A priori convergence of the Generalized Empirical Interpolation Method

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***Abstract*—In an effort to extend the classical Lagrangian interpolation tools, new interpolating methods that use general interpolating functions are explored. The Generalized Empirical Interpolation Method (GEIM) belongs to this class. It generalizes the plain Empirical Interpolation Method [1] by replacing the evaluation at interpolating points by application of a class of interpolating linear functions. Since its efficiency depends critically on the choice of the interpolating functions (that are chosen by a Greedy selection procedure), the purpose of this paper is therefore to provide a priori convergence rates for the Greedy algorithm that is used to build the GEIM interpolating spaces.**

I. Introduction

The extension of the Lagrangian interpolation process is an old problem that is still currently subject to active research (see, e.g. [1] and also the activity concerning the kriging [2], [3] in the stochastic community). While this classical method approximates general functions by finite sums of well chosen, linearly independent interpolating functions (e.g. polynomial functions) and the optimal location of the interpolating points is well documented (and completely solved in one dimension), the question remains on how to approximate general functions by finite expansions involving general interpolating functions and how to optimally select the interpolation points in this case.

One step in this direction is the Empirical Interpolation Method (EIM, [4], [5], [1]) that has been developed in the broad framework where the functions f to approximate belong to a compact set F of a Banach space $\mathcal{X}$. The set F is supposed to be such that any $f \in F$ is approximable by linear combinations of small size. In particular, this is the case when the Kolmogorov $n-$width of F in $\mathcal{X}$ is small. Indeed, the Kolmogorov $n-$width of F in $\mathcal{X}$ is defined by $d_n(F,\mathcal{X}) := \inf_{\substack{X_n\subset\mathcal{X}\\ dim(X_n)=n}} \sup_{x\in F} \inf_{y\in X_n} \|x-y\|_{\mathcal{X}}$ (see [6]) and measures the extent to which F can be approximated by finite dimensional spaces $X_n \subset \mathcal{X}$ of dimension n. The Empirical Interpolation Method builds simultaneously the set of interpolating functions and the associated interpolating points by a greedy selection procedure (see [4]).

A recent generalization of this interpolation process consists in replacing the evaluation at interpolating points by application of a class of interpolating continuous linear functions chosen in a given dictionary $\Sigma \subset \mathcal{L}(F)$ and this gives rise to the so-called Generalized Empirical Interpolation Method (GEIM, [7]). In this newly developed method, the particular case where the space $\mathcal{X} = L^2(\Omega)$ is considered, with Ω being a bounded spatial domain of $\mathbb{R}^d$ and F being a compact set of $L^2(\Omega)$.

In the present work, we analyze the quality of the finite dimensional subspaces X_n contained in the span of F built by the greedy selection procedure of GEIM together with the properties of the associated interpolation operator. For this purpose, the accuracy of the approximation in X_n of the elements of F will be compared to the best possible performance which is the Kolmogorov $n-$ width $d_n(F, L^2(\Omega))$.

The methodology developed in this paper is in the spirit of the greedy reduced basis method. Alternative approaches exist like POD and gappy POD or even Adaptive Cross Approximation. We refer to the review paper [8] for a comparative presentation of all these sampling approaches.

The proceeding is organized as follows: after a brief recall of GEIM's Greedy algorithm (section II), we will analyze in sections III and IV some convergence decay rates of the generalized empirical interpolation error as the dimension n of X_n increases and when $d_n(F, L^2(\Omega))$ has a polynomial or an exponential decreasing behavior.

II. The generalized Empirical Interpolation Method

In the following, we assume that the dimension of the vectorial space spanned by F is of dimension $\geq \mathcal{N}$.

In a similar procedure as in the Empirical Interpolation Method (EIM) [4], [5], [1], the Generalized EIM allows to define simultaneously the set of interpolating functions recursively chosen in F together with the associated linear functions selected from a dictionary of continuous linear forms $\Sigma \subset \mathcal{L}(F)$, with norm 1 in $L^2(\Omega)$. The dictionary has the additional property that if $\varphi \in F$ is such that $\sigma(\varphi) = 0$ for any $\sigma \in \Sigma$, then $\varphi = 0$. The selection of the interpolating functions and linear forms is based on a greedy selection procedure as outlined in [7].

The first interpolating function is, e.g.: $\varphi_0 = \arg\sup_{\varphi \in F} \|\varphi\|_{L^2(\Omega)}$. The first interpolating linear form is $\sigma_0 = \arg\sup_{\sigma\in\Sigma} |\sigma(\varphi_0)|$. We then define the first basis function as $q_0 = \dfrac{\varphi_0}{\sigma_0(\varphi_0)}$. The second interpolating function is $\varphi_1 = \arg\sup_{\varphi\in F} \|\varphi - \sigma_0(\varphi)q_0\|_{L^2(\Omega)}$. The second interpolating linear form is $\sigma_1 = \arg\sup_{\sigma\in\Sigma} |\sigma(\varphi_1 - \sigma_0(\varphi_1)q_0)|$ and the second basis function is defined as $q_1 = \dfrac{\varphi_1 - \sigma_0(\varphi_1)q_0}{\sigma_1(\varphi_1 - \sigma_0(\varphi_1)q_0)}$.

We then proceed by induction : assume that we have built the set of interpolating functions $\{q_0, q_1, \dots, q_{N-1}\}$ and the set of associated interpolating linear forms $\{\sigma_0, \sigma_1, \dots, \sigma_{N-1}\}$, for $1 \le N \le N_{max}$, with $N_{max} \le \mathcal{N}$ being an upper bound fixed *a priori*. For $N \le 1$, we first solve the interpolation problem: find $\{\alpha_j^N(\varphi)\}_j$ such that: $\forall i = 0, \dots, N-1, \quad \sigma_i(\varphi) = \sum_{j=0}^{N-1} \alpha_j^N(\varphi)\sigma_i(q_j)$. We then compute $\mathcal{J}_N[\varphi] = \sum_{j=0}^{N-1} \alpha_j^N(\varphi) q_j$ and evaluate $\varepsilon_N(\varphi) = \|\varphi - \mathcal{J}_N[\varphi]\|_{L^2(\Omega)}, \ \forall\varphi \in F$. We define $\varphi_N = \arg\sup_{\varphi\in F} \varepsilon_N(\varphi)$ and $\sigma_N = \arg\sup_{\sigma\in\Sigma} |\sigma(\varphi_N - \mathcal{J}_N[\varphi_N])|$. The next basis function is then $q_N = \dfrac{\varphi_N - \mathcal{J}_N[\varphi_N]}{\sigma_N(\varphi_N - \mathcal{J}_N[\varphi_N])}$

We finally set $X_{N+1} \equiv \text{span}\,\{q_j,\ j \in [0, N]\} = \text{span}\,\{\varphi_j,\ j \in [0, N]\}$. It has been proven in [7]:

Lemma 1: For any $N \le \mathcal{N}$, the set $\{q_j,\ j \in [0, N-1]\}$ is linearly independent and X_N is of dimension N. The generalized empirical interpolation procedure is well-posed in $L^2(\Omega)$ and $\forall\varphi \in F$, the interpolation error satisfies:

$$\|\varphi - \mathcal{J}_N[\varphi]\|_{L^2(\Omega)} \le (1 + \Lambda_N) \inf_{\psi_N\in X_N} \|\varphi - \psi_N\|_{L^2(\Omega)}$$

where Λ_N is the Lebesgue constant in the L^2 norm: $\Lambda_N := \sup_{\varphi\in F} \dfrac{\|\mathcal{J}_N[\varphi]\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}}$.

Remark 1: In a similar way as in the classical Lagrangian interpolation, the Lebesgue constant Λ_N defined in our generalized interpolation procedure depends both on set F and on the choice of the dictionary of continuous linear forms Σ but no detailed analysis of the behavior of Λ_N as a function of F or Σ has been carried out so far.

Remark 2: In practice the selection of the interpolation functions in F and the interpolating elements in the dictionary can be done by discretizing both F and Σ as is the case for standard greedy approximations like in [5], [6]; an alternative approach is [9] where the selection is done through a continuous algorithm based on an iterative sequence of optimization problems (solved by Newton methods) that seek to maximize the error between the RB approximation and the underlying true solution. The interpolants can be efficiently computed recursively as outlined in [10].

III. Preliminary notations and basic properties

In what follows, we denote by $(\varphi_n^*)_{n\ge0}$ the orthonormal system obtained from $(\varphi_n)_{n\ge0}$ by Gram-Schmidt orthogonalization.

For any $n \ge 1$, we define the orthogonal projector P_n from $\mathcal{X}$ onto X_n which is given by $P_n(f) = \sum_{j=0}^{n-1} < f, \varphi_j^* > \varphi_j^*$, $\forall f \in F$, where $< .,. >$ is the $L^2(\Omega)$ scalar product. In particular: $\varphi_n = P_{n+1}(\varphi_n) = \sum_{j=0}^{n} a_{n,j}\varphi_j^*$, with $a_{n,j} :=< \varphi_n, \varphi_j^* >, 0 \le j \le n$.

Finally, let us denote $\tau_0(F)_{L^2(\Omega)} := d_0(F, L^2(\Omega))$ and, for any $n \ge 1$: $\tau_n := \tau_n(F)_{L^2(\Omega)} := \max_{f\in F} \|f - P_n(f)\|_{L^2(\Omega)}$ and by γ_n the constant $\gamma_n = 1/(1 + \Lambda_n)$.

We begin by proving the two following lemmas:

Lemma 2: For any $n \ge 1$, $\|\varphi_n - P_n(\varphi_n)\|_{L^2(\Omega)} \ge \gamma_n\tau_n(F)$.

Proof: From lemma 1 applied to $\varphi = \varphi_n$ we have $\|\varphi_n - P_n(\varphi_n)\|_{L^2(\Omega)} \ge \gamma_n\|\varphi_n - \mathcal{J}_n(\varphi_n)\|_{L^2(\Omega)}$. But $\|\varphi_n - \mathcal{J}_n(\varphi_n)\|_{L^2(\Omega)} \ge \|\varphi - \mathcal{J}_n(\varphi)\|_{L^2(\Omega)}$ for any $\varphi \in F$ according to the definition of φ_n. Thus $\|\varphi_n - P_n(\varphi_n)\|_{L^2(\Omega)} \ge \gamma_n\|\varphi - \mathcal{J}_n(\varphi)\|_{L^2(\Omega)} \ge \gamma_n\|\varphi - P_n(\varphi)\|_{L^2(\Omega)}$. ∎

Lemma 3: Let A be the lower triangular matrix defined by $A := (a_{i,j})_{i,j=0}^{\infty}$ $(a_{i,j} := 0,\ j > i)$. A has two important properties:

- P1: $\gamma_n\tau_n \le |a_{n,n}| \le \tau_n$.
- P2: For every $m \ge n$, $\sum_{j=n}^{m} a_{m,j}^2 \le \tau_n^2$.

Proof:

- P1: $\forall f \in F:\ P_n(f) = \sum_{j=0}^{n-1} < f, \varphi_j^* > \varphi_j^*$. In particular: $\varphi_n - P_n(\varphi_n) = a_{n,n}\varphi_n^* \Rightarrow \|\varphi_n - P_n(\varphi_n)\|_{L^2(\Omega)}^2 = a_{n,n}^2$. The upper bound is thus obvious and Lemma 2 gives the lower bound.
- P2: For every $m \ge n$: $\sum_{j=n}^{m} |a_{m,j}|^2 = \|\varphi_m - P_n(\varphi_m)\|_{L^2(\Omega)}^2 \le \max_{f\in F} \|f - P_n(f)\|^2 = \tau_n^2$.

∎

IV. A priori convergence rates of the GEIM Greedy method

In order to get convergence decay rates in the generalized interpolation error of our method, we first note that lemma 2 shows that the GEIM's Greedy algorithm is what is called in [11] a "weak Greedy algorithm" of parameter $\gamma_n = 1/(1+\Lambda_n)$ that depends on the dimension of X_n.

Thanks to this observation, we shall derive convergence decay rates in the sequence $(\tau_n)_{n\ge0}$. This task consists in extending the proofs of [11] where the constant case $\gamma_n = \gamma$ was addressed and where the following two results were proven in Corollary 3.3:

i) If $d_n(F) \le C_0 n^{-\alpha}$ for $n \ge 1$, then $\tau_n \le C_0 2^{5\alpha+1}\gamma^{-2}n^{-\alpha}$ for $n \ge 1$.
ii) If $d_n(F) \le C_0 e^{-c_0 n^{\alpha}}$ for $n \ge 1$, then $\tau_n \le \sqrt{2C_0}\gamma^{-1}e^{-c_1 n^{\alpha}}$ for $n \ge 1$, where $c_1 := 2^{-1-2\alpha}c_0$.

In order to extend $i)$ and $ii)$ to the more general case where γ depends on the dimension n, the following preliminary theorem is required:

***Theorem** 4:* For any $N \geq 0$, consider the weak Greedy algorithm with constant γ_N in $L^2(\Omega)$ associated with the compact set F. We have the following inequalities between τ_N and $d_N := d_N(F, L^2(\Omega))$: for any $K \geq 1$, $1 \leq m < K$

$$\prod_{i=1}^{K} \tau_{N+i}^2 \leq \frac{1}{\prod_{i=1}^{K} \gamma_{N+i}^2} \left(\frac{K}{m}\right)^m \left(\frac{K}{K-m}\right)^{K-m} \tau_{N+1}^{2m} d_m^{2(K-m)}.$$

Proof: This result is an extension of Theorem 3.2 of [11] to the case where the parameter of the weak Greedy algorithm (γ_N) depends on the dimension of the reduced space X_N. Its proof is a slight modification to the one provided in [11] using γ_N and the properties P1 and P2 stated in Lemma 3. ■

Using theorem 4, convergence rates in the sequence $(\tau_n)_{n\geq 0}$ when $(d_n)_{n\geq 0}$ has a polynomial or an exponential decay can be inferred and lead to lemmas 5 and 6:

***Lemma** 5 (Polynomial decay of $(d_n)_{n\geq 0}$):* For any $n \geq 1$, let $n = 4\ell + k$ (where $\ell \in \{0, 1, \dots\}$ and $k \in \{0, 1, 2, 3\}$). Assume that there exists a constant $C_0 > 0$ such that $\forall n \geq 1$, $d_n(F, L^2(\Omega)) \leq C_0 n^{-\alpha}$, then $\tau_n \leq C_0 \beta_n n^{-\alpha}$, where $\beta_1 = 2$ and for $n \geq 2$: $\beta_n = \beta_{4\ell+k} := \sqrt{2\beta_{\ell_1}} \frac{1}{\prod_{i=1}^{\ell_2} \gamma_{\ell_1 - \lceil \frac{k}{4} \rceil + i}^{\frac{1}{\ell_2}}} (2\sqrt{2})^{\alpha}$ and $\ell_1 = 2\ell + \lfloor \frac{2k}{3} \rfloor$, $\ell_2 = 2\left(\ell + \lceil \frac{k}{4} \rceil\right)$.

Proof: The proof is done by recurrence over n. We initialize the reasoning by proving that $\tau_1 \leq 2C_0$ and then prove the general statement for $n \geq 2$.

Case $n = 1$: We recall that $\varphi_0 = \arg\sup_{\varphi \in F} \|\varphi\|_{L^2(\Omega)}$ and that P_1 is the projector operator onto span $\{\varphi_0\}$. We set: $f_1 = \arg \tau_1 = \arg\max_{f\in F} \|f - P_1(f)\|_{L^2(\Omega)}$ and let $\mu \in F$ span the one dimensional subspace of F for which $d_1 \geq \|f - P_\mu(f)\|_{L^2(\Omega)}$ for any $f \in F$ (P_μ being the projector operator onto span $\{\mu\}$). We have: $\tau_1 = \|f_1 - P_1(f_1)\|_{L^2(\Omega)} = \|f_1 - P_\mu(f_1) + P_\mu(f_1) - P_1(f_1)\|_{L^2(\Omega)} = \|f_1 - P_\mu(f_1) - P_1(f_1 - P_\mu(f_1)) + P_\mu(f_1) - P_1 P_\mu(f_1)\|_{L^2(\Omega)} \leq d_1 + \|P_\mu(f_1) - P_1 P_\mu(f_1)\|_{L^2(\Omega)}$.

We have: $\|P_\mu(f_1) - P_1 P_\mu(f_1)\|_{L^2(\Omega)} = \|\frac{< f_1, \mu > \mu}{\|\mu\|^2_{L^2(\Omega)}} - \frac{\langle < f_1, \mu > \mu, \varphi_0 \rangle \varphi_0}{\|\mu\|^2_{L^2(\Omega)} \|\varphi_0\|^2_{L^2(\Omega)}}\|_{L^2(\Omega)} = \frac{|< f_1, \mu >|}{\|\mu\|_{L^2(\Omega)}} \|\frac{\mu}{\|\mu\|_{L^2(\Omega)}} - \frac{< \varphi_0, \mu > \varphi_0}{\|\mu\|_{L^2(\Omega)} \|\varphi_0\|^2_{L^2(\Omega)}}\|_{L^2(\Omega)}$.

Since for any $x, y \in F$ with norm 1 we have $\|x- < x, y > y\|_{L^2(\Omega)} = \|y- < x, y > x\|_{L^2(\Omega)}$, we deduce that : $\|P_\mu(f_1) - P_1 P_\mu(f_1)\|_{L^2(\Omega)} = \frac{|< f_1, \mu >|}{\|\mu\|_{L^2(\Omega)}} \|\frac{\varphi_0}{\|\varphi_0\|_{L^2(\Omega)}} - \frac{< \varphi_0, \mu > \mu}{\|\mu\|^2_{L^2(\Omega)} \|\varphi_0\|_{L^2(\Omega)}}\|_{L^2(\Omega)}$.

Hence: $\tau_1 \leq d_1 + \frac{|< f_1, \mu >|}{\|\mu\|_{L^2(\Omega)} \|\varphi_0\|_{L^2(\Omega)}} \|\varphi_0 - \frac{< \varphi_0, \mu > \mu}{\|\mu\|^2_{L^2(\Omega)}}\|_{L^2(\Omega)} \leq d_1 \left(1 + \frac{|< f_1, \mu >|}{\|\mu\|_{L^2(\Omega)} \|\varphi_0\|_{L^2(\Omega)}}\right) \leq 2d_1$.

***Remark** 3:* In the case where $\|\varphi_0\|_{L^2(\Omega)} \geq \gamma_0 \|f\|_{L^2(\Omega)}$ for any $f \in F$ ($0 < \gamma_0 \leq 1$), we would have obtained $\tau_1 \leq d_1 \left(1 + \frac{1}{\gamma_0}\right)$.

Case $n \geq 2$: Let $n = N + K$ for any $N \geq 0$, $K \geq 2$. If $i \leq K$, we have $\tau_n = \tau_{N+K} \leq \tau_{N+i}$ from the monotonicity of $(\tau_n)_{n\geq 0}$. By combining this inequality with theorem 4, if $1 \leq m < K$, we derive that $\tau_n \leq \frac{1}{\prod_{i=1}^{K} \gamma_{N+i}^{\frac{1}{K}}} \sqrt{\left(\frac{K}{m}\right)^{\frac{m}{K}} \left(\frac{K}{K-m}\right)^{1-\frac{m}{K}}} \tau_{N+1}^{\frac{m}{K}} d_m^{1-\frac{m}{K}} \leq \frac{1}{\prod_{i=1}^{K} \gamma_{N+i}^{\frac{1}{K}}} \sqrt{2} \tau_{N+1}^{\frac{m}{K}} d_m^{1-\frac{m}{K}}$, since $x^{-x}(1-x)^{x-1} \leq 2$ for any x, $0 < x < 1$. We now use that $d_m \leq C_0 m^{-\alpha}$ and the recurrence hypothesis in $N + 1 < n$: $\tau_{N+1} \leq C_0 \beta_{N+1} (N + 1)^{-\alpha}$ which yields: $\tau_{N+K} \leq C_0 \sqrt{2} \frac{1}{\prod_{i=1}^{K} \gamma_{N+i}^{\frac{1}{K}}} \beta_{N+1}^{\frac{m}{K}} \xi(n)^\alpha (N+K)^{-\alpha}$ where $\xi(n) = \frac{n}{m} \left(\frac{m}{N+1}\right)^{\frac{m}{K}}$.

Any $n \geq 2$ can be written as $n = 4\ell + k$ with $\ell \in \{0, 1, \dots\}$ and $k \in \{0, 1, 2, 3\}$. If $k = 1, 2$ or 3, it can easily be proven that $\xi(n) \leq 2\sqrt{2}$ by setting $N = 2\ell - 1$, $K = 2\ell + 2$, $m = \ell + 1$ if $k = 1$ and $\ell \geq 1$, $N = 2\ell$, $K = 2\ell + 2$, $m = \ell + 1$ if $k = 2$ and $\ell \geq 0$ and $N = 2\ell + 1$, $K = 2\ell + 2$, $m = \ell + 1$ if $k = 3$ and $\ell \geq 0$. These choices of N, K and m combined with the upper bound of ξ yield the result $\tau_n \leq C_0 \beta_n n^{-\alpha}$ in the case $k = 1, 2$ or 3.

In the case $n = 4\ell$ ($\ell \geq 1$), using the fact that $\tau_{N+1} \leq \tau_N$, we can derive that $\tau_n \leq \frac{1}{\prod_{i=1}^{K} \gamma_{N+i}^{\frac{1}{K}}} \sqrt{2} \tau_N^{\frac{m}{K}} d_m^{1-\frac{m}{K}}$. If we choose $N = K = 2\ell$ and $m = \ell$, the previous inequality directly yields $\tau_{4\ell} \leq C_0 \sqrt{2\beta_{2\ell}} \frac{1}{\prod_{i=1}^{2\ell} \gamma_{2\ell+i}^{\frac{1}{2\ell}}} (2\sqrt{2})^\alpha (4\ell)^{-\alpha}$.

■

***Lemma** 6 (Exponential decay in $(d_n)_{n\geq 0}$):* Assume that there exists a constant $C_0 > 0$ such that $\forall n \geq 1$, $d_n(F, L^2(\Omega)) \leq C_0 e^{-c_1 n^\alpha}$, then $\tau_n \leq C_0 \beta_n e^{-c_2 n^\alpha}$, where $\beta_n := \frac{1}{\prod_{i=1}^{\lceil \frac{n}{2} \rceil} \gamma_{\lfloor \frac{n}{2} \rfloor + i}^{\frac{1}{\lceil \frac{n}{2} \rceil}}} \sqrt{2\beta_{\lfloor \frac{n}{2} \rfloor}}$ for $n \geq 2$, $\beta_1 = 2$ and $c_2 := 2^{-1-3\alpha} c_1$.

Proof: The proof is done by recurrence over n.

The case $n = 1$ is addressed by following the same lines as in lemma 5.

In the case $n = 2$, we have: $\tau_2 \leq \tau_1 \leq 2C_0$.

For $n \geq 3$, we start from $\tau_{N+K} \leq \frac{1}{\prod_{i=1}^{K} \gamma_{N+i}^{\frac{1}{K}}} \sqrt{2} \tau_{N+1}^{\frac{m}{K}} d_m^{1-\frac{m}{K}}$ and treat the cases $n = N + K = 2\ell$ and $n = N + K = 2\ell + 1$ separately ($\ell \geq 1$).

If $n = N + K = 2\ell$, we choose $N = K = \ell$ and $m = \lfloor \frac{K}{2} \rfloor$.

The inequality yields $\tau_{2\ell} \leq \frac{1}{\prod_{i=1}^{\ell} \gamma_{\ell+i}^{\frac{1}{\ell}}} \sqrt{2\tau_\ell} e^{-c_2(2\ell)^\alpha}$.

In a similar procedure, the desired result can be inferred for $n = N + K = 2\ell + 1$ if we choose $N = \ell$, $K = \ell + 1$ and $m = \lfloor \frac{K}{2} \rfloor$. ■

Remark 4: 1) In the case where γ_n is constant $\gamma_n = \gamma$, lemmas 5 and 6 yield results that are similar to the ones obtained in [11] (see results i) and ii) above).

2) In the case where $(\gamma_n)_{n\geq 1}$ is a monotonically decreasing sequence, the following bounds can be derived for τ_n:
   - If $d_n(F, L^2(\Omega)) \leq C_0 n^{-\alpha}$ for any $n \geq 1$, then $\tau_n \leq C_0 \beta n^{-\alpha}$ for $n \geq 1$, with $\beta := 2^{3\alpha+1} \left(\min_{1\leq j\leq n} \gamma_j\right)^{-2} = 2^{3\alpha+1} \gamma_n^{-2}$.
   - If $d_n(F, L^2(\Omega)) \leq C_0 e^{-c_1 n^\alpha}$ for any $n \in \{1, 2, \dots\}$, then $\tau_n \leq C_0 \beta e^{-c_2 n^{-\alpha}}$ for $n \geq 1$, with $\beta := 2 \left(\min_{1\leq j\leq n} \gamma_j\right)^{-2} = 2\gamma_n^{-2}$.

Lemmas 5 and 6 are the keys to derive the decay rates of the interpolation error of the GEIM Greedy algorithm. This is the purpose of the following theorem:

Theorem 7: 1) Assume that $d_n(F, L^2(\Omega)) \leq C_0 n^{-\alpha}$ for any $n \geq 1$, then the interpolation error of the GEIM Greedy selection process satisfies for any $\varphi \in F$ the inequality $\|\varphi - \mathcal{J}_n[\varphi]\|_{L^2(\Omega)} \leq C_0(1 + \Lambda_n)\beta_n n^{-\alpha}$, where the parameter β_n is defined as in lemma 5.

2) Assume that $d_n(F, L^2(\Omega)) \leq C_0 e^{-c_1 n^\alpha}$ for any $n \geq 1$, then the interpolation error of the GEIM Greedy selection process satisfies for any $\varphi \in F$ the inequality $\|\varphi - \mathcal{J}_n[\varphi]\|_{L^2(\Omega)} \leq C_0(1 + \Lambda_n)\beta_n e^{-c_2 n^\alpha}$, where β_n and c_2 are defined as in lemma 6.

Proof: It can be inferred from lemma 1 that, $\forall \varphi \in F$, $\|\varphi - \mathcal{J}_n[\varphi]\|_{L^2(\Omega)} \leq (1 + \Lambda_n)\|\varphi - P_n(\varphi)\|_{L^2(\Omega)} \leq (1 + \Lambda_n)\tau_n$ according to the definition of τ_n. We conclude the proof by bounding τ_n thanks to lemmas 5 and 6. ■

Remark 5: If $(\Lambda_n)_{n\geq 1}$ is a monotonically increasing sequence, then the sequence $(\gamma_n)_{n\geq 1}$ in the GEIM procedure is monotonically decreasing. Using remark 4, the following decay rates in the generalized interpolation error can be derived:

- For any $\varphi \in F$, if $d_n(F, L^2(\Omega)) \leq C_0 n^{-\alpha}$ for any $n \geq 1$, then the interpolation error of the GEIM Greedy selection process can be bounded as $\|\varphi - \mathcal{J}_n[\varphi]\|_{L^2(\Omega)} \leq C_0 2^{3\alpha+1}(1 + \Lambda_n)^3 n^{-\alpha}$.
- For any $\varphi \in F$, if $d_n(F, L^2(\Omega)) \leq C_0 e^{-c_1 n^\alpha}$ for any $n \geq 1$, then the interpolation error of the GEIM Greedy selection process can be bounded as $\|\varphi - \mathcal{J}_n[\varphi]\|_{L^2(\Omega)} \leq C_0 2(1 + \Lambda_n)^3 e^{-c_2 n^\alpha}$.

Remark 6: The evolution of the Lebesgue constant Λ_N as a function of N is a subject of great interest. From the theoretical point of view, crude estimates exist and provide an exponential upper bound that is far from being what we get in the applications. As is shown in ([4], [5], [1]), the growth is lower than linear in N in the EIM situations. Our first numerical experiments with the GEIM reveal cases where it is uniformly bounded when evaluated in the $\mathcal{L}(L^2)$ norm (see [7], [10] for an illustration of this topic as well as for an application of the method to data assimilation coupled with simulation). We do not pretend that this is universal, but it only shows that the theoretical exponentially increasing upper bound is far from being optimal in a class of sets F that have a small Kolmogorov n-width.

V. Conclusion

In this work, it has been proven that the approximation properties of the generalized interpolating spaces X_n lead to an error that has the same qualitative decay as the best possible choice and that is distant by a (multiplicative) factor $(1 + \Lambda_n)\beta_n$ from it. This has been proven in the case of a polynomial or exponential convergence in the $n-$width and is a first step towards the explanation of efficiency of this method in practice (as outlined in [7]).

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