# NOVEL SEARCH METHOD FOR STRUCTURED CODE-MATCHED INTERLEAVERS

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## ABSTRACT

*This paper presents a new structured code-matched interleaver selection algorithm based on the distance spectrum of the specific turbo code. The proposed method is described in depth and applied to several high performance interleavers (DRP, ARP and QPP) using both dual and post-interleaver trellis termination strategies.*

**Keywords:** *turbo codes, interleaved codes, Hamming distance, Hamming weight*

## 1. INTRODUCTION

Turbo codes provide remarkable performance at low signal to noise ratio scenarios, typical to those existing in wireless communications. This feature can be explained through the spectral thinning phenomenon, which appears due to the degree of randomness inserted by the interleaver [1]. However, turbo codes display a flare at high signal to noise ratios. This error floor limitation is caused by several factors including low free distance of the code words, improper trellis termination of the constituent convolutional encoders, poor interleaver design, sub-optimal iterative decoding, imperfect early stopping criteria and sub-block overlap.

This paper presents a method of constructing good turbo codes with low error flares, through maximizing the free distance of the turbo codes. This increase of the free distance is a result of the fact that not only both convolutional encoders are terminated, but also the selected interleavers are the best code-matched interleavers in terms of the first line of the distance spectrum.

At a high signal to noise ratio, the BER (Bit Error Rate) and FER (Free Error Rate) of a turbo code with BPSK (Binary Phase Shift Keying) modulation in an AWGN (Additive White Gaussian Noise) channel can asymptotically be expressed as:

$$BER\_AWGN \to \frac{1}{2}\frac{w_{free}}{L} erfc\left(\sqrt{d_{free} * R_c * SNR}\right) \quad (1)$$

$$FER\_AWGN \to \frac{1}{2} n_{free} erfc\left(\sqrt{d_{free} * R_c * SNR}\right) \quad (2)$$

In the above relations, L is the length of the interleaver, $d_{free}$ is the free distance, $n_{free}$ is the total number of code words that have a Hamming distance equal to $d_{free}$, $w_{free}$ is the total weight of the $n_{free}$ information sequences that generate code words of weight equal to $d_{free}$, SNR is the signal to noise ratio, $R_c$ is the code rate and erfc (.) is the error function complement.

At a high signal to noise ratio, the BER and FER of a turbo code with BPSK modulation in an RMF (Rayleigh Multiplicative Fading) channel can asymptotically be expressed as:

$$BER\_RMF \to \frac{1}{2}\frac{w_{free}}{L}\left(1+R_c * SNR\right)^{-d_{free}} \quad (3)$$

$$FER\_AWGN \to \frac{1}{2} n_{\ free}\left(1+R_c * SNR\right)^{-d_{free}} \quad (4)$$

From equations (1-4) it can be concluded that low error rates are obtained by turbo codes with high free distances and low free multiplicities, not only in case of the AWGN, but also in case of the RMF.

This paper presents a new search algorithm suitable for structured code-matched interleavers. The proposed method takes into account not only the trellis termination, but also some important parameters such as dispersion and spread.

## 2. TRELLIS TERMINATION

This section introduces some popular trellis termination strategies. Trellis termination of the constituent convolutional encoders is an important issue, which influences the decoding performance of the turbo code near to the end of a data sequence. Furthermore, the performance degradation due to the terminated trellis depends also on the type of the interleaver used, phenomenon known as interleaver edge effects. These edge effects lead to low weight code words, in cases in which at least one trellis is truncated to an unknown state. This inconvenient can be overcome by using interleavers with a large corner of merit [2].

There are several trellis termination methods such as truncation, termination only of the first constituent encoder, termination of both encoders by imposing restrictions on the interleaver, post-interleaver termination [3], dual termination [4] and tailbiting.

Truncation leaves both trellises terminated in unknown states, thus leading to the worst performance. Termination of the first encoder is based on appending m tail bits (m is the memory of the convolutional encoder) to the information sequence that force the first encoder in the zero state. Both encoders can be terminated using this method, provided that the interleaver $\pi$ respects the condition:

$$\pi(i) = i \mod (2^m - 1) \quad (5)$$

Tailbiting termination modifies both the encoder and the decoder, so that both encoders begin and end in the same state. This is done by encoding the data sequence in order to determine the circular state and then re-encoding the same data

sequence, which substantially increases the complexity of the turbo code.
Post-interleaver trellis termination ensures that both encoders are independently forced to the zero state. After the data sequence, for each encoder m flush bits and m parity bits are computed. The tail bits are not interleaved, thus low weight code words can be generated if a weight one sequence with the one bit near the end is mapped by the interleaver into a sequence with the one bit also near the end. Because, usually the m parity bits of the m tail bits of the second encoder are not transmitted, the code rate is:

$$R_{c,post} = \frac{L}{3(L+m)} \quad (6)$$

The dual termination determines specific interleaver dependent positions in which 2*m tail bits can be inserted, so that both encoders are independently forced to the zero state. In this case the tail bits are interleaved, so there are no interleaver edge effects. This termination method leads to a higher free distance then the post-interleaver termination method, but the code rate is lower than in the previous situation. The code rate in this case is:

$$R_{c,dual} = \frac{L-2m}{3L} \quad (7)$$

## 3. STRUCTURED CODE - MATCHED INTERLEAVERS

This paragraph describes two important interleaver parameters, namely dispersion and spread. Furthermore, three structured code-matched interleavers are introduced. Interleavers play an essential role in the performance of turbo codes. There are two important interleaver parameters dispersion and spread. The normalized dispersion is defined as:

$$\gamma = \frac{cardinality\ of\ \{j-i, \pi(j)-\pi(i)\}}{L(L-1)/2} \in (0,1]; i<j \quad (8)$$

The higher the dispersion, the higher is the degree of randomness of the interleaver, which leads to lower multiplicities in the distance spectrum. There are several definitions of the spread of the interleaver, but the most restrictive one is:

$$D = \min_{i,j}\left(|\pi(i)-\pi(j)|_L + |i-j|_L\right) \in (0, \sqrt{2L}]; i \neq j \quad (9)$$

Where:

$$|x-y|_L = \min_{x \neq y}\left((x-y) \bmod L, (y-x) \bmod L\right) \quad (10)$$

The higher the spread, the higher is the free distance of the turbo code.
The best interleavers in terms of performance are the structured code - matched ones. Because of their deterministic generation manner, their dispersion is low, but their structure can provide a very high spread and free distance. There are three important structured code – matched interleavers: DRP (Dithered Relative Prime) [5], ARP (Almost Regular Permutation) [6] and QPP (Quadratic Permutation Polynomial) [2].
The DRP has three design stages. The first stage consists of permuting the input vector using a small read dither vector r of size R. In the second stage scrambles the permutation obtained in the first stage using two constants s and p, where p and L are relative primes. In the final stage, the obtained vector is again permuted using a small write dither vector w of size W. The relations that describe the DRP interleaver are:

$$\pi_a(i) = R\lfloor i/R \rfloor + r(i \bmod R) \quad (11)$$

$$\pi_b(i) = (s+ip) \bmod L \quad (12)$$

$$\pi_c(i) = W\lfloor i/W \rfloor + w(i \bmod W) \quad (13)$$

$$\pi(i) = \pi_a\left(\pi_b\left(\pi_c(i)\right)\right); i = \overline{0, L-1} \quad (14)$$

In most cases the lengths of the dither read and write vectors are equal R=W=M. For small interleaver lengths (L<300), M=4. A further simplification can be made, if the read and write vectors are identical r=w.
The ARP interleaver is derived from a circular permutation and is described with the help of a constant P, which is relative prime to L, a vector Q of size C and an offset index $i_0$. The equation that generates this interleaver is:

$$\pi(i) = \left(Pi + Q(i \bmod C) + i_0\right) \bmod L \quad (15)$$

The vector Q is determined from the equation:

$$Q(j) = CPA(j \bmod C) + CB(j \bmod C) \quad (16)$$

Where A and B are two vectors of length C, with elements ranging from 0 to 8. For small interleaver lengths (L<300), the length C=4. Furthermore, some simplifications can be made if the offset index $i_0$=0 and the A and B vectors are equal to $B=[0,b_1,b_2,b_3]$ and $A=[0,1,0,1]$ or $A=[0,0,1,1]$.
The QPP interleaver is based on a quadratic polynomial and is generated by the following equation:

$$\pi(i) = \left(q_0 + q_1 i + q_2 i^2\right) \bmod L \quad (17)$$

Where $q_0$, $q_1$ and $q_2$ are the coefficients of the quadratic polynomial. If $\gcd(2q_2, L) \neq L$, then the quadratic polynomial is nonlinear. Furthermore, if L is divisible by 4, then relation (17) describes a valid permutation provided that $\gcd(q_1, L) = 1$ and all the prime numbers that divide L, divide $q_2$ as well. A simplification can be made if the shift coefficient $q_0$=0. Also, the pairs of coefficients ($q_1$, $q_2$) and ($q_1$+L/2 mod L, $q_2$+L/2 mod L) lead to identical permutations.

## 4. PROPOSED SEARCH ALGORITHM

In this paragraph the proposed search algorithm is presented. The structured code – matched interleaver generation algorithms presented in the previous section lead to some very good interleavers, not only in terms of performance, but also in terms of memory efficiency. Because of their code – matched nature, the DRP, ARP and QPP interleavers have lower flares than the HS-Random or swap S-Random interleavers. Furthermore, the structured code – matched designs do not have large memory requirement, because their generation method is fully deterministic. This means that for a given set of component convolutional encoders, with a given termination method and a given interleaver length, only a few parameters are stored in memory, unlike the pseudo-random permutations, where the whole interleaver has to be stored in memory.
The main problem that arises with this kind of interleavers is the search algorithm that selects from the whole interleaver family the best possible interleaver for a specific length, termination and code.

| L | p | $[a_0,a_1,a_2,a_3]$ | $[b_0,b_1,b_2,b_3]$ | $[d_{free},n_{free},w_{free}]$ | D | γ | FER_G | FER_F |
|---|---|---|---|---|---|---|---|---|
| **m=2, G=[1 5/7] and post-interleaver termination** | | | | | | | | |
| 40 | 13 | [0,1,0,1] | [0,2,4,6] | [14,17,33] | 4 | 0.1949 | 2.8e-8 | 9.9e-5 |
| 80 | 31 | [0,0,1,1] | [0,6,4,2] | [17,1,2] | 4 | 0.1041 | 1.7e-11 | 3.8e-7 |
| 160 | 71 | [0,0,1,1] | [0,2,2,1] | [21,4,7] | 10 | 0.0766 | 2.3e-13 | 4.5e-8 |
| **m=2, G=[1 5/7] and dual termination** | | | | | | | | |
| 40 | 23 | [0,1,0,1] | [0,2,2,4] | [16,25,100] | 6 | 0.1397 | 7.2e-9 | 4.1e-5 |
| 80 | 49 | [0,1,0,1] | [0,0,1,1] | [21,55,275] | 10 | 0.0703 | 8.2e-12 | 9.4e-7 |
| 160 | 61 | [0,0,1,1] | [0,8,5,2] | [24,35,140] | 16 | 0.0748 | 4.9e-14 | 3.6e-8 |
| **m=3, G=[1 15/13] and post-interleaver termination** | | | | | | | | |
| 40 | 19 | [0,1,0,1] | [0,1,1,2] | [17,15,41] | 4 | 0.1295 | 1.0e-9 | 1.0e-5 |
| 80 | 9 | [0,1,0,1] | [0,3,5,8] | [20,4,8] | 8 | 0.0820 | 2.0e-12 | 1.5e-7 |
| 160 | 17 | [0,1,0,1] | [0,7,0,7] | [23,2,4] | 16 | 0.0366 | 1.0e-14 | 4.7e-9 |
| **m=3, G=[1 15/13] and dual termination** | | | | | | | | |
| 40 | 17 | [0,1,0,1] | [0,1,3,4] | [19,9,45] | 6 | 0.1346 | 2.3e-10 | 2.5e-6 |
| 80 | 67 | [0,1,0,1] | [0,6,7,0] | [24,25,100] | 4 | 0.1231 | 1.7e-13 | 5.2e-8 |
| 160 | 23 | [0,0,1,1] | [0,1,5,2] | [31,97,485] | 8 | 0.0760 | 2.2e-17 | 3.7e-10 |

Table 1: *ARP interleavers with FER bounds computed for SNR=4dB.*

| L | r | w | s | p | $[d_{free},n_{free},w_{free}]$ | D | γ | FER_G | FER_F |
|---|---|---|---|---|---|---|---|---|---|
| **m=2, G=[1 5/7] and post-interleaver termination** | | | | | | | | | |
| 40 | [1,2,3,4] | [1,2,4,3] | 1 | 33 | [14,1,1] | 7 | 0.1949 | 1.6e-9 | 5.8e-6 |
| 80 | [3,1,4,2] | [4,2,3,1] | 0 | 49 | [17,1,2] | 4 | 0.1028 | 1.7e-11 | 3.8e-7 |
| 160 | [2,1,4,3] | [3,4,1,2] | 3 | 73 | [21,2,4] | 12 | 0.0756 | 1.1e-13 | 2.2e-8 |
| **m=2, G=[1 5/7] and dual termination** | | | | | | | | | |
| 40 | [3,1,4,2] | [3,1,4,2] | 1 | 23 | [16,23,92] | 6 | 0.1372 | 6.6e-9 | 3.8e-5 |
| 80 | [2,1,4,3] | [2,1,4,3] | 2 | 51 | [21,54,270] | 10 | 0.0718 | 8.1e-12 | 9.2e-7 |
| 160 | [2,1,3,4] | [3,1,2,4] | 1 | 13 | [24,32,64] | 13 | 0.0734 | 4.4e-14 | 3.3e-8 |
| **m=3, G=[1 15/13] and post-interleaver termination** | | | | | | | | | |
| 40 | [2,1,4,3] | [2,1,4,3] | 2 | 19 | [17,13,43] | 4 | 0.1282 | 8.8e-10 | 8.9e-6 |
| 80 | [2,1,4,3] | [1,2,3,4] | 1 | 59 | [20,4,8] | 8 | 0.0715 | 2.0e-12 | 1.5e-7 |
| 160 | [3,1,2,4] | [2,1,4,3] | 3 | 73 | [24,3,6] | 11 | 0.0763 | 4.1e-15 | 3.0e-9 |
| **m=3, G=[1 15/13] and dual termination** | | | | | | | | | |
| 40 | [1,2,3,4] | [1,3,4,2] | 1 | 7 | [19,3,15] | 5 | 0.2667 | 7.9e-11 | 8.3e-7 |
| 80 | [2,4,1,3] | [4,2,3,1] | 1 | 11 | [25,93,367] | 8 | 0.1244 | 1.8e-13 | 8.7e-8 |
| 160 | [3,4,2,1] | [2,4,1,3] | 1 | 61 | [31,61,450] | 12 | 0.0748 | 1.4e-17 | 2.3e-10 |

Table 2*: DRP interleavers with FER bounds computed for SNR=4dB.*

| L | $[q_0,q_1,q_2]$ | $[d_{free},n_{free},w_{free}]$ | D | γ | FER_G | FER_F |
|---|---|---|---|---|---|---|
| **m=2, G=[1 5/7] and post-interleaver termination** | | | | | | |
| 40 | [37,3,10] | [11,2,4] | 4 | 0.1359 | 1.5e-7 | 1.3e-4 |
| 80 | [70,9,20] | [14,1,2] | 10 | 0.0712 | 9.2e-10 | 4.5e-6 |
| 160 | [39,19,40] | [21,2,4] | 16 | 0.0361 | 1.1e-13 | 2.2e-8 |
| **m=2, G=[1 5/7] and dual termination** | | | | | | |
| 40 | [7,3,10] | [12,13,52] | 4 | 0.1359 | 5.2e-7 | 5.0e-4 |
| 80 | [5,11,20] | [21,54,270] | 10 | 0.0712 | 8.1e-12 | 9.2e-7 |
| 160 | [6,19,40] | [24,284,1136] | 16 | 0.0365 | 3.9e-13 | 2.9e-7 |
| **m=3, G=[1 15/13] and post-interleaver termination** | | | | | | |
| 40 | [38,19,10] | [15,6,14] | 4 | 0.1231 | 4.9e-9 | 2.0e-5 |
| 80 | [55,51,20] | [20,8,16] | 10 | 0.0693 | 4.1e-12 | 3.0e-7 |
| 160 | [0,89,40] | [23,7,19] | 10 | 0.0362 | 3.6e-14 | 1.6e-8 |
| **m=3, G=[1 15/13] and dual termination** | | | | | | |
| 40 | [0,13,10] | [18,17,102] | 4 | 0.1372 | 1.4e-9 | 1.0e-5 |
| 80 | [27,9,20] | [23,45,225] | 10 | 0.0712 | 1.1e-12 | 2.1e-7 |
| 160 | [8,61,40] | [31,245,1225] | 16 | 0.0362 | 5.7e-17 | 9.3e-10 |

Table 3: *QPP interleavers with FER bounds computed for SNR=4dB.*

The proposed search algorithm is based on the distance spectrum of the interleaver, which is computed with the true distance measurement method [7]. The algorithm can be synthesised using the following steps:

a). From the all the possible parameters select those which lead to valid permutations. In the case of the DRP and ARP interleavers, gcd(p,L)=1 and gcd(P,L)=1, respectively. In the case of the QPP interleaver, $\gcd(2q_2, L) \neq L$, so that the permutation is not reducible to a monomial permutation. Furthermore, because the considered lengths are divisible by 4, $\gcd(q_1, L) = 1$ and all the prime numbers that divide L, divide $q_2$ as well. In order to avoid identical permutations, only a pair from ($q_1$, $q_2$) and ($q_1$+L/2 mod L, $q_2$+L/2 mod L) is retained.

b). For all the previously retained parameter combinations, compute the first spectral line. Select those parameter combinations that have the largest free distance $d_{free}$. The distance spectrum computation takes into account the termination method used for the trellises of the constituent convolutional encoders.

c). If there are more than one selected permutation, keep that which leads to the fewest multiplicities $w_{free}$ (for BER optimization) or $n_{free}$ (for FER optimization).

d). Provided there is more than one retained interleaver, select that which has the largest D spread parameter.

e). If there is more than one selected interleaver, keep that which has the largest $\gamma$ normalized dispersion.

f). Finally, among the remaining interleavers, make Monte Carlo simulations under AWGN and RMF conditions and select that with the best error rates.

The proposed search algorithm has several advantages. First of all, the search method is done in a systematic and exhaustive manner, unlike the methods presented in [2] for QPP interleavers or [6] for ARP interleavers. Secondly, the algorithm uses a real distance spectrum measurement method, which can determine not only the distances, but also the multiplicities, unlike the method presented in [8] for DRP interleavers. Thirdly, besides the distance spectrum, the proposed algorithm also takes into account some important interleaver parameter such as dispersion and spread. Furthermore, the algorithm leads to better interleavers in terms of error flares than the ones found using the search algorithm for QPP interleavers presented in [9]. Finally, the search algorithm can be applied to all three types of structured code-matched interleavers and was validated using both the post-interleaver and the dual trellis termination methods.

## 5. SPECTRAL DISTANCES AND ERROR BOUNDS

This section presents the error bounds of the selected code-matched interleavers. The proposed search algorithm was applied to three different interleaver lengths L=40, L=80 and L=160, using two generator polynomials G=[1, 5/7] of memory m=2 and G=[1, 15/13] of memory m=3. Furthermore, two trellis termination methods have been considered, namely the post-interleaver and the dual termination. The selected interleavers are depicted in Tables 1, 2 and 3, together with the first spectral line, dispersion and spread. Additionally, the tables contain the theoretical FER bounds computed at SNR=4dB for both AWGN (FER_G) and RMF (FER_F) channels, which were computed using equations (2), (4), (6) and (7). The results yield that in terms of the distance spectrum, the dual termination provides better error rates than the post-interleaver termination. Furthermore, DRP and ARP interleavers have the largest free distances, whereas QPP interleavers have the lowest, but still larger than the ones found through other search methods [10].

## 6. CONCLUSIONS AND FUTURE WORK

This paper proposes a new search method for selecting the best structured code – matched interleaver for a given length, convolutional constituent code and termination method. The method has been validated for DRP, ARP and QPP interleavers for post-interleaver and dual trellis termination methods. The obtained results are confirmed in terms of free distance for the DRP and ARP interleavers and have given better permutations in the case of the QPP interleavers than the best previously known permutations of this type.

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