Approximate FRI with Arbitrary Kernels

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***Abstract*—In recent years, several methods have been developed for sampling and exact reconstruction of specific classes of non-bandlimited signals known as signals with finite rate of innovation (FRI). This is achieved by using adequate sampling kernels and reconstruction schemes, for example the exponential reproducing kernels of [1]. Proper linear combinations of this type of kernel with its shifted versions may reproduce polynomials or exponentials exactly.**

In this paper we briefly review the ideal FRI sampling and reconstruction scheme and some of the existing techniques to combat noise. We then present an alternative perspective of the FRI retrieval step, based on moments [1] and approximate reproduction of exponentials. Allowing for a controlled model mismatch, we propose a unified reconstruction stage that addresses two current limitations in FRI: the number of degrees of freedom and the stability of the retrieval. Moreover, the approach is universal in that it can be used with any sampling kernel from which enough information is available.

***Index Terms*—FRI, Sampling, Noise, Matrix Pencil, Approximation**

I. Introduction

Sampling, or the conversion of signals from analog to digital, provides the connection between the continuous-time and discrete-time worlds. The acquisition process is usually modelled as a filtering stage of the input $x(t)$ with a smoothing function $\varphi(t)$ (or sampling kernel), followed by uniform sampling at a rate $f_s = \frac{1}{T}$[Hz]. According to this setup, the measurements are given by

$$y_n = \int_{-\infty}^{\infty} x(t)\varphi\left(\frac{t}{T}-n\right)\mathrm{d}t = \left\langle x(t), \varphi\left(\frac{t}{T}-n\right)\right\rangle.$$

The fundamental problem of sampling is to recover the original waveform $x(t)$ using the samples y_n. When the signal is bandlimited, the answer due to the Nyquist-Shannon theorem is well known. Recently, however, it has been shown [2], [1], [3] that it is possible to sample and perfectly reconstruct specific classes of non-bandlimited signals, known as signals with finite rate of innovation (FRI). Perfect reconstruction is achieved by using a variation of Prony's method, called the annihilating filter method [3], [4].

In this paper we introduce the approximate recovery of FRI signals, from noisy samples taken by an arbitrary kernel. Our analysis follows the setup of [1], where the key to FRI reconstruction is exact reproduction of exponentials. We introduce the new property of approximate reproduction of exponentials by finding proper linear combinations of the sampling kernel. The main advantages of our method are that we can increase the number of measurements, improve the stability of the recovery and generalise the reconstruction stage.

The outline of the paper is as follows. In Section II we review the noiseless scenario of [1] and then give an overview of existing denoising techniques [3], [5]. In Section III we introduce the approximate FRI scenario. We first study the approximate reproduction of exponentials, and then apply this property to the recovery of FRI signals. We also propose an iterative algorithm to refine the accuracy of the reconstruction. Finally, in Section IV we show simulation results, to then conclude in Section V.

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II. Sampling signals with FRI

A. Perfect reconstruction of a stream of Diracs

We first summarise the main steps needed to sample and perfectly recontruct a train of K Diracs

$$x(t) = \sum_{k=0}^{K-1} a_k \delta(t-t_k), \tag{1}$$

where $t_k \in [0, \tau)$, from the samples

$$y_n = \left\langle x(t), \varphi\left(\frac{t}{T}-n\right)\right\rangle = \sum_{k=0}^{K-1} a_k \varphi\left(\frac{t_k}{T}-n\right), \tag{2}$$

for $n = 0, 1, \ldots, N-1$. Here we assume that the sampling period T is such that $\tau = NT$. Moreover, $\varphi(t)$ is an exponential reproducing kernel [1], [6] of compact support that satisfies

$$\sum_{n\in\mathbb{Z}} c_{m,n}\varphi(t-n) = \mathrm{e}^{\alpha_m t}, \tag{3}$$

for proper coefficients $c_{m,n}$, with $m = 0, \ldots, P$ and $\alpha_m \in \mathbb{C}$.

To begin, we linearly combine the samples y_n with the coefficients $c_{m,n}$ of (3) and obtain the new measurements (exponential moments):

$$s_m = \sum_{n=0}^{N-1} c_{m,n} y_n, \tag{4}$$

for $m = 0, \ldots, P$. Then, given that the signal $x(t)$ is a stream of Diracs (1) and combining (4) with (2) we have [1]:

$$s_m = \left\langle x(t), \sum_{n=0}^{N-1} c_{m,n}\varphi\left(\frac{t}{T}-n\right)\right\rangle = \sum_{k=0}^{K-1} x_k u_k^m, \tag{5}$$

with $x_k = a_k \mathrm{e}^{\alpha_0 \frac{t_k}{T}}$ and $u_k = \mathrm{e}^{\lambda \frac{t_k}{T}}$. In order for (5) to hold, we have restricted our analysis to parameters of the form $\alpha_m = \alpha_0 + m\lambda$, where $m = 0, \ldots, P$ and $\alpha_0, \lambda \in \mathbb{C}$. The reason we use these parameters is that they are needed for the values s_m to have a power sum series form (5), which is key to the recovery stage.

The new pairs of unknowns (u_k, x_k) for $k = 0, \ldots, K-1$ can then be retrieved from the measurements s_m using the annihilating filter method [2], [1], [3]. Let h_m with $m = 0, \ldots, K$ denote the filter with z-transform $\hat{h}(z) = \sum_{m=0}^{K} h_m z^{-m} = \prod_{k=0}^{K-1}\left(1 - u_k z^{-1}\right)$. Then, h_m annihilates the series s_m:

$$h_m * s_m = \sum_{i=0}^{K} h_i s_{m-i} = \sum_{k=0}^{K-1} x_k u_k^m \underbrace{\sum_{i=0}^{K} h_i u_k^{-i}}_{\hat{h}(u_k)} = 0. \tag{6}$$

The zeros of this filter uniquely define the values u_k provided the locations t_k are different. Interestingly, identity (6) can be written in matrix-vector form as:

$$\mathbf{S}\mathbf{h} = 0 \tag{7}$$

which reveals that the Toeplitz matrix $\mathbf{S}$ is rank deficient. The annihilating filter is therefore in the null space of $\mathbf{S}$. By solving the above system, we find the coefficients h_m, and then retrieve u_k from the roots of $\hat{h}(z)$. Finally, we determine the weights x_k by solving the first K consecutive equations in (5). Notice that the problem can be solved only when $N \geqslant P+1 \geqslant 2K$.

An exponential reproducing kernel is any function $\varphi(t)$ that, together with its shifted versions, can reproduce exponentials, that is, it satisfies (3). The coefficients $c_{m,n}$ are given by

$$c_{m,n} = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha_m t} \tilde{\varphi}(t-n)\,\mathrm{d}t = c_{m,n}\mathrm{e}^{\alpha_m n},$$

where the function $\tilde{\varphi}(t)$ is such that $\langle \tilde{\varphi}(t-n), \varphi(t-m)\rangle = \delta_{m-n}$, and with $c_{m,0} = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha_m x}\tilde{\varphi}(x)\mathrm{d}x$.

Any exponential reproducing kernel can be written as $\varphi(t) = \gamma(t) * \beta_{\vec{\alpha}}(t)$ [1], [6], where $\gamma(t)$ is an arbitrary function, even a distribution, $\beta_{\vec{\alpha}}(t)$ is an E-Spline and $\vec{\alpha} = \{\alpha_m\}_{m=0}^{P}$. The Fourier domain representation of an E-Spline of order $P+1$ is:

$$\hat{\beta}_{\vec{\alpha}}(\omega) = \prod_{m=0}^{P} \frac{1-\mathrm{e}^{\alpha_m - j\omega}}{j\omega - \alpha_m}.$$

In this paper we work with real valued sampling kernels characterised by $\gamma(t)$ and $\beta_{\vec{\alpha}}(t)$ being real. Since $\alpha_m = \alpha_0 + m\lambda$ with $m = 0, \ldots, P$ this implies $\lambda = (\alpha_0^* - \alpha_0)/P$. Note that this condition makes α_m and s_m exist in complex conjugate pairs.

B. Sampling signals with FRI in the presence of noise

Noise is generally present in data acquisition, making the solution explained so far ideal. Assume the noiseless samples y_n are corrupted by additive noise such that we have access to the measurements $\tilde{y}_n = y_n + \epsilon_n$ for $n = 0, \ldots, N-1$. In this situation, the moments, given by the linear combination of samples (4), become noisy:

$$\tilde{s}_m = \underbrace{\sum_{k=0}^{K-1} x_k u_k^m}_{s_m} + \underbrace{\sum_{n=0}^{N-1} c_{m,n}\epsilon_n}_{b_m}, \tag{8}$$

and perfect reconstruction is no longer possible. If ϵ_n are i.i.d. Gaussian, then b_m are samples of Gaussian noise, but not necessarily white.

Note that now (7) becomes approximate due to $\tilde{\mathbf{S}} = \mathbf{S} + \mathbf{B}$, where $\mathbf{B}$ is a Toeplitz matrix formed from the values b_m of (8). Thus, we need alternative ways of solving (7), for instance by using total least squares and Cadzow [3] or the matrix pencil method [7], [5]. The latter can be summarised as follows: obtain the SVD decomposition $\tilde{\mathbf{S}} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$, keep the K columns of $\mathbf{U}$ corresponding to the dominant singular values and estimate u_k as the eigenvalues of $\underline{\mathbf{U}}_K^+ \overline{\mathbf{U}}_K$. Here, $\underline{(\cdot)}$ and $\overline{(\cdot)}$ are operations to omit the last and first rows of $(\cdot)$.

In addition, note that the covariance matrix of the noise $\mathbf{R}_\mathbf{B} = \mathrm{E}\{\mathbf{B}^H\mathbf{B}\}$ may not be a multiple of the identity. In order for SVD to operate properly it is necessary to pre-whiten the noise [8], for instance by using a linear transform $\mathbf{W} = \mathbf{R}_\mathbf{B}^{\dagger/2}$ [9] for $\mathbf{A} = \mathbf{B}\mathbf{W}$ to satisfy that $\mathbf{R}_\mathbf{A} = \mathrm{E}\{\mathbf{A}^H\mathbf{A}\} = \mathbf{I}$. Here, $(\cdot)^{\dagger/2}$ denotes the square root of the pseudoinverse of $(\cdot)$. In our simulations, we apply pre-whitening on $\tilde{\mathbf{S}}$ such that $\tilde{\mathbf{S}}\mathbf{W}$ is now contaminated by white noise $\mathbf{A}$. We then directly use matrix pencil on $\tilde{\mathbf{S}}\mathbf{W}$.

In order to analyse the effect of noise on the accuracy with which FRI signals can be recovered we use the Cramér–Rao lower bound (CRB). This is a lower bound on the mean square error (MSE) that applies to any unbiased estimator [4]. A stream of K Diracs is completely characterised by the vector $\Theta = (t_0, \ldots, t_{K-1}, a_0, \ldots, a_{K-1})^T$, of K locations and amplitudes. And the goal of FRI reconstruction is to estimate Θ either from the vector of N samples $\tilde{\mathbf{y}} = (\tilde{y}_0, \ldots, \tilde{y}_{N-1})^T$ or the vector of $P+1$ noisy moments $\tilde{\mathbf{s}} = (\tilde{s}_0, \ldots, \tilde{s}_P)^T$.

The analysis of the CRB for the estimation problem given $\tilde{\mathbf{y}}$ is detailed in [3]. In our simulations, we compare the estimation accuracy with this bound, but we consider the CRB for the estimation from $\tilde{\mathbf{s}}$. Given values s_m that exist in complex conjugate pairs, then any unbiased estimate $\hat{\Theta}(\tilde{\mathbf{s}}) = (\hat{t}_0, \ldots, \hat{t}_{K-1}, \hat{a}_0, \ldots, \hat{a}_{K-1})^T$ has a covariance matrix that is lower bounded by [10]

$$\mathrm{cov}(\hat{\Theta}(\tilde{\mathbf{s}})) \geqslant (\Phi^H \mathbf{R}^{-1} \Phi)^{-1}. \tag{9}$$

Here, $(\cdot)^H$ denotes Hermitian transpose. Moreover, provided ϵ_n are samples of additive white Gaussian noise, of zero mean and variance σ^2, then $\mathbf{R} = \mathrm{E}\{\mathbf{b}\mathbf{b}^H\} = \sigma^2\mathbf{C}\mathbf{C}^H$, because $\mathbf{b}$ is the vector of noise values b_m of (8). The matrix Φ in (9) takes the form:

$$\left(\begin{array}{ccc|ccc} a_0\alpha_0\mathrm{e}^{\alpha_0 t_0} & \ldots & a_{K-1}\alpha_0\mathrm{e}^{\alpha_0 t_{K-1}} & \mathrm{e}^{\alpha_0 t_0} & \ldots & \mathrm{e}^{\alpha_0 t_{K-1}} \\ a_0\alpha_1\mathrm{e}^{\alpha_1 t_0} & \ldots & a_{K-1}\alpha_1\mathrm{e}^{\alpha_1 t_{K-1}} & \mathrm{e}^{\alpha_1 t_0} & \ldots & \mathrm{e}^{\alpha_1 t_{K-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_0\alpha_P\mathrm{e}^{\alpha_P t_0} & \ldots & a_{K-1}\alpha_P\mathrm{e}^{\alpha_P t_{K-1}} & \mathrm{e}^{\alpha_P t_0} & \ldots & \mathrm{e}^{\alpha_P t_{K-1}} \end{array}\right).$$

III. Universal sampling of signals with FRI

In many practical circumstances we may not be able to choose the sampling kernel $\varphi(t)$, or even know its exact shape. In such cases there might not be an easy way of finding coefficients $c_{m,n}$ for the linear combination of samples (4) to yield a power sum series (5). And this is key in the FRI setting to map the signal reconstruction problem to Prony's method in spectral-line estimation theory.

In this section we consider any function $\varphi(t)$ for which the exponential reproduction property (3) is only approximate. We propose to use the coefficients $c_{m,n}$ for approximate reproduction to build (4) such that they yield a power sum series (5) from which the FRI parameters can be retrieved.

A. Approximate reproduction of exponentials

Assume we want to use a function $\varphi(t)$ and its integer shifts to approximate the exponential $\mathrm{e}^{\alpha t}$. In other words, we seek the coefficients c_n that best fit:

$$\sum_{n\in\mathbb{Z}} c_n \varphi(t-n) \cong \mathrm{e}^{\alpha t}. \tag{10}$$

In order to do so, we directly use $c_n = c_0\mathrm{e}^{\alpha n}$. Then, equation (10) is equivalent to approximating $g_\alpha(t) = c_0 \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\alpha(t-n)}\varphi(t-n)$ by the constant value 1. We also note that $g_\alpha(t)$ is a 1-periodic function, because $g_\alpha(t) = g_\alpha(t+1)$. It can therefore be decomposed using the Fourier series as

$$g_\alpha(t) = \sum_{l\in\mathbb{Z}} g_l \mathrm{e}^{j2\pi l t}, \tag{11}$$

where

$$g_l = \int_0^1 g_\alpha(t)\mathrm{e}^{-j2\pi l t}\mathrm{d}t = c_0 \sum_{k\in\mathbb{Z}} \int_0^1 \mathrm{e}^{-\alpha(t-k)}\varphi(t-k)\mathrm{e}^{-j2\pi l t}\mathrm{d}t$$
$$\stackrel{(a)}{=} c_0 \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x}\varphi(x)\mathrm{e}^{-j2\pi l x}\mathrm{d}x = c_0\hat{\varphi}(\alpha + j2\pi l).$$

Here, (a) is due to using $x = t-k$ and combining the sum over $k \in \mathbb{Z}$ and the integral dependent on k. Also $\hat{\varphi}(s) = \int_{-\infty}^{\infty} \varphi(x)\mathrm{e}^{-sx}\mathrm{d}x$ denotes the Laplace transform of $\varphi(x)$.

In general $\varphi(t)$ may be any function and we can find different sets of coefficients c_n for (10) to hold. The accuracy of our approximation is given by:

$$\varepsilon(t) = \mathrm{e}^{\alpha t}\left[1 - c_0 \sum_{l\in\mathbb{Z}} \hat{\varphi}(\alpha + j2\pi l)\mathrm{e}^{j2\pi l t}\right]. \quad (12)$$

Note that if the Laplace transform of $\varphi(t)$ decays sufficiently quickly, very few terms are needed to have an accurate bound for the error.

A natural choice of the coefficients $c_n = c_0\mathrm{e}^{\alpha n}$ is obtained by discarding every term in (11) for $l \neq 0$ and making $g_0 = 1$, hence $c_0 = \hat{\varphi}(\alpha)^{-1}$. Interestingly, this is a simplified version of the least-squares coefficients [11] for the approximation in (10). The main advantage of using coefficients with $c_0 = \hat{\varphi}(\alpha)^{-1}$ is that they are very easy to compute, because they only require the knowledge of the Laplace transform of $\varphi(t)$ at α.

We conclude with an example. Consider a linear spline that reproduces polynomials of orders 0 and 1 exactly, as shown in Figure 1 (a). We want to approximate the complex exponentials $\mathrm{e}^{j-\frac{\pi}{16}(2m-7)t}$ for $m = 3$ and $m = 0$ by using linear combinations of the spline. This can be done by selecting coefficients $c_{m,n} = \hat{\varphi}(\alpha_m)^{-1}\mathrm{e}^{\alpha_m n}$ where $\alpha_m = j\frac{\pi}{16}(2m - 7)$. We illustrate the reproduction of the real part of the complex exponentials in Figure 1 (b-c). Note how the one with lower frequency is better approximated. Moreover, we have seen experimentally that higher order splines tend to improve the quality of the approximation. Also note there is no fixed number of exponentials that may be well approximated.

B. Approximate FRI recovery

Consider again the stream of Diracs (1) and samples of the form (2), now taken by an arbitrary sampling kernel $\varphi(t)$. In order to retrieve the locations t_k and amplitudes a_k for $k = 0, \ldots, K-1$, we first obtain the coefficients $c_{m,n} = \hat{\varphi}(\alpha_m)^{-1}\mathrm{e}^{\alpha_m n}$ for $m = 0, \ldots, P$. We only need to know the Laplace transform of $\varphi(t)$ at α_m. Note that P is a free parameter, subject to $P + 1 \geqslant 2K$.

We proceed in the same way as in the case of exact reproduction of exponentials, but now the exponential moments take the form

$$s_m = \left\langle x(t), \underbrace{\sum_{n=0}^{N-1} c_{m,n}\varphi(t-n)}_{\mathrm{e}^{\alpha_m t} - \varepsilon_m(t)} \right\rangle = \sum_{k=0}^{K-1} x_k u_k^m - \zeta_m$$

where $x_k = a_k\mathrm{e}^{\alpha_0 t_k}$ and $u_k = \mathrm{e}^{\lambda t_k}$. Here we have used $T = 1$ and $\alpha_m = \alpha_0 + m\lambda$, with $m = 0, \ldots, P$, and $\alpha_0, \lambda \in \mathbb{C}$. There is a model mismatch due to the approximation error $\varepsilon_m(t)$ of (12), equal to $\zeta_m = \sum_{k=0}^{K-1} a_k\varepsilon_m(t_k)$.

The model mismatch depends on the quality of the approximation, and depends on the coefficients $c_{m,n}$ and the values α_m and P. We treat this error as noise, and retrieve the parameters of the signal using the methods of Section II-B. In close-to-noiseless settings, the estimation of the Diracs can be refined using the iterative procedure shown in Algorithm 1.

C. How to select the approximation parameters α_m

In order to simplify the problem, we restrict the exponential parameters to be of the form:

$$\alpha_m = j\omega_m = j\frac{\pi}{L}(2m - P) \quad m = 0, \ldots, P. \quad (13)$$

Purely imaginary parameters allow for a more stable retrieval of the pairs (t_k, a_k) from (5). The values to be determined are, therefore, P and L. We choose the values that minimise the first diagonal term

Algorithm 1 Recovery of a train of K Diracs using approximation of exponentials

1: Compute the moments $s_m^0 = \sum_n c_{m,n}y_n$ and set $s_m^i = s_m^0$, for $m = 0, \ldots, P$.
2: Build the system of equations (6) using s_m^i and retrieve the annihilating filter h_m.
3: Calculate u_k^i from the roots of h_m, and $t_k^i = \frac{1}{\lambda}\ln u_k^i$, for the ith iteration.
4: Find x_k^i from the first K consecutive equations in (5), and the amplitudes $a_k^i = x_k^i\mathrm{e}^{-\alpha_0 t_k}$.
5: Recalculate the moments for the next iteration by removing the model mismatch:
$$s_m^{i+1} = s_m^0 + \sum_{k=0}^{K-1} a_k^i\varepsilon_m(t_k^i),$$
where $\varepsilon_m(t)$ is given by (12).
6: Repeat steps 2 to 5 until convergence of the values (a_k^i, t_k^i).

of (9) when $K = 1$, which corresponds to the error in the recovery of the location t_0. In most cases we have analysed, the best P is greater or equal than the support of the sampling kernel $\varphi(t)$ and L is in the range $P + 1 \leqslant L \leqslant 4(P + 1)$.

IV. Simulations

We take $N = 31$ samples of a train of K Diracs using a B-Spline kernel, and we corrupt the measurements with additive white Gaussian noise of variance σ^2. This is chosen according to the required signal-to-noise ratio $\mathrm{SNR(dB)} = 10\log(\|\mathbf{y}\|^2/N\sigma^2)$. We then obtain the approximation coefficients $c_{m,n} = \hat{\varphi}(\alpha_m)^{-1}\mathrm{e}^{\alpha_m n}$, where α_m is as in (13) with $L = 2(P + 1)$ and $m = 0, \ldots, P$. Finally, we compute the noisy $P + 1$ moments and retrieve the innovation parameters (t_k, a_k), for $k = 0, \ldots, K-1$, using the matrix pencil method. We calculate the standard deviation of the error in the estimation of the location, over 1000 realisations of the noise, and compare it to the sample-based and moment-based CRBs of Section II-B.

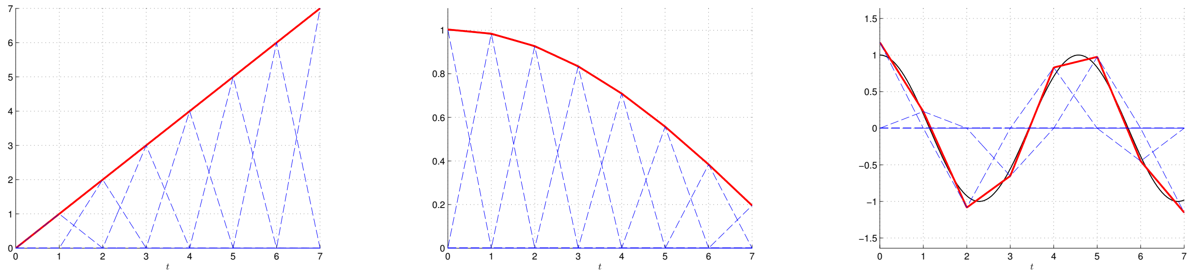
Figure 2 shows the deviation in the location of $K = 6$ Diracs. We compare the performance (a) when we sample with a B-Spline of order 26 and use the default retrieval based on the reproduction of polynomials [1], with (b) when we sample with a B-Spline of order 6 and apply the retrieval based on approximation of exponentials, with 26 moments; both aided with pre-whitening. The SNR is 20dB. It is only in the latter case that we can recover all the Diracs. Moreover, the accuracy with which the Diracs are recovered is one order of magnitude better for the approximated FRI.

We show further results when we use the approximate method to retrieve $K = 2$ Diracs from the samples taken by a B-Spline kernel of order 6. Even when we fix the order of the kernel, we can use any number of moments $P + 1$ to improve the performance. Figures 3 (a-d) are for parameters (13) with $L = \frac{3}{2}(P+1)$ and $m = 0, \ldots, P$. As the number of moments $P+1$ increases, the performance is better and eventually reaches the sample-based CRB.

V. Conclusions

We have presented an alternative FRI retrieval approach, based on the approximate reproduction of exponentials. Allowing for a controlled model mismatch, we propose a standard reconstruction stage that is able to increase the stability of existing FRI schemes.

Moreover, in many practical circumstances we may not be able to choose the sampling kernel or even know its exact shape. However,

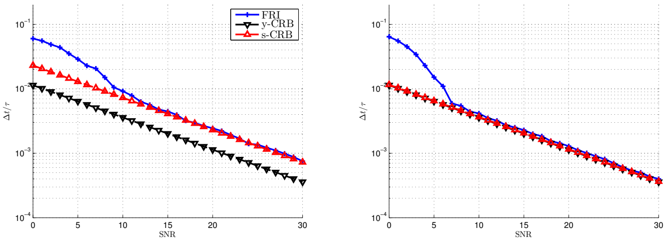


(a) Reproduction of t (b) Approximation of $\mathrm{Re}\{\mathrm{e}^{-j\frac{\pi}{16}t}\}$ (c) Approximation of $\mathrm{Re}\{\mathrm{e}^{-j\frac{7\pi}{16}t}\}$

Figure 1. *B-Spline kernel reproduction capabilities*. Figure (a) shows the exact reconstruction of a polynomial of order 1, by using a proper combination of shifted versions of a linear spline. Figures (b-c) show the approximation of the real parts of 2 complex exponentials: $\mathrm{e}^{j\frac{\pi}{16}(2m-7)t}$ for $m = 3, 0$, by using a proper combination of shifted versions of the linear spline. The coefficients are $c_{m,n} = \hat{\varphi}(\alpha_m)^{-1}\mathrm{e}^{\alpha_m n}$ where $\alpha_m = j\frac{\pi}{16}(2m-7)$. We plot the weighted and shifted versions of the splines with dashed blue lines, the reproduced polynomial and exponentials with red solid lines, and the exact functions with solid black lines.

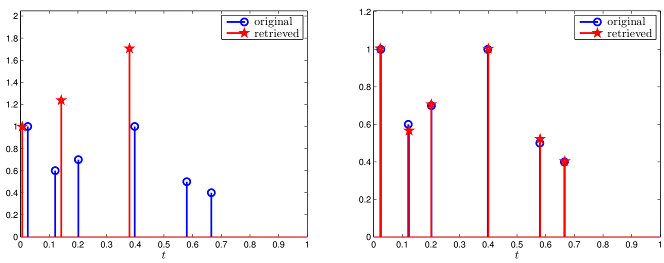
we have seen that if we know the Laplace transform of the kernel at values α_m, we can find coefficients for the linear combination of shifted versions of the sampling kernel to approximate exponentials $\mathrm{e}^{\alpha_m t}$. Equipped with this property we can sample a stream of K Diracs and retrieve it from $2K$ measurements. The accuracy of the reconstruction depends on the quality of the approximation and the level of noise.

Future work includes FRI retrieval with partial information on the sampling kernel, with more challenging existing FRI kernels (such as the Gaussian), and extensions to more dimensions and non-uniform sampling. In addition, approximate reconstruction may also be generalised when we have access to measurements taken by different kernels, each of which is capable of approximating certain exponentials.



(a) $P+1=6$ (b) $P+1=31$

Figure 3. *Approximated retrieval using a B-Spline*. These figures show the error in the estimation of the first Dirac out of $K = 2$ retrieved using the approximated FRI recovery. We show how, even when we fix the order of the kernel to 6, we can reconstruct any number of moments $P+1$ and improve the performance. In fact, with the appropriate choice $L = \frac{3}{2}(P+1)$ the performance improves until the sample-based CRB is reached.



(a) Default (b) Approximation

Figure 2. *B-Spline kernel behaviour*. We retrieve $K = 6$ Diracs from $N = 31$ noisy samples: (a) using the polynomial recovery of [1], with a kernel of order 26 and also $P+1 = 26$ moments; (b) using the approximated recovery with parameters (13) where $L = 2(P+1)$ and $m = 0, \ldots, P$, with a kernel of order 6 and $P+1 = 26$ moments. The SNR in both cases is 20dB.

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