BUILDING RIESZ BASES WITH THE AID OF LOW-PASS FILTERS

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ABSTRACT

We present a new method for the generation of Riesz bases with the aid of low-pass filters. The Riesz bases obtained can be used for the implementation of sampling systems for non-bandlimited signals.
Keywords: *Frames, Riesz bases, ideal sampling*.

1. INTRODUCTION

In linear algebra, a *frame of a vector space V* with an inner product can be seen as a generalization of a basis to sets whose elements may be linearly dependent. The key issue related to the generation of a frame appears when we have a sequence of vectors $\{\mathbf{e}_k\}$, with each $\mathbf{e}_k \in V$ and we want to express an arbitrary element **v** as a linear combination of the vectors $\{\mathbf{e}_k\}$: $\mathbf{v}=\sum_k c_k \mathbf{e}_k$, and to determine the coefficients c_k. If the set does not span V, then these coefficients cannot be determined for all such **v**. If $\{\mathbf{e}_k\}$, spans V and also is linearly independent, this set forms a basis of V, and the coefficients c_k are uniquely determined by **v**: they are the coordinates of **v** relative to this basis. If, however, $\{\mathbf{e}_k\}$ spans V but is not linearly independent, the question of how to determine the coefficients becomes less apparent, in particular if V is of infinite dimension.

Given that $\{\mathbf{e}_k\}$ spans V and is linearly dependent, it may appear obvious that we should remove vectors from the set until it becomes linearly independent and forms a basis. There are some problems with this strategy:

1. By removing vectors randomly from the set, it may lose its possibility to span V before it becomes linearly independent.
2. Even if it is possible to devise a specific way to remove vectors from the set until it becomes a basis, this approach may become infeasible in practice if the set is large or infinite.
3. In some applications, it may be an advantage to use more vectors than necessary to represent **v**. This means that we want to find the coefficients c_k without removing elements in $\{\mathbf{e}_k\}$.

In 1952, Duffin and Schaeffer [1] gave a solution to this problem, by describing a condition on the set $\{\mathbf{e}_k\}$ that makes it possible to compute the coefficients c_k in a simple way. More precisely, a *frame* is a set of elements of V which satisfy the so-called *frame condition:*

There exist two real numbers, A and B such that

$$0 < A \le B < \infty \quad (1)$$

and

$$A\|\mathbf{v}\|^2 \le \sum_k \left|\langle \mathbf{v},\mathbf{e}_k \rangle\right|^2 \le B\|\mathbf{v}\|^2 \quad (2)$$

for all $\mathbf{v}\in V$.

The numbers A and B are called lower and upper *frame bounds*.

The sequence of vectors $\{\mathbf{e}_k\}$ is named *Riesz sequence* of the Hilbert space H if there are two real numbers A and B such that condition (1) and the following condition:

$$A\sum_k |a_k|^2 \le \left\|\sum_k a_k \mathbf{e}_k\right\|^2 \le B\sum_k |a_k|^2 \quad (2')$$

are satisfied for all sequences of scalars $\{a_k\}$ in the space l^2. A Riesz sequence is called Riesz basis if:

$$\overline{\mathrm{span}\{\mathbf{e}_k\}} = H .$$

Hence a Riesz basis is a frame of vectors that are linearly independent [2]. This concept is very important for the construction of multiresolution analyses [2].

In the following we will construct Riesz bases of the form:

$$\mathbf{e}_k(t)=\varphi(t-k) \text{ with } k \in Z \text{ and } \varphi(t) \in L^2(R), \quad (3)$$

for Hilbert spaces included in the space of finite energy signals. An equivalent form for (2') in the spectral domain is [2]:

$$A' = B^{-1} \le \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 \le B' = A^{-1}. \quad (4)$$

The series in the middle represents the spectrum of the digital signal obtained by ideal sampling with unitary step the autocorrelation of the signal $\varphi(t)$, $R_{\varphi\varphi}(t)$. This spectrum will be denoted by ${}_s\hat{R}_{\varphi\varphi}(\omega)$ and is a periodic function with period 2π as a Fourier transform in discrete time of the signal $R_{\varphi\varphi}[n]$. So, the frame bounds can be identified with the following equations:

$$A' = \min_{\omega}\left\{ {}_s\hat{R}_{\varphi\varphi}(\omega)\right\} \text{and} \quad B' = \max_{\omega}\left\{ {}_s\hat{R}_{\varphi\varphi}(\omega)\right\}. \qquad (5)$$

We will prove in the following that the function $\varphi(t)$ can be the impulse response of a low-pass filter with arbitrary order.

2. FIRST EXAMPLE

A first order low-pass filter has the impulse response:

$$_1\varphi(t) = (A_0/\tau)e^{-t/\tau}\sigma(t). \qquad (6)$$

Its autocorrelation is:

$$R_{1\varphi 1\varphi}(t) = \left(A_0^2/2\tau\right)e^{-|t|/\tau}. \qquad (7)$$

By ideal sampling it with unitary step we obtain the discrete time signal with the spectrum:

$${}_s\hat{R}_{1\varphi 1\varphi}(\omega) = \frac{(A_0^2/2\tau)(1-e^{-2/\tau})}{(1-2e^{-1/\tau}\cos\omega + e^{-2/\tau})}. \qquad (8)$$

Hence the values of the frame bounds are in this example:

$$A'_1 = {}_s\hat{R}_{1\varphi_1\varphi}(\pi) = \frac{A_0^2}{2\tau}\cdot\frac{1\text{-}e^{-1/\tau}}{1+e^{-1/\tau}} \quad \text{and } B'_1 = {}_s\hat{R}_{1\varphi_1\varphi}(0) = \frac{A_0^2}{2\tau}\cdot\frac{1+e^{-1/\tau}}{1-e^{-1/\tau}}. \qquad (9)$$

3. SECOND EXAMPLE

The transfer function of a second order low-pass filter is:

$$_2\hat{\varphi}(s) = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}. \qquad (10)$$

Let us consider that this system has real distinct poles ($\xi > 1$):

$$s_{1,2} = -\omega_n\left(\xi \mp \sqrt{\xi^2-1}\right). \qquad (11)$$

The expression of the impulse response of the filter is:

$$_2\varphi(t) = \frac{1}{2\omega_n\sqrt{\xi^2-1}}\left(e^{s_1 t} - e^{s_2 t}\right)\sigma(t). \qquad (12)$$

Its autocorrelation is given by equation (13). The spectrum of the signal obtained by ideal sampling this autocorrelation with unitary step is expressed in equation (14). The values of the frame bounds in this second example are computed in equation (15).

Similar considerations can be made for second order low-pass filters with real double poles or with complex conjugate poles.

$$R_{2\varphi_2\varphi}(t) = \frac{2\xi - \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi - \sqrt{\xi^2-1}\right)}\cdot e^{s_1|t|} + \frac{2\xi + \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi + \sqrt{\xi^2-1}\right)} e^{s_2|t|}. \qquad (13)$$

$${}_s\hat{R}_{2\varphi_2\varphi}(\omega) = \frac{2\xi - \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi - \sqrt{\xi^2-1}\right)}\cdot\frac{1-e^{2s_1}}{1-2e^{s_1}\cos\omega + e^{2s_1}} + \frac{2\xi + \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi + \sqrt{\xi^2-1}\right)}\cdot\frac{1-e^{2s_2}}{1-2e^{s_2}\cos\omega + e^{2s_2}}. \qquad (14)$$

$$A'_2 = {}_s\hat{R}_{2\varphi_2\varphi}(\pi) = \frac{2\xi - \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi - \sqrt{\xi^2-1}\right)}\cdot\frac{1\text{-}e^{s_1}}{1+e^{s_1}} + \frac{2\xi + \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi + \sqrt{\xi^2-1}\right)}\cdot\frac{1\text{-}e^{s_2}}{1+e^{s_2}}$$
$$\text{and } B'_2 = {}_s\hat{R}_{2\varphi_2\varphi}(0) = \frac{2\xi - \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi - \sqrt{\xi^2-1}\right)}\cdot\frac{1+e^{s_1}}{1-e^{s_1}} + \frac{2\xi + \sqrt{\xi^2-1}}{8\omega_n^3\xi\left(\xi^2-1\right)\left(\xi + \sqrt{\xi^2-1}\right)}\cdot\frac{1+e^{s_2}}{1-e^{s_2}}. \qquad (15)$$

4. MAIN RESULT

The two examples already presented can be unified in the framework of the following:

Remark. *Each continuous in time low-pass filter impulse response generates a Riesz basis which corresponds to a Hilbert subspace of the space of finite energy signals.*

Proof.

The single limitation of the algorithm for the construction of Riesz bases proposed comes from equation (1), $A > 0$. The sum of the series $\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega+2k\pi)|^2$ is positive, because its terms are positive, but it must be strictly positive. This condition is satisfied by each continuous in time low-pass filter impulse response because the corresponding frequency responses does not have transmission zeros. Indeed the transfer function of an N^{th} order low-pass filter is of the form:

$$ {}_N\hat{\varphi}(s)=\frac{1}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + .. a_N} \quad (16)$$

These Riesz bases can be transformed into orthogonal bases with the aid of the well known procedure described in equations (17) and (18), [2]. Denoting by:

$$\left|{}_k\hat{m}(\omega)\right|^2 = \sum_{l=-\infty}^{\infty} \left|{}_k\hat{\varphi}(\omega+2l\pi)\right|^2 = {}_s\hat{R}_{{}_k\varphi_k\varphi}(\omega) \quad (17)$$

we can compute the Fourier transform:

$$ {}_k\hat{g}(\omega) = \frac{{}_k\hat{\varphi}(\omega)}{{}_k\hat{m}(\omega)} \quad (18)$$

of the function ${}_k g(t)$ which generates by translation with integers the corresponding orthonormal basis. For our first example:

$$ {}_1\hat{g}(\omega) = [\alpha/(1+j\omega\tau)](1-e^{-1/\tau}e^{-j\omega}) \quad (19)$$

with:

$$\alpha = \sqrt{2/\left[\tau\left(1-e^{-2/\tau}\right)\right]}. \quad (20)$$

So:

$$ {}_1 g(t) = \alpha[e^{-t/\tau}(\sigma(t)-\sigma(t-1))] \quad (21)$$

5. AN APPLICATION

The reduction of the aliasing is a difficult problem encountered in the design of sampling systems. The WKS sampling theorem supposes the limitation of the bandwidth of the signal which must be sampled. Unfortunately, there are situations when the bandwidth of the input signal it is not priory known and the aliasing could appear. For input signals band limited at π, the system which implements the WKS theorem is presented in Fig. 1, [3]. The proof of this theorem is based on the fact that the set $\{\mathrm{sinc}(\pi(t-k))\}_{k\in Z}$ represents an orthonormal basis of the Hilbert space of signals with finite energy and band limited at π, denoted by B_π^2. In figure 2 is presented a sampling system for non-bandlimited signals.

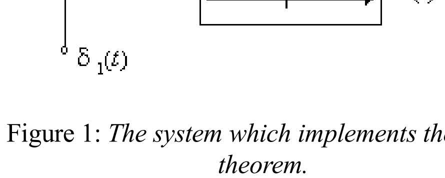

Figure 1: *The system which implements the WKS theorem.*

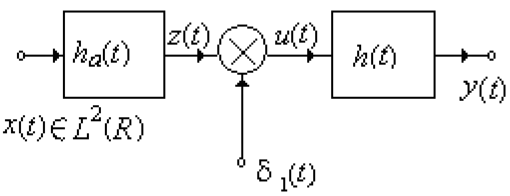

Figure 2: *The structure of a system for the sampling of non bandlimited signals.*

The input system is an antialiasing filter and the output system is a reconstruction filter. The signal $z(t)$ must be bandlimited having the maximal pulsation in spectrum equal with π. The best mean square approximation of the input signal with bandlimited signals is obtained by projecting the input signal on the space B_π^2. The antialiasing filter which projects the input singnal on this space has the impulse response $h_a(t)=h^v(t)=h(-t)$. Neither the reconstruction system with the impulse response $h(t)$, nor the antialiasing filter with the impulse response $h^v(t)$ are not causal. The identity system proposed in [4] is presented in Fig. 3, where g could be the function defined in (18). The WKS sampling theorem was generalized in [4] for non bandlimited input signals:

P1. *If $\{g(t-k)\}_{k\in Z}$ represents a Riesz basis of a closed Hilbert subspace of the finite energy signals V_0, then any signal from V_0 can be perfectly reconstructed using the system in figure 3.*

The proof can be found in [4]. The output signal $y(t)$ represents the decomposition of the input signal $x(t)$ in the Riesz basis of V_0, $\{g(t-k)\}_{k\in Z}$. The elements of the space V_0 could be non bandlimited signals. These signals can be perfectly reconstructed from samples with the aid of the system in Fig. 3. The aliasing phenomenon is avoided. The filter with impulse response $g^v(t)$ will be named in the following antialiasing filter, taking into consideration the similarity of figures 2 and 3. If the finite energy signal $x(t)$ does not belong to the space V_0, then it will be only approximated by its projection on V_0, $y(t)$, which represents its best mean square approximation. The mean square approximation error is given by the difference of the energies of the input and output signals. If $g(t)$ represents the impulse response of a causal system then $g^v(t)=g(-t)$ is the impulse response of an anti causal system. So, it is very important that $g(t)$ to have compact support. This is the case for the function ${}_1g(t)$ derived from our first example. If $g(t)$ has compact support then the system in figure 4, very similar with the system in figure 3, can be used for the treatment of non bandlimited signals, elements of V_0.

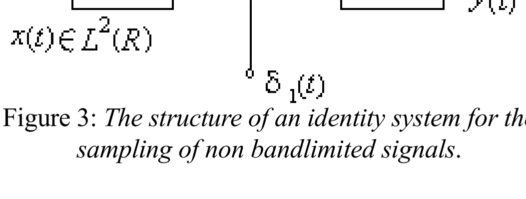

Figure 3: *The structure of an identity system for the sampling of non bandlimited signals.*

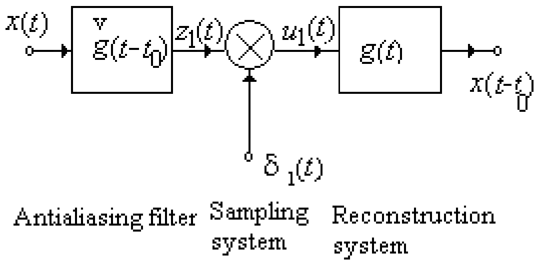

Figure 4: *A causal identity system.*

Any signal $x(t)$ from V_0 can be perfectly reconstructed using the system in Fig. 4, if the delay t_0 is neglected. If this delay is longer than the duration of the impulse response $g(t)$, then the filters which compose the system in Fig. 4 are causal. New sampling theorems, corresponding to different Hilbert spaces V_0 can be formulated using different functions g. The corresponding antialiasing filters could be obtained. For our first example, the expression of the function ${}_1g(t)$ is given in equation (21), the value of the delay is $t_0 = 1$, the reconstruction filter can be implemented with the system in Fig. 5 and the antialiasing filter with the system in Fig. 6 [4]. All the sub-systems in figures 5 and 6 are realisable. An excellent paper on the same subject is [5]. Its section 5 entitled "Regular Sampling and Frames" treats the reconstruction of signals from their samples. The analysis reported in this section refers only at bandlimited signals and the causality condition of the reconstruction filters is not imposed.

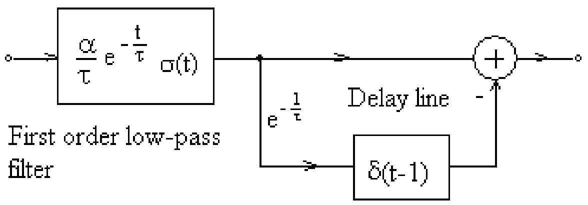

Figure 5: *The architecture of the reconstruction filter in example 1.*

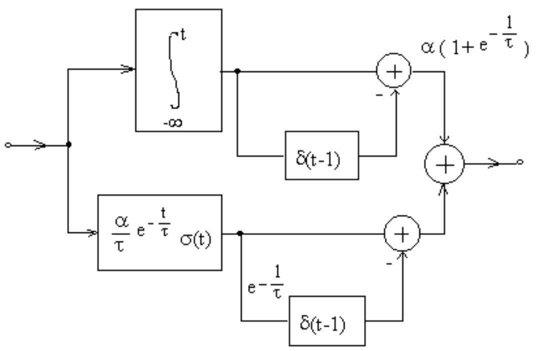

Figure 6: A*ntialiasing filter-example 1.*

6. CONCLUSION

We propose a new algorithm for the generation of Riesz bases of some closed Hilbert sub-spaces of $L^2(R)$ which contain non-bandlimited signals. The proposed algorithm starts from the expression of the impulse response of a continuous in time low-pass filter. It can be applied in sampling theory or for the generation of new multiresolution analyses of the space of finite energy signals.

This paper represents a continuation of the last chapter of our book [4] where we have proved proposition P1. In our knowledge, the content of the second example and the Remark from the begining of section 4 are original. This remark is quite general because it refers to low-pass filters of any order. We have also presented the procedure required for the transformation of those Riesz bases into orthonormal bases. It seems, analyzing the first example, that the elements of those orthonormal bases have compact support, because the case of higher order low-pass filters with real distinct poles can be reduced to the case of few cascaded first order low-pass filters. We will investigate in the future this property. Another research direction is to establish some connections with the mother wavelets families already known, via the corresponding scaling functions.

We have generalized the WKS sampling thorem for the case of some classes of non bandlimited signals. The identity systems proposed have the generic architecture presented in Fig. 4. They are composed by realizable sub-systems. The corresponding classes of non bandlimited signals can appear in practice. The signals belonging to the Hilbert space ${}_1V_0$, which correspond to our first example, are produced at the output of a differentiator realized with a capacitor and a resistor when at its input is connected a train of rectangular impulses. It is dificult to sample such signals due to their discontinuities.

7. ACKNOWLEDGEMENT

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