Recovery of bilevel causal signals with finite rate of innovation using positive sampling kernels

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***Abstract*—Bilevel signal x with maximal local rate of innovation R is a continuous-time signal that takes only two values 0 and 1 and that there is at most one transition position in any time period of $1/R$. In this note, we introduce a recovery method for bilevel causal signals x with maximal local rate of innovation R from their uniform samples $x*h(nT), n \geq 1$, where the sampling kernel h is causal and positive on $(0,T)$, and the sampling rate $\tau := 1/T$ is at (or above) the maximal local rate of innovation R. We also discuss stability of the bilevel signal recovery procedure in the presence of bounded noises.**

I. Introduction

Let $T > 0$ and N be a nonnegative integer or infinity, and denote by χ_E the indicator function on a set E. In this note, we consider *bilevel causal signals*

$$x(t) := \sum_{i=1}^{N} \chi_{[t_{2i-1}, t_{2i})}(t) \tag{1}$$

with unknown transition values (positions) $t_i, 1 \leq i \leq 2N$, satisfying

$$t_i < t_{i+1}, \ 1 \leq i < 2N; \tag{2}$$

and also a uniform generalized sampling process

$$x(t) \longmapsto x * h(t) \longmapsto \{x * h(nT)\}_{n \geq 1} \tag{3}$$

with sampling kernel h being causal and uniform sampling taken every T seconds. For the bilevel causal signal x in (1), define its *maximal local rate of innovation* R by reciprocal of the maximal positive number σ_0 such that there is at most one transition position $t_i, 1 \leq i \leq 2N$, in any time period $[t, t+\sigma_0), t \geq 0$, that is,

$$R = \sup_{1 \leq i < 2N} \frac{1}{t_{i+1} - t_i}. \tag{4}$$

The concept of signals with finite rate of innovation was introduced by Vetterli, Marziliano and Blu [1]. Examples of signals with finite rate of innovation include streams of Diracs, piecewise polynomials, band-limited signals, and signals in a finitely-generated shift-invariant space [1]–[4]. In the past ten years, the paradigm for reconstructing signals with finite rate of innovation from their samples has been developed, see for instance [1], [2] and [4]–[13] and references therein.

Precise identification of transition positions is important to reach meaningful conclusions in many applications. Vetterli, Marziliano and Blu show in [1] that a bilevel signal x defined in (1) can be reconstructed from its samples (3) when the sampling kernel h is the box spline $\chi_{[0,T)}$ (or the hat spline $(T-|t|)\chi_{[-T,T)}(t)$) and the sample rate $\tau := 1/T$ is at (or above) the maximal local rate of innovation R of the signal x. In this note, we show that bilevel causal signals x defined in (1) are uniquely determined from their samples $x * h(nT), n \geq 1$, in (3) if the sampling kernel h is causal and positive on $(0,T)$, and the sample rate τ is at (or above) the maximal local rate of innovation R, see Theorem 1. Our numerical simulations indicate that the bilevel signal recovery procedure from noisy samples $x*h(nT)+\epsilon_n, n \geq 1$, is stable when there are limited numbers of transition positions for the bilevel signal x.

II. Recovery of bilevel causal signals

In this section, we provide a necessary condition on the sampling kernel h such that bilevel signals x in (1) are uniquely determined from their samples $\{x * h(nT)\}$ in (3). Also in this section, we propose an algorithm for the bilevel signal recovery.

The main theorem of this note is as follows:

Theorem 1: Let $T > 0$ and set $\tau = 1/T$. If h is a causal sampling kernel with $h(t) > 0$ on $(0,T)$, then any bilevel causal signal x in (1) with maximal local rate of innovation R being less than or equal to the sampling rate τ can be recovered from its samples $x * h(nT), n \geq 1$.

Proof: Let

$$H(t) = \int_0^t h(s)ds, \ 0 \leq t \leq T. \tag{5}$$

Then $H(0) = 0$ and H is a strictly increasing function on $[0,T)$ as h is strictly positive on $(0,T)$. Denote its inverse function on $[0,T]$ by $H^{-1} : [0, H(T)] \longmapsto [0,T]$.

Let x be a bilevel causal signal in (1) with transition positions $t_i, 1 \leq i \leq 2N$, satisfying (2). Then its first sample $y_1 = x * h(T)$ is given by

$$\begin{aligned} y_1 &= \int_0^\infty x(t) h\big(T-t\big)dt = \int_0^T x(t)h\big(T-t\big)dt \\ &= \int_0^T \chi_{[t_1,t_2)}(t)h\big(T-t\big)dt = H\big(\max\{T-t_1, 0\}\big), \end{aligned}$$

where the first two equalities hold by the causality of the signal x and the sampling kernel h, and the fourth equality follows from (1) and the observation that

$$t_i \geq t_2 = (t_2 - t_1) + t_1 \geq 1/R + 0 \geq 1/\tau = T, \ i \geq 2$$

by (2), (4) and the assumption that $R \leq \tau$. Recall that H is strictly increasing on $[0, T)$. Then there exists a transition position in the time range $[0, T)$ if and only if $y_1 = x*h(T) > 0$. Moreover, if it exists, it is given by

$$t_1 = T - H^{-1}(y_1). \tag{6}$$

Thus for a bilevel causal signal, we may determine from its first sample $x * h(T)$ the (non-)existence of its transition position in the time period $[0, T)$ and further its transition position in that time period if there is one.

Inductively, we assume that all transition positions of the bilevel signal x in the time range $[0, nT)$ have been determined from its samples $y_k = x * h(kT), 1 \leq k \leq n$. We examine four cases to determine its transition position in the time period $[nT, (n+1)T)$ from the sample $y_{n+1} = x * h((n+1)T)$.

Case 1: There is no transition position in $[0, nT)$.

In this case, following the above argument to determine transition positions in the time range $[0, T)$, we have that there exists a transition position in $[nT, (n+1)T)$ if and only if $y_{n+1} > 0$. If there is, the transition position is the first transition position t_1 of the bilevel causal signal x, and

$$t_1 = (n+1)T - H^{-1}(y_{n+1}). \tag{7}$$

Case 2: The last transition position in $[0, nT)$ is t_{2i_0-1} for some $i_0 \geq 1$.

In this case, $t_{2i_0} \geq nT$ and $t_i \geq (n+1)T$ for all $i > 2i_0$. Thus

$$\begin{aligned} y_{n+1} &= \int_0^{(n+1)T} x(t) h\big((n+1)T - t\big) dt \\ &= \int_0^{(n+1)T} h\big((n+1)T - t\big) \\ &\quad \times \Big(\sum_{i=1}^{i_0 - 1} \chi_{[t_{2i-1}, t_{2i})}(t) + \chi_{[t_{2i_0-1}, (n+1)T)}(t) \Big) dt \\ &\quad - \int_{nT}^{(n+1)T} h\big((n+1)T - t\big) \\ &\quad \times \chi_{[\min(t_{2i_0}, (n+1)T), (n+1)T)}(t) dt. \end{aligned}$$

Hence there exists a transition position t_{2i_0} in the time range $[nT, (n+1)T)$ if and only if

$$\tilde{y}_{n+1} := -y_{n+1} + \int_0^{(n+1)T} h\big((n+1)T - t\big) \times \Big(\sum_{i=1}^{i_0 - 1} \chi_{[t_{2i-1}, t_{2i})}(t) + \chi_{[t_{2i_0-1}, (n+1)T)}(t) \Big) dt \tag{8}$$

is positive. Moreover if $\tilde{y}_{n+1} > 0$, the transition position t_{2i_0} in the time range $[nT, (n+1)T)$ is determined by

$$t_{2i_0} = (n+1)T - H^{-1}(\tilde{y}_{k+1}). \tag{9}$$

Case 3: The last transition position in $[0, nT)$ is t_{2i_0} for some $1 \leq i_0 < N$.

In this case, the $(n+1)$-th sample $y_{n+1} = x * h((n+1)T)$ is given by

$$\begin{aligned} y_{n+1} &= \int_0^{nT} \Big(\sum_{i=1}^{i_0} \chi_{[t_{2i-1}, t_{2i})(t)} \Big) h\big((n+1)T - t\big) dt \\ &\quad + \int_{\min(t_{2i_0+1}, (n+1)T)}^{(n+1)T} h\big((n+1)T - t\big) dt. \end{aligned} \tag{10}$$

Thus there exists a transition value $t_{2i_0+1} \in [nT, (n+1)T)$ if and only if

$$\tilde{y}_{n+1} := y_{n+1} - \int_0^{nT} \Big(\sum_{i=1}^{i_0} \chi_{[t_{2i-1}, t_{2i})(t)} \Big) h\big((n+1)T - t\big) dt \tag{11}$$

is positive. Also we see that if $\tilde{y}_{n+1} > 0$, then the transition value t_{2i_0+1} can be obtained by

$$t_{2i_0+1} = (n+1)T - H^{-1}(\tilde{y}_{n+1}). \tag{12}$$

Case 4: The last transition position in $[0, nT)$ is t_{2N}.

In this case, all transition positions of the bilevel signal x have been recovered already. Hence the bilevel signal x is fully recovered.

This completes our inductive proof. ■

From the above argument of Theorem 1, we can use the following algorithm to recover a bilevel causal signal x in (1) from its samples $x * h(nT), 1 \leq n \leq K$,where $K > t_{2N}\tau$:

Bilevel Signal Recovery Algorithm:

Step 1: If all samples $y_n = x * h(nT), 1 \leq n \leq K$, are zero, then set $x = 0$ and stop; else find the first nonzero sample, say $y_{n_0} > 0$, the first transition position of the bilevel signal x is located at $t_1 := n_0 - H^{-1}(y_{n_0})$, and set $n = n_0$.

Step 2: Do Step 2a if the last transition position in the time range $[0, nT)$ is t_{2i_0-1} for some $i_0 \geq 1$; do Step 2b elseif the last transition position in the time range $[0, nT)$ is t_{2i_0} for some $1 \leq i_0 < N$; and do Step 4 else.

- Step 2a: Define t_{2i_0} as in (9) if $\tilde{y}_{n+1}$ in (8) is positive, else do Step 3.
- Step 2b: Define t_{2i_0+1} as in (12) if $\tilde{y}_{n+1}$ in (11) is positive, else do Step 3.

Step 3: Set $n = n + 1$. Do Step 2 if $n < K$, and Step 4 if $n = K$.

Step 4: Stop as all transition positions $t_i, 1 \leq i \leq 2N$, of the bilevel signal x are recovered.

We finish this section with a remark that the requirement $R \leq \tau$ in Theorem 1 can be relaxed to the following: There is at most one transition position $t_i, 1 \leq i \leq 2N$, in each sampling range $[nT, (n+1)T), n \geq 1$.

III. Stable recovery of bilevel causal signals

In this section, we consider the maximal sampling error $\sup_n |x * h(nT) - \tilde{x} * h(nT)|$ of two bilevel signals x and $\tilde{x}$ when maximal error of their transition positions are small. We then present some numerical simulations on recovery of a bilevel signal x in (1) from its noisy samples $\{x*h(nT)+\epsilon_n\}$ in (3), where $\epsilon_n, n \ge 1$, are bounded noises of low levels.

First we notice that sampling procedure from bilevel signals x to their samples $\{x * h(nT)\}$ are stable in bounded norm.

Theorem 2: Let $T > 0$, h be a bounded filter supported in $[0, MT)$, $x(t) = \sum_{i=1}^{N} \chi_{[t_{2i-1}, t_{2i})}(t)$ be a bilevel causal signal with maximal local innovation rate $R \le \tau := 1/T$, and $\tilde{x}(t) = \sum_{i=1}^{N} \chi_{[t_{2i-1}+\delta_{2i-1}, t_{2i}+\delta_{2i})}$ be a perturbation of the bilevel signal x with perturbed transition positions $\{t_i+\delta_i\}_{i=1}^{2N}$ satisfying

$$\delta := \sup_{1\le i\le 2N} |\tilde{t}_i - t_i| < \frac{1}{2R}.$$

Then the sample errors between $x*h(nT)$ and $\tilde{x}*h(nT), n \ge 1$, are dominated by $(\lfloor MRT \rfloor + 2)\|h\|_\infty \delta$, i.e.,

$$|x*h(nT)-\tilde{x}*h(nT)| \le \big(\lfloor MRT\rfloor+2\big)\|h\|_\infty\delta,\ n\ge 1, \quad (13)$$

where $\|h\|_\infty$ is the L^∞ norm of the sampling kernel h.

Proof: By the assumption on maximal local innovation rate R of the bilevel signal x and the maximal transition position perturbation δ between bilevel signals x and $\tilde{x}$, we have that

$$|x(t) - \tilde{x}(t)| = \sum_{i=1}^{2N} \chi_{t_i+[\min(\delta_i,0),\max(\delta_i,0))}(t).$$

This together with the support assumption for the sampling kernel h gives that

$$\begin{aligned} & |x * h(nT) - \tilde{x} * h(nT)| \\ = & \Big|\int_0^{nT} (x(t) - \tilde{x}(t)) h(kT-t)dt\Big| \\ \le & \|h\|_\infty \int_{(n-M)T}^{nT} \sum_{i=1}^{2N} \chi_{t_i+[\min(\delta_i,0),\max(\delta_i,0))}(t)dt. \end{aligned}$$

Therefore

$$\begin{aligned} & |x * h(nT) - \tilde{x} * h(nT)| \\ \le & \delta\|h\|_\infty \#\{t_i : t_i \in [(n-M)T - \delta, nT + \delta)\} \\ \le & \delta\|h\|_\infty (\lfloor (MT + 2\delta)/(1/R)\rfloor + 1) \\ \le & \delta\|h\|_\infty (\lfloor MRT\rfloor + 2), \end{aligned}$$

where the first inequality holds as $t_i \in [(n-M)T - \delta, nT + \delta)$ if $t_i + [\min(\delta_i, 0), \max(\delta_i, 0))$ and $[(n-M)T, nT)$ have nonempty intersection, the second inequality is true as $t_{i+1} - t_i \ge 1/R$ for all $1 \le i < 2N$, and the last inequality follows from the assumptions that $\delta < 1/(2R)$ and $R \le \tau$. This proves the sampling error estimate (13) between the bilevel causal signals x and $\tilde{x}$. ■

Now we consider the corresponding nonlinear inverse problem how to recover a bilevel signal x from its noisy samples $\{x*h(nT)+\epsilon_n\}$ in (3), where $\epsilon_n, n \ge 1$, are bounded noises. Let us start by looking at two examples.

Example 1: Take $x_1(t) = \sum_{i=1}^{\infty} \chi_{[2i-1,2i)}(t)$ as the original bilevel signal and $h_1(t) = \chi_{[0,2)}(t)$ as the sampling kernel. For sufficiently small $\epsilon > 0$, define $x_{1,\epsilon} = \sum_{i=1}^{\infty} \chi_{[(1+\epsilon)(2i-1),2(1+\epsilon)i)}(t)$. Then for every $i \ge 1$, the i-th transition positions of bilevel signals x_1 and $x_{1,\epsilon}$ are i and $i(1+\epsilon)$ respectively (hence their difference is $i\epsilon$ that could be arbitrary large for sufficiently large i), but on the other hand, maximal sampling errors for those two bilevel signals x_1 and $x_{1,\epsilon}$ are bounded by ϵ,

$$|x_{1,\epsilon} * h_1(n) - x_1 * h_1(n)| = |x_{1,\epsilon} * h_1(n) - 1| \le \epsilon,\ n \ge 1.$$

This leads to instability of the recovery procedure from samples $\{x_1 * h_1(n)\}$ to the bilevel signal x_1 in the presence of bounded noises.

Example 2: Take x_1 and h_1 in Example 1 as the original bilevel signal and the sampling kernel respectively. Define $x_{2,\epsilon} = \sum_{i=1}^{\infty} \chi_{[2i-1+\epsilon,2i+\epsilon)}(t)$ for sufficiently small $\epsilon > 0$. Then for every $i \ge 1$ the difference between i-th transition positions of bilevel signals x_1 and $x_{2,\epsilon}$ is always ϵ, and there is no difference between their n-th samples except for $n = 1$. This suggests that the recovery procedure from samples $\{x_1 * h_1(n)\}$ to the bilevel signal x_1 is not locally-behaved and the reconstruction error on transition positions could disseminate.

From the above two examples, we see that the nonlinear recovery procedure from samples $\{x*h(n)\}$ to bilevel signals x is *unstable* in the presence of bounded noises and that it is *globally-behaved* in general. In this note, we present some initial numerical simulations with small numbers of transition positions, sampling rate over maximal local rate of innovation and very low levels of noise. Detailed noise performance analysis and stable recovery in the presence of other types of noises will be discussed in the coming paper.

Take a sampling kernel $h_0(t) = \frac{t+1}{2}\chi_{[0,1)}(t)+(2t-1)\chi_{[1,2)}$, and a bilevel signal

$$\begin{aligned} x_0(t) \;=\; & \chi_{[0.3791,1.9885)}(t) + \chi_{[3.1306,4.3440)}(t) \\ & +\chi_{[5.7552,7.1820)}(t) + \chi_{[8.7423,10.1052)}(t) \\ & +\chi_{[11.4200,12.6884)}(t) \quad (14) \end{aligned}$$

containing 10 transition positions, see Figure 1. Here transition

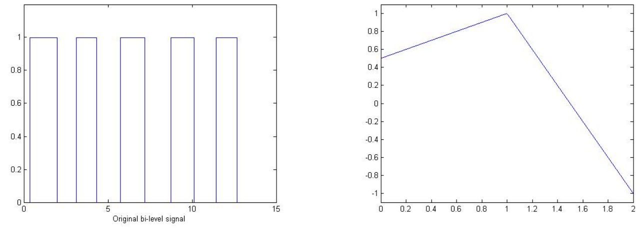

Fig. 1. Bi-level signal x_0 (left) and sampling kernel h_0 (right)

positions $t_i^0, 1 \le i \le 10$, of the bilevel signal x_0 are randomly selected so that $t_i^0 - t_{i-1}^0 \in [1.1, 1.9], 2 \le i \le 10$. The bilevel signal x_0 in (14) has 0.8756 as its maximal local rate of innovation. We sample the convolution $x_0 * h_1$ between x_0 and h_1 every second, which generates the sampling vector $Y_0 = (x_0 * h(1), \ldots, x_0 * h(14))$, and then we add bounded random noise to the sampling vector

$$Y_\delta = Y_0 + \delta(\epsilon_1, \ldots, \epsilon_{14})$$

with noise level $\delta \ge 0$, where $\epsilon_i \in [-1, 1], 1 \le i \le 14$, are random noises. We apply the bilevel signal recovery algorithm in Section II with some technical adjustment when the reconstructed transition position is approximately located at some sampling positions, and denote the first ten transition positions of the reconstructed bilevel signal x_δ by $t_{1,\delta}, \ldots, t_{10,\delta}$. Define maximal error of first ten transition positions by

$$P(\delta) = \max_{1 \le i \le 10} |t_{i,\delta} - t_i^0|.$$

We perform the bilevel signal recovery algorithm in Section II 50 times for every noise level $\delta \in [0, 0.03]$. The maximal value of $P(\delta)$ after performing the algorithm 50 times is plotted in Figure 2 with solid line, while the average value of $P(\delta)$ plotted with dashed line. Notice that $\max_{1 \le n \le 14} |x_0 * h_1(n)| = 0.9796$. Thus the maximal error $P(\delta)$ of transition positions is less than 10% when the noise level $\epsilon = \max_{n \ge 1} |\epsilon_n|$ is at (or below) 2% of the maximal sample value $\max_{n \ge 1} |x_0 * h_0(nT)|$, while some transition positions could not be recovered approximately when the noise level is above 3%. This indicates that our algorithm to recover the bilevel signal from its noisy samples is "reliable" only for low level of bounded noises. We doubt that it is because of the instability of the nonlinear recovery procedure in the presence of bounded noises. We will do the detailed noise performance analysis in the coming paper.

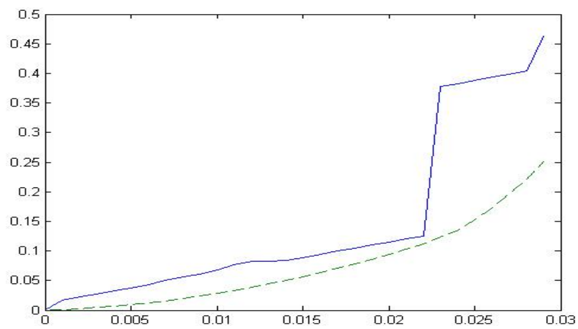

Fig. 2. Maximal transition position error

IV. Conclusion

In this note, we show that bilevel causal signals x could be reconstructed from their samples $x * h(nT), n \ge 1$, if the sampling kernel h is causal and positive on $(0, T)$ and if the sample rate is at (or above) the maximal local rate of innovation of the bilevel signal x. We also propose a stable bilevel signal recovery algorithm in the presence of bounded noise if the number of transition positions of bilevel signals is not large. We remark that the bilevel signal recovery algorithm proposed in this note is applicable when uniform sampling $x*h$ every T second replaced by nonuniform sampling $\{x * h(s_n)\}$ with sampling density $\sup_{n \ge 1} |s_{n+1} - s_n| \le T$, and bilevel causal signal $x = \sum_{i=1}^N \chi_{[t_{2i-1}, t_{2i})}(t)$ with maximal local rate of innovation $R \le 1/T$ replaced by box causal signals $x = \sum_{i=1}^N c_i \chi_{[t_{2i-1}, t_{2i})}(t)$ with maximal local rate of innovation $R \le 1/(2T)$, where for every $1 \le i \le N$, c_i is height of the box located on the time period $[t_{2i-1}, t_{2i})$.

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