Reconstruction of solutions to the Helmholtz equation from punctual measurements

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***Abstract*—We analyze the sampling of solutions to the Helmholtz equation (e.g. sound fields in the harmonic regime) using a least-squares method based on approximations of the solutions by sums of Fourier-Bessel functions or plane waves. This method compares favorably to others such as Orthogonal Matching Pursuit with a Fourier dictionary. We show that using a significant proportion of samples on the border of the domain of interest improves the stability of the reconstruction, and that using cross-validation to estimate the model order yields good reconstruction results.**

I. Introduction

Sampling an acoustical field (i.e. the spatial and temporal behavior of sound pressure) or a mechanical field (e.g. distribution of velocities on a vibrating membrane) is an ubiquituous task in experimental acoustics and mechanics. Usually, these fields are sampled on a uniform grid with density chosen according to the sampling theorem. However, in the particular cases mentionned above, the fields are known to satisfy the wave equation

$$\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad (1)$$

or, in the harmonic regime, the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad (2)$$

in two or three dimensions, where c is the wave velocity, and k the wavenumber. This fact allows to sample such fields with a reduced number of samples, with a least-squares method described in section 2. Of interest here is the choice of the repartition of the sampling points on the domain of interest, and the choice of the order of approximation used in the least-squares reconstruction. In section 3, we recall the results given in [1] on the stability of the reconstruction in function of the sampling scheme for the case of the disk, and extend it to the 3D-ball. We also gives numerical evidence for the case of the square, further showing that sampling on the border of the domain as well as inside improves the stability of the reconstruction. Finally, we give results of numerical simulations using cross-validation for the determination of the model order in section 4.

II. Reconstruction method

Our goal here, given a solution to the Helmholtz equation (2) in a domain $D \subset \mathbf{R}^d, d = 2$ or 3, is to reconstruct it in a domain $\Omega \subset D$ from a limited number of punctual measurements, without knowing the shape of D or the boundary conditions on ∂D. The reconstruction scheme we use is based on the Vekua theory and least-squares approximations, and has already been shown to compare favorably with existing methods such as OMP using sparsity in the Fourier domain [1], and to give good results in experimental settings [2].

The Vekua theory [3], in its general formulation, allows to build approximations of solutions to general elliptic partial differential equations, by building operators mapping these solutions to harmonic functions and reciprocally. Approximation of harmonic functions by harmonic polynomials can then be translated as approximation of solutions of the PDE by the images of the polynomials. The particular case of the Helmholtz equation in 2 and 3 dimensions has been analyzed by Moiola *et al.* [4]. In this case, the images of the polynomials are the so-called *generalized harmonic polynomials*. In two dimensions, the space of generalized harmonic polynomials of order L is given in polar coordinates (r, θ) by

$$HP_{k,L} = \underset{l=-L,\cdots,L}{\operatorname{span}} e^{il\theta} J_l(kr)$$

where J_l is the l-th Bessel function. In three dimensions, these spaces are defined in spherical coordinates (r, θ, ϕ)

by

$$HP_{k,L} = \operatorname*{span}_{\substack{l=0,\cdots,L \\ m=-l,\cdots,l}} Y_{lm}(\theta,\phi) j_l(kr)$$

where Y_{lm} are the spherical harmonics, and j_l the spherical Bessel functions. Note that in two dimensions, the dimension of $HP_{k,L}$ is $2L+1$, while it is $(L+1)^2$ in three dimensions.

Their main result, given here in a simplified form and for convex domains, is as follows:

Theorem 1. *[4] Let $u \in H^K(\Omega), K \geq 1$ be a solution to the Helmholtz equation in the convex domain $\Omega \in \mathbf{R}^d$, $d = 2, 3$. Then, for $j < K$, there exists a generalized harmonic polynomial $\tilde{u}_L$ of order L such that, in two dimensions,*

$$\|u - \tilde{u}_L\|_{H^j} \leq C \left(\frac{L}{\log L} \right)^{K-j} \|u\|_{H^K},$$

and in three dimensions,

$$\|u - \tilde{u}_L\|_{H^j} \leq C L^{\lambda(K-j)} \|u\|_{H^K},$$

where λ depends only on the shape of Ω.

The result also holds for star-shaped domains, with a slower convergence. Identical results are also available for approximation by plane waves.

To reconstruct a solution u to the Helmholtz equation using n samples, we fix an order of approximation L such that $m = \dim HP_{k,L} \leq n$, and estimate u by the function $\tilde{u} \in HP_{k,L}$ minimizing the sum of the squares of the errors between values u_j sampled at the points x_j and $\tilde{u}(x_j)$, the sampling points being drawn using a predefined density on Ω:

$$\tilde{u} = \min_{\hat{u} \in HP_{k,L}} \sum_{j=1}^{n} |\hat{u}(x_j) - u_j|^2.$$

Such a reconstruction scheme is not always stable. A theorem, from Cohen et al [5], gives indication whether the reconstruction $\tilde{u}$ in a m-dimensional subspace using n samples drawn with probability density ν is stable. With $(L_j)_{j=1\cdots m}$ an orthogonal basis(with respect to the scalar product defined by the density ν) of the subspace, we define

$$K(m) = \max_{x \in \Omega} \sum_{j=1}^{m} |L_j(x)|^2 .$$

The result is as follows:

Theorem 2. *[5] Let $r > 0$ be arbitrary but fixed and let $\kappa := \frac{1-\log 2}{2+2r}$. If m is such that*

$$K(m) \leq \kappa \frac{n}{\log n}, \tag{3}$$

then, one has

$$E(\|u - \tilde{u}\|^2) \leq (1 + \epsilon(n)) \sigma_m(u)^2 + 8M^2 n^{-r}, \tag{4}$$

where $\epsilon(n) := \frac{4\kappa}{\log n} \to 0$ as $n \to +\infty$, $\sigma_m(u)$ is the best approximation error, M a upper bound of $|u|$ and $\tilde{u}$ the least square approximation of u thresholded such that $|\tilde{u}| \leq M$.

This suggests that the slowest $K(m)$ increases, the largest m can be, allowing a better reconstruction. The choice of the density ν is here important, as K is dependent on it. This means that choosing a adequate density allows to use a lower number of samples that, e.g. the uniform density. Note however that the choice of the density ν also affects the norm used in theorem 2 to measure the error, which can be different than the norm we are interested in. We are here interested in the stability for the standard L^2 norm. We thus choose a density of the form $\nu = (1-\alpha)\lambda + \alpha\nu'$ where λ is the uniform density, and ν' an arbitrary fixed density. The choice of the density ν' and the parameter α is discussed in the next section for some particular cases.

III. Choice of the sample distribution

Here, we will concentrate on measures $\nu = (1-\alpha)\lambda + \alpha\sigma$, where the support of σ is the boundary of Ω. This heuristic is supported by the following results on the disk and the ball. For these two cases, we will give estimations of K for particular values of m, i.e. the size of the spaces $HP_{k,L}$, $m = 2L+1$ in 2D and $m = (L+1)^2$ in 3D.

For the case of the disk with densities $\nu_\alpha = (1-\alpha)\lambda + \alpha\sigma$ where σ is the uniform measure on the circle, the Fourier-Bessel functions, after normalization, form an orthogonal basis for these measures. Using properties of the Bessel functions, we can estimate the behaviour of $K(m)$ in function of α:

Theorem 3. *[1] For the approximation by generalized harmonic polynomials on the unit disk, one has for sufficiently large m*

$$K(2L+1) \geq c_0 + c_1 L^2$$

when $\alpha = 0$ for any $c_1 < 1/4$ and where c_0 depends on c_1 and λ, and

$$K(2L+1) \leq C + \frac{2L+1}{\alpha}$$

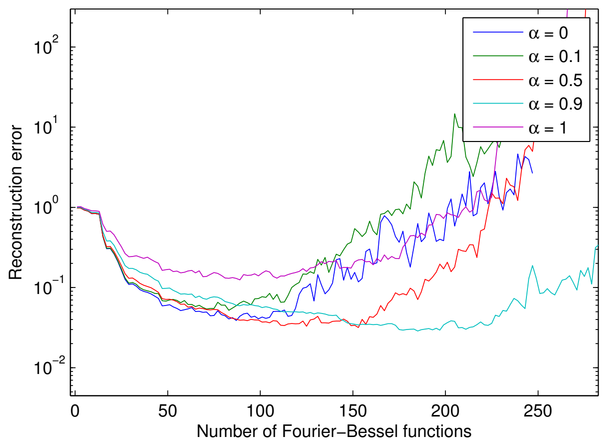

Fig. 1. Reconstruction error in function of the order model for 400 measurements, with different proportions α of samples on the border.

when $\alpha > 0$, and C depends on λ and α.

A similar result is available for plane waves approximations. This theorem shows that using samples on the border needs a number of samples proportionnal to the dimension of $HP_{k,L}$ to ensure stability, while sampling only inside the disk needs more samples.

The effect of the coefficient α is shown on figure 1, where the approximation error in function of α and the order of the model is given, for the recovery of a solution of the Helmholtz equation with $k = 12$, using $n = 400$ measurements. We see that a large proportion of samples on the border allows a large order model, which improve the reconstruction result. However, using samples on the border only ($\alpha = 1$) is detrimental to the reconstruction error, as in this case, theorem 2 controls the error in the norm defined by ν_1 which is the L_2-norm on the circle only.

We compare on figure 2 the results of the least-squares method with Fourier-Bessel functions, OMP with a large dictionary of Fourier modes defined on a square containing the disk, and the least-squares method with a smallest dictionary. The reconstruction of the least-squares method combined with the Fourier-Bessel approximation are clearly better than the two other tested methods.

A slightly modified proof, using properties of the spherical Bessel functions and of the spherical harmonics, yields the following result for the 3D case, with σ the uniform measure on the sphere:

Theorem 4. *For the approximation by generalized harmonic polynomials on the unit ball, one has for sufficiently large L*

$$K((L+1)^2) \geq c_0 + c_1 L^3$$

when $\alpha = 0$ for any $c_1 < 1/9$ and where c_0 depends on c_1 and λ, and

$$K((L+1)^2) \leq C + \frac{(L+1)^2}{\alpha}$$

when $\alpha > 0$, where C depends on λ and α.

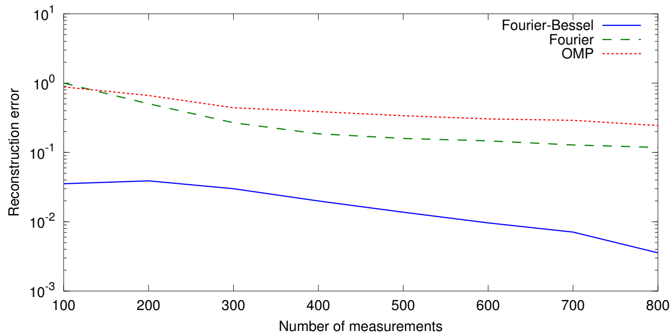

Fig. 2. Reconstruction error in function of the number of measurements for least square with Fourier-Bessel function, Fourier modes, and Orthogonal Matching Pursuit with a dictionnary of Fourier modes

In this case, the number of measurements needed to ensure stability grows faster than the dimension of $HP_{k,L}$ for the uniformly dense sampling, while being proportional to this dimension when using additional samples on the border.

We now turn to the case of the square. As neither the Fourier-Bessel functions, nor the plane waves, form an orthogonal basis, we constuct one by orthogonalizing the plane waves, using the Gram matrix of the plane waves families which can be computed exactly in the case of the measures described below.

We numerically compute $K(m)$ for three differents distributions:

- $\nu_0 = \lambda$, the uniform distribution on the square
- $\nu_\alpha = (1-\alpha)\lambda + \alpha\sigma$, where σ is the uniform distribution on the boundary of the square
- $\nu'_\alpha = (1-\alpha)\lambda + \alpha\sigma'$, where σ' is the measure on the boundary with weight $1/4\pi\sqrt{1-s^2}$ where $s = \min(x, y)$.

The estimated values of $K(m)$ for ν_0, $\nu_{1/2}$ and $\nu'_{1/2}$ are given on figure 3. Here, sampling on the border of the square improves the stability of the reconstruction compared to the uniform case, but still needs a high number of samples.

Using the non-uniform sampling on the border, with more samples in the sections of the boundary furthest

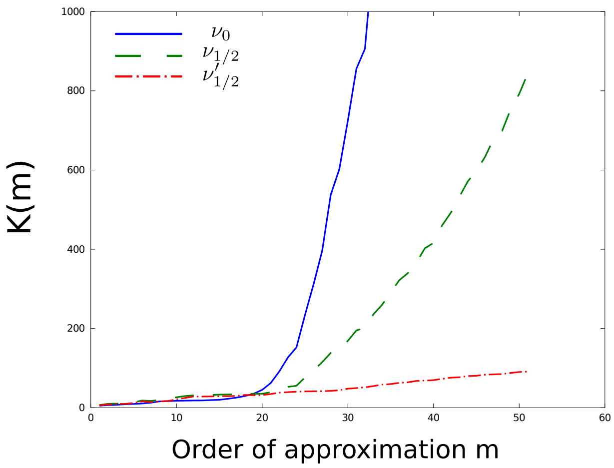

Fig. 3. Numerical evaluation of K for three different samples distribution on the square.

from the origin, makes the behaviour of $K(m)$ comparable to m, which is the best case possible.

IV. Choice of the model order

Once the number of samples and their distribution are fixed, the model order used in the reconstruction has to be chosen. While a sufficient number of plane waves or Fourier-Bessel functions are needed (physical arguments recommend a number proportionnal to the product of the wavenumber and the diameter of the domain), using a too large order can result in overfitting, as visible figure 1.

A way to estimate the best order m to use is the cross-validation. Given m samples, we reconstruct f from a subset of m' samples, and compute the reconstruction error on the remaining $m - m'$ samples. We then repeat with different subsets, and chose the order for which the average error is minimal. Figure 4 compares the best reconstruction error knowing f, and the reconstruction error using the order estimated using the cross-validation.

V. Conclusion

The sampling of solutions to the Helmholtz equation is interesting both for its experimental applications as well as for theoretical developments. We showed here that a careful choice of the density of the samples can improve the stability of the reconstruction, with theoretical results in simple cases, and numerical simulations in more general settings. We also show that using cross-validation to estimate the model order yields good results.

A general sampling strategy, *i.e.* a choice of the sample density, dependent on the shape of the domain of interest and of the frequency k, and possibly on the order m is yet to be designed.

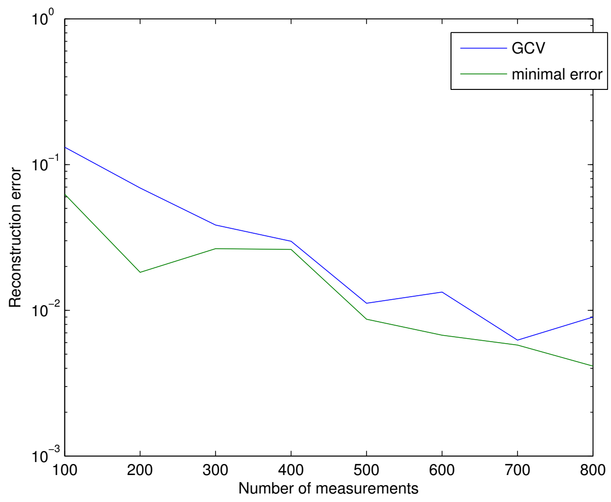

Fig. 4. Comparison of the reconstruction using the generalized cross validation to estimate the order model, and the best reconstruction error.

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