FRI-based Sub-Nyquist Sampling and Beamforming in Ultrasound and Radar

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***Abstract*—Signals consisting of short pulses are present in many applications including ultrawideband communication, object detection and navigation (radar, sonar) and medical imaging. The structure of such signals, effectively captured within the finite rate of innovation (FRI) framework, allows for significant reduction in sampling rates, required for perfect reconstruction. In this work we consider two applications, ultrasound imaging and radar, where the FRI signal structure allows to reduce both sampling and processing rates. Furthermore, we show how the FRI framework inspires new processing techniques, such as beamforming in the frequency domain and Doppler focusing. In both applications a pulse of a known shape or a stream of such pulses is transmitted into the respective medium, and the received echoes are sampled and digitally processed in a way referred to as beamforming. Applied either spatially or temporally, beamforming allows to improve signal-to-noise ratio. In radar applications it also allows for target Doppler frequency estimation. Using FRI modeling both for detected and beamformed signals, we are able to reduce sampling rates and to perform digital beamforming directly on the low-rate samples.**

I. Introduction

When sampling an analog signal, we aim to represent it by discrete-time coefficients, while capturing its important features. According to the classic Shannon-Nyquist theorem the minimal sampling rate required for perfect reconstruction of bandlimited signals is twice the the maximal frequency. The required sampling rate can be significantly reduced when additional information about the signal structure is available. An interesting class of structured signals was suggested by Vetterli et al. [1], who considered signals with a finite number of degrees of freedom per unit time - signals with finite rate of innovation (FRI). One of the most studied cases of FRI signals is a stream of pulses, namely, a signal consisting of a stream of short pulses where the pulse shape is known. Such signals are presented in abundance in ultrawideband communication, object detection and navigation (radar, sonar) and medical imaging.

In this work we consider two applications where the FRI signal structure allows to reduce both sampling and processing rates and inspires new processing techniques. In particular, we show how different forms of beamforming, used to improve resolution and increase signal-to-noise-ratio (SNR), can be implemented directly on reduced rate samples. This is achieved by replacing the standard time-domain beamforming by a frequency domain approach and relying on previous FRI sampling techniques in frequency [2]–[4].

The first application is medical ultrasound, where the known pulse shape is transmitted into the tissue and the echoes reflected off scatterers form a stream of pulses signal detected by the elements of the transducer. Signals detected at each element are sampled and digitally processed by beamforming in time, exploiting the array geometry. Such a beamformed signal forms a line in the image. Treating both detected and beamformed signals in the FRI framework and performing beamforming in frequency allows to reduce the sampling rate far below standard rates that are required to improve the system's beamforming resolution.

The second application is radar. Similar to ultrasound, a stream of known pulses is transmitted into space and reflected off any targets. Whereas in ultrasound digital beamforming is performed spatially, i.e. combining a single pulse from different transducers, in the single transceiver radar model we consider beamforming is performed temporally between different pulses on the same transceiver. This beamforming process, besides improving SNR, allows for target Doppler frequency estimation as well. Here again we show how beamforming, and consequently, radar detection, can be performed efficiently at sub-Nyquist rates by using sub-Nyquist sampling methods in frequency [4], [5].

II. Ultrasound

Modern imaging systems use multiple transducer elements to transmit and receive acoustic pulses. The imaging process is described as follows: An energy pulse is transmitted along a narrow beam. During its propagation echoes are scattered by acoustic impedance perturbations in the tissue, and detected by the elements of the transducer. Collected data are sampled and digitally beamformed, resulting in an image line.

Rates up to 4 times the Nyquist rate, dictated by the bandwidth of the individual signal, are required in order to improve the system's beamforming resolution and to avoid artifacts caused by digital implementation. From now on we will denote this sampling rate as the beamforming rate f_s.

To get a sense of the sampling and processing rates involved in ultrasound imaging, we can evaluate the number of samples taken at each transducer element based on the imaging setup used to acquire in vivo cardiac data. The acquisition was performed with a GE breadboard ultrasonic scanner of 64 acquisition channels. The radiated depth $r = 16$ cm and the speed of the sound $c = 1540$ m/sec yield a signal of duration $T = 2r/c \simeq 210$ μsec. The acquired signal is characterized by a narrow bandpass bandwidth of 2 MHz, centered at the carrier frequency $f_0 \approx 3.1$ MHz, leading to a beamforming rate of $f_s \approx 16$ MHz and $Tf_s = 3360$ real-valued samples.

We now show that the number of samples can be reduced significantly since the oversampling dictated by the digital implementation of beamforming in time can be bypassed, when the beamformed signal is treated within the FRI framework and beamforming is performed in the frequency domain.

A. Signal Model

According to [2], [4], the beamformed signal in ultrasound imaging obeys an FRI model:

$$\Phi(t;\theta) \simeq \sum_{l=1}^{L} \tilde{b}_l h(t - t_l), \tag{1}$$

where $h(t)$ is the transmitted pulse-shape, L is the number of scattering elements in direction θ, $\{\tilde{b}_l\}_{l=1}^{L}$ are the unknown amplitudes of the reflections and $\{t_l\}_{l=1}^{L}$ denote the unknown delays. Sampling both sides of (1) at the rate f_s and quantizing the delays $\{t_l\}_{l=1}^{L}$ with quantization step $1/f_s$, such that $t_l = q_l/f_s, q_l \in \mathbb{Z}$, we can rewrite (1) as follows:

$$\Phi[n;\theta] \simeq \sum_{l=1}^{L} \tilde{b}_l h[n - q_l] = \sum_{l=0}^{N-1} b_l h[n - l], \tag{2}$$

where

$$b_l = \begin{cases} \tilde{b}_l & \text{if } l = q_l \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Calculating the Discrete Fourier Transform (DFT) using (2):

$$c_k = \sum_{n=0}^{N-1} \Phi[n;\theta] e^{-i\frac{2\pi}{N}kn} = h_k \sum_{l=0}^{N-1} b_l e^{-i\frac{2\pi}{N}kl}, \tag{4}$$

where h_k is the DFT coefficient of $h[n]$. The transmitted pulse $h(t)$ is a narrowband baseband waveform, $g(t)$, modulated by a carrier at frequency f_0. When such a pulse is sampled at rate f_s, most of its DFT coefficients are zero. Obviously, (4) implies that the only non-zero DFT coefficients are in the bandwidth of the transmitted pulse. When a set κ of these non-zero DFT coefficients is known we can reconstruct the signal perfectly by zero-padding and then performing an inverse DFT (IDFT). In the cardiac imaging setup described above the bandwidth of $g(t)$ is equal to 2 MHz, the modulation frequency $f_0 = 3.1$ MHz, and the sampling rate $f_s = 16$ MHz, leading to $K = |\kappa| \approx 360$.

As we show further in Section II-C, sampling rates are proportional to the number of DFT coefficients of the beamformed signal that we want to calculate. Hence, to reduce the sampling rates, we aim to obtain only a subset $\mu \subset \kappa$, $|\mu| = M < K = |\kappa|$, of non-zero DFT coefficients of the beamformed signal and propose a method to reconstruct κ from its subset μ.

B. Beamformed Signal Reconstruction

Defining a K-length vector $\mathbf{c}$ with k-th entry $c_k/h_k, k \in \kappa$, we can rewrite (4) in matrix form:

$$\mathbf{c} = \mathbf{D}\mathbf{b}, \tag{5}$$

where $\mathbf{D}$ is a $K \times N$ matrix formed by taking the set κ of rows from an $N \times N$ DFT matrix, and vector $\mathbf{b}$ is of length N with l-th entry b_l. Since from now on only subset μ is given, define an M-length vector $\mathbf{c}_\mu$ with k-th entry $c_k/h_k, k \in \mu$ and rewrite (5) as follows:

$$\mathbf{c}_\mu = \mathbf{A}\mathbf{D}\mathbf{b}, \tag{6}$$

where $\mathbf{A}$ is an $M \times K$ measurement matrix which picks the subset μ of rows from $\mathbf{D}$.

We propose an analysis approach [6], namely, we aim to reconstruct the set κ from its subset μ, while assuming that the analyzed vector $\mathbf{D}^*\mathbf{c}$ is compressible. This assumption is justified as follows: A typical beamformed ultrasound signal is comprised of a relatively small number of strong reflections and a bunch of much weaker scattered echoes. It is, therefore, natural to assume that $\mathbf{b}$ from (5) is compressible, implying that $\mathbf{c}$ has a compressible expansion in $\mathbf{D}$. Since $\mathbf{D}$ is a partial DFT matrix, its Gram matrix is nearly diagonal, implying that $\mathbf{D}^*\mathbf{c}$ is also compressible [6]. The analysis approach can be translated into the l_1 optimization problem:

$$\min_{\mathbf{c}} \|\mathbf{D}^*\mathbf{c}\|_1 \quad \text{subject to} \quad \|\mathbf{A}\mathbf{c} - \mathbf{c}_\mu\|_2 \leq \varepsilon. \tag{7}$$

Under certain conditions which are satisfied in our ultrasound setup [6], [7] the solution of (7) yields the set $\tilde{\kappa}$ of non-zero DFT coefficients of the beamformed signal which is sufficiently close to the true values of κ.

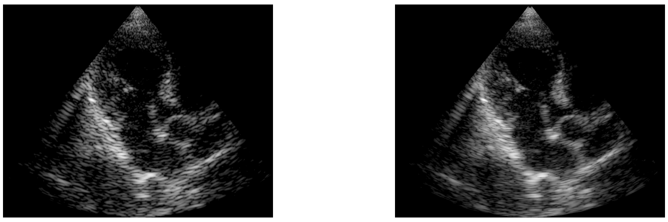
C. Sampling Scheme and Beamforming in Frequency

We now address the following question: how many samples of the individual signals should be taken in order to compute the subset μ of non-zero DFT coefficients of the beamformed signal?

To answer this question we introduce a recently developed technique, referred to as beamforming in frequency. This method was proposed in [4] and [7], where it was shown that beamforming can be performed directly in the frequency domain, namely, a set μ of the DFT coefficients of the beamformed signal can be calculated as a linear combination of a set ν of the DFT coefficients of each individual signal. Experimental results show that $|\nu| \approx |\mu|$, implying that we can calculate the desired set of beamformed DFT coefficients from a small number of DFT coefficients of each individual signal.

As it was shown in [2], [4], [7], a set ν of the DFT coefficients of each individual signal can be obtained by the sub-Nyquist Xampling ("compressed sampling") [8] method, an analog-to-digital conversion (ADC) which performs analog prefiltering of the signal before taking low-rate point-wise samples. The number of samples taken from the individual signal in this case is $|\nu| \approx |\mu|$.

To demonstrate the proposed method and evaluate the rate reduction, a subset μ of 100 DFT coefficients corresponding to the central frequency samples in the bandwidth of the transmitted pulse were chosen. To calculate μ we need approximately $|\mu| = 100$ samples per individual signal, implying 30 fold reduction in sampling rate. The result is shown in Fig. 1 (a). We compare it with an image created by a standard technique using 3360 samples per individual signal in Fig. 1 (b). As can be seen, we obtain sufficient image quality with more than 30 fold reduction in sampling rate.



(a) (b)

Fig. 1: Cardiac images. (a) Proposed method, 100 samples per image line. (b) Standard method, 3360 samples per image line.

III. RADAR

We next consider target detection and feature extraction in a single transceiver, monostatic, narrow-band pulse-train radar system. We show that both sampling and processing can be performed at sub-Nyquist rates, when an appropriate signal model is used. Targets are non-fluctuating point targets, sparsely populated in the radar's unambiguous time-frequency region: delays up to the Pulse Repetition Interval (PRI) and Doppler frequencies up to its reciprocal the Pulse Repetition Frequency (PRF). We propose a recovery method which can detect and estimate targets' delay and Doppler, using a linear, non-adaptive sampling technique at a rate significantly lower than the radar signal's Nyquist frequency, assuming the number of targets L is small.

Current state-of-the-art radar systems sample at the signal's Nyquist rate, which can be hundreds of MHz and higher. Similarly to the ultrasound application, the goal of our approach, breaking the link between the signal bandwidth and sampling rate, is achieved by using FRI signal model and the Xampling method. The latter yields compressed samples ("Xamples"), containing the information needed to recover the desired signal parameters. This work expands [5], adding Doppler to the target model and proposing a new digital recovery method to estimate it by relying on beamforming ideas operating on sub-Nyquist samples, as we showed in the context of ultrasound imaging.

A. Signal Model

We consider a radar transceiver that transmits a pulse train

$$x_T(t) = \sum_{p=0}^{P-1} h(t - p\tau), \qquad 0 \leq t \leq P\tau \tag{8}$$

consisting of P equally spaced pulses $h(t)$. The pulse-to-pulse delay τ is referred to as the PRI. The pulse $h(t)$ is a known time-limited baseband function with continuous-time Fourier transform (CTFT) $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$. We assume that $H(\omega)$ has negligible energy at frequencies beyond $B_h/2$ and we refer to B_h as the bandwidth of $h(t)$. The target scene is composed of L non-fluctuating point targets, where we assume that L is known, although this assumption can easily be relaxed. The pulses reflect off the L targets and propagate back to the transceiver. Each target l is defined by three parameters: a delay τ_l, a Doppler frequency ν_l and a complex amplitude α_l, proportional to the target's radar cross section (RCS) and all propagation factors.

Under several assumptions [9], we can write the received signal as

$$x(t) = \sum_{p=0}^{P-1} \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l - p\tau)e^{-j\nu_l p\tau}. \tag{9}$$

It will be convenient to express the signal as a sum of single frames

$$x(t) = \sum_{p=0}^{P-1} x_p(t), \tag{10}$$

where

$$x_p(t) = \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l - p\tau)e^{-j\nu_l p\tau}. \tag{11}$$

It is evident from (9) that we are dealing with an FRI signal, since it can be described by $3L$ parameters spanning an interval of duration $P\tau$, yielding a rate of innovation of $3L/P\tau$. Our goal is to accurately detect the L targets, i.e. to estimate the $3L$ parameters $\{\alpha_l, \tau_l, \nu_l\}_{l=0}^{L-1}$ in (9), using the least possible number of digital samples.

B. Doppler Focusing

The Doppler Focusing processing technique uses target echoes from different pulses to create a single superimposed pulse, improving SNR for robustness against noise and implicitly estimating targets' Doppler in the process. Using (11), we define the following time shift and modulation operation on the received signal:

$$\begin{aligned}\Phi(t;\nu) &= \sum_{p=0}^{P-1} x_p(t + p\tau)e^{j\nu p\tau} \\ &= \sum_{p=0}^{P-1} \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l)e^{j(\nu-\nu_l)p\tau} \\ &= \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l) \sum_{p=0}^{P-1} e^{j(\nu-\nu_l)p\tau}.\end{aligned} \tag{12}$$

We now analyze the sum of exponents in (12). For any given ν, targets with Doppler frequency ν_l in a band of width $2\pi/P\tau$ around ν, i.e. in $\Phi(t;\nu)'s$ "focus zone", will achieve coherent integration and an SNR boost of approximately

$$g(\nu|\nu_l) = \sum_{p=0}^{P-1} e^{j(\nu-\nu_l)p\tau} \overset{|\nu-\nu_l|<2\pi/P\tau}{\cong} P \tag{13}$$

compared with a single pulse. On the other hand, since the sum of P equally spaced points covering the unit circle is generally close to zero, targets with ν_l not "in focus" will approximately cancel out. Thus $g(\nu|\nu_l) \cong 0$ for $|\nu - \nu_l| > 2\pi/P\tau$. Hence we can approximate (12) by

$$\Phi(t;\nu) \cong P \sum_{l:|\nu-\nu_l|<2\pi/P\tau} \alpha_l h(t-\tau_l). \tag{14}$$

Instead of trying to estimate delay and Doppler together, we have reduced our problem to delay only estimation for a small range of Doppler frequencies, with increased amplitude for improved performance against noise.

C. Delay-Doppler Recovery Using Doppler Focusing

Calculating the DFT of each of the pulses $x_p(t)$ of the multi-pulse signal (9), and since $x_p(t)$ is confined to the interval $t \in [p\tau, (p+1)\tau]$, we obtain

$$c_p[k] = \frac{1}{\tau} H(2\pi k/\tau) \sum_{l=0}^{L-1} \alpha_l e^{-j\nu_l p\tau} e^{-j2\pi k\tau_l/\tau}, \tag{15}$$

where we used the fact that since both $k, p \in \mathbb{Z}$ we have $e^{-j2\pi kp} \equiv 1$. From (15) we see that all $3L$ unknown parameters $\{\alpha_l, \tau_l, \nu_l\}_{l=0}^{L-1}$ are embodied in the Fourier coefficients $c_p[k]$ in the form of a complex sinusoid problem.

Having acquired $c_p[k]$ using a framework similar to one introduced in section II-C, we now perform the Doppler focusing operation for a specific frequency ν

$$\begin{aligned}\Psi_\nu[k] &= \sum_{p=0}^{P-1} c_p[k] e^{j\nu p\tau} \\ &= \frac{1}{\tau} H(2\pi k/\tau) \sum_{l=0}^{L-1} \alpha_l e^{-j2\pi k\tau_l/\tau} \sum_{p=0}^{P-1} e^{j(\nu-\nu_l)p\tau}.\end{aligned} \tag{16}$$

Following the same arguments as in (13), for any target l satisfying $|\nu - \nu_l| < 2\pi/P\tau$ we have

$$\sum_{p=0}^{P-1} e^{j(\nu-\nu_l)p\tau} \cong P. \tag{17}$$

Therefore, Doppler focusing can be performed on the low rate sub-Nyquist samples:

$$\Psi_\nu[k] \cong \frac{P}{\tau} H(2\pi k/\tau) \sum_{l:|\nu-\nu_l|<2\pi/P\tau} \alpha_l e^{-j2\pi k\tau_l/\tau}. \tag{18}$$

Equation (18) is scaled by P compared with a single pulse, increasing SNR for improved performance with noise. Furthermore, we reduced the number of active delays. For each ν we now have a delay estimation problem, which can be written in vector form as

$$\mathbf{\Psi}_\nu = \frac{P}{\tau} \mathbf{H}\mathbf{V}\mathbf{x}_\nu, \tag{19}$$

where

$$\mathbf{\Psi}_\nu = [\Psi_\nu[k_0] \; ... \; \Psi_\nu[k_{|\kappa|-1}]]^T \in \mathbb{C}^{|\kappa|}. \tag{20}$$

This is a CS problem which has already been solved [3], [9], [10]. We emphasize that the Doppler focusing technique is a continuous operation on ν, and can be performed for any Doppler frequency. Since the focus zone for each ν is of width $2\pi/P\tau$, we can find various finite sets of ν's spanning $[0, 2\pi/\tau]$. For any such set, define its size as N_ν. For each ν in the set, we solve (19) assuming $\mathbf{x}_\nu$'s support is of size L. This problem can be solved using an abundance of CS algorithms [11]–[13]. After solving N_ν separate CS problems with dictionary of size $|\kappa| \times N_\tau$, we hold at most LN_ν estimated amplitudes. Since the absolute value of amplitudes recovered in the support is indicative of true target existence as opposed to noise, we take the L strongest ones as true target locations.

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