrections and suggestions which help to improve the presentation of the manuscript and to Diana Stoeva for her reading of the final version. Supported by European program COMPETE/FEDER via FCT project PTDC/MAT/114394/2009, by Austrian Science Foundation (FWF) project "Frames and Harmonic Analysis" and START-project FLAME ('Frames and Linear Operators for Acoustical Modeling and Parameter Estimation').

References

[1] L. D. Abreu, *Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions*, Appl. Comp. Harm. Anal., 29 (2010), 287-302.
[2] L. D. Abreu, *Wavelet frames with Laguerre functions,* C. R. Acad. Sci. Paris, Ser. I, 349 (2011), 255-258.
[3] L. D. Abreu, *Super-wavelets versus poly-Bergman spaces,* Int. Eq. Oper. Theory, 73 (2012), 177–193.
[4] L. D. Abreu, *Wavelets (super)frames with Laguerre functions and sampling in polyanalytic spaces*, manuscript.
[5] L. D. Abreu, A. Bandeira, *Landau's necessary density conditions for the Hankel transform*, J. Funct. Anal. 162 (2012), 1845-1866.
[6] L. D. Abreu, A. Bandeira, *Landau's necessary density conditions for the Airy transform*, work in progress.
[7] L. D. Abreu, K. Gröchenig, *Banach Gabor frames with Hermite functions: polyanalytic spaces from the Heisenberg group*, Appl. Anal., 91 (2012), 1981-1997.
[8] L. D. Abreu, H. G. Feichtinger, *Function spaces of polyanalytic functions*, HCAA special volume, 32pp, to appear.
[9] I. Daubechies, *"Ten Lectures On Wavelets"*, CBMS-NSF Regional conference series in applied mathematics, (1992).
[10] D. E. Dutkay, P. Jorgensen, *Oversampling generates super-wavelets.* Proc. Amer. Math. Soc. 135 (2007), 2219–2227.
[11] K. Gröchenig, Y. Lyubarskii, *Gabor frames with Hermite functions*, C. R. Acad. Sci. Paris, Ser. I 344 (2007), 157-162.
[12] K. Gröchenig, Y. Lyubarskii, *Gabor (Super)Frames with Hermite Functions*, Math. Ann., 345 (2009), 267-286.
[13] K. Gröchenig, J. Stoeckler, *Gabor frames and totally positive functions,* Duke Math. J., to appear.
[14] Y. Lyubarskii, P. G. Nes, *Gabor frames with rational density,* Appl. Comp. Harm. Anal., 34 (2013), 488494.
[15] G. Kuttyniok, *Affine Density in Wavelet Analysis,* Lecture Notes in Mathematics 1914 , Springer-Verlag, Berlin, 2007.
[16] H. J. Landau: *Necessary Density Conditions for Sampling and Interpolation of Certain Entire Functions*, Acta Math., 117 (1967), 37-52.
[17] K. Seip, *Beurling type density theorems in the unit disc*, Invent. Math., 113 (1993), 21-39.
[18] C. A. Tracy, H. Widom, *Level-spacing distributions and the Airy kernel*, Commun. Math. Phys. 159 (1994), 151-174.
[19] C. A. Tracy, H. Widom, *Level-spacing distributions and the Bessel kernel*, Commun. Math. Phys. 161 (1994), 289-309.