PHASE APPROXIMATIONS FROM GAIN SAMPLES - THE LOGARITHMIC FREQUENCY DOMAIN - A COMPARISON

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ABSTRACT

The aim of this paper is to compare different phase approximation methods from gain samples, in nepers, in the logarithmic frequency domain, for different applications in signals reconstruction. In general, our attention was focused on those methods that required finally only a small number of samples to approximate the unknown component. For each type of data different ways to approximate the unknown component is suitable.

Keywords: *Kramers-Kronig transform, phase approximation, Bode relations.*

1. INTRODUCTION

Hilbert transform (HT) and the related Bode relations have been recognized as very important methods in circuit theory, communications and control sciences [1]. HT arises also in many applications from optics, where it is known as Kramers-Kronig transform (KKT).

The logarithmic sampling was not a very often used way in the transform domain; the usual way was the uniform sampling. The recent spread of the filter banks and wavelet transform reconsiders more frequently other types of sampling besides the uniform one [2]. Taking into account the weights values, one can see that the borderer samples contribute most to the real value of the signal. Hence, it is more efficient to concentrate on the neighbors samples near the interest point and to omit the farther ones; this was an important reason for the developing of some nonuniform sampling methods.

The goal of this paper is to compare different phase approximation methods from gain samples, in nepers, in the logarithmic frequency domain. Our attention was focused on those methods that required finally only a small number of samples to approximate the unknown component.

The paper is organized as follows. The most important theoretical aspects regarding the subject of this paper are presented in Section 2: HT, KKT, and Bode relations. Section 3 deal with the phase approximation methods: logarithmic derivative, logarithmic difference, Newton-Cotes and Simpson approaches. In Section 4 the used test data are described: ideal test data and also variants of them affected by perturbations. Section 5 is dedicated to the phase approximations performances evaluation, respectively in Section 6 conclusions are presented.

2. HILBERT TRANSFORM

Let us consider $H(\omega)$ the Fourier transform of a causal function $h(t)$:

$$H(\omega) = \int_0^\infty h(t)e^{-j\omega t}dt = R(\omega) + jI(\omega), \qquad (1)$$

then we have

$$R(\omega) = R(\infty) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{I(\omega)}{y-\omega}dy,$$
$$I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{y-\omega}dy,$$

which establish the Hilbert pair of $R(\omega)$ and $I(\omega)$. By taking logarithms, after fulfilling the requirements needed to satisfy the right half plane analyticity conditions of the HT, i.e. the stable and minimum phase conditions, the Bode (or gain-phase) relationships are obtained [3]. Under the assumption that $H(s)$ is not only analytic, but has no zeros for $Re(s) \geq 0$, the phase $\beta(\omega)$ will be uniquely determined from the gain (in nepers) $\alpha(\omega)$:

$$\beta(\omega) = \frac{2\omega}{\pi}\int_0^\infty \frac{\alpha(y)-\alpha(\omega)}{y^2-\omega^2}dy. \qquad (2)$$

According to KK theory, the attenuation/amplification of light is always connected with a phase shift. The interaction between a light wave and matter can be described with a complex susceptibility of the material, whose real and imaginary parts are connected with the phase shift and the amplitude variation of the wave, respectively [4].

3. PHASE APPROXIMATION METHODS

Considering the sampling points $\left\{\omega_j \middle| \omega_j = \omega_c\Delta^j, j \in \mathbf{Z}\right\}$, with $\Delta > 1$, the phase can be expressed as:

$$\beta(\omega_c) \approx \sum_{n\in\mathbf{N}} \Gamma(n,\Delta,\omega_c)\left[\alpha\left(\omega_c\Delta^n\right) - \alpha\left(\omega_c\Delta^{-n}\right)\right], \qquad (3)$$

where $\Gamma(n,\Delta,\omega_c)$ is the function to be determined.

3.1 Approximation by Use of Quadrature Formulae

For numerical computational support, it is of interest to use a quadrature formula where the phase to be determined by gain samples [5]. The condition of equally spaced abscissa leads to one of the Newton-Cotes or Simpson's quadrature approaches. At the beginning we select the trapezoidal formula (Newton-Cotes) and we obtain the first approximation $\beta_T(\omega)$:

$$\beta_T(\omega) = \sum_{p\in\mathbf{Z}} T_p \alpha(\omega\Delta^p), \tag{4}$$

where $T_p = T_{-p}$

$$T_p = \begin{cases} \frac{1}{\pi}\left(1+\frac{\ln\Delta}{\Delta-\Delta^{-1}}\right), & p=1 \\ \frac{2\ln\Delta}{\pi\left(\Delta^p-\Delta^{-p}\right)}, & p=\overline{2,k-1} \\ \frac{\ln\Delta}{\pi\left(\Delta^p-\Delta^{-p}\right)}, & p=k \\ 0, & \text{otherwise} \end{cases}$$

The parabolic rule (Simpson), for $k = 2m+1$ gives the second proposed quadrature approach $\beta_S(\omega)$:

$$\beta_S(\omega) = \sum_{p\in\mathbf{Z}} S_p \alpha(\omega\Delta^p), \tag{5}$$

where $S_p = S_{-p}$

$$S_p = \begin{cases} \frac{1}{\pi}\left(1+\frac{2/3\ln\Delta}{\Delta-\Delta^{-1}}\right), & p=1 \\ \frac{8\ln\Delta}{3\pi\left(\Delta^p-\Delta^{-p}\right)}, & p=\pm2,\ldots,\pm(k-1) \\ \frac{4\ln\Delta}{3\pi\left(\Delta^p-\Delta^{-p}\right)}, & p=\pm3,\ldots,\pm(k-2) \\ \frac{2\ln\Delta}{3\pi\left(\Delta^p-\Delta^{-p}\right)}, & p=\pm k \\ 0, & \text{otherwise} \end{cases}$$

3.2 Logarithmic Approximation

A popular way for phase approximation consists in using of gain logarithmic derivative:

$$\beta(\omega_c) \approx \frac{\pi}{2}\frac{\partial\alpha(\omega_c,\Delta)}{\partial\Delta}. \tag{6}$$

Another method for phase evaluation uses an approximative formula (with first order logarithmic difference):

$$\beta(\omega_c) \approx \frac{\pi}{2}\frac{\alpha(\omega_c\Delta)-\alpha\left(\omega_c\Delta^{-1}\right)}{\Delta-\Delta^{-1}}.$$

The relationship using four terms (second order logarithmic difference) was proposed in [6]:

$$\beta(\omega_c) = \sum_{n=1}^{2} a_n\left[\alpha\left(\omega_c 2^{-n}\right)-\alpha\left(\omega_c 2^{n}\right)\right]. \tag{7}$$

Other methods implement the phase evaluation using fourth order logarithmic difference:

$$\beta(\omega_c) = \sum_{n=1}^{4} a_n\left[\alpha\left(\omega_c 2^{-n}\right)-\alpha\left(\omega_c 2^{n}\right)\right]. \tag{8}$$

Parameters values a_n are given in Table 1.

Table 1: Values for a_n parameters

n	a_n		
	Set I [6]	Set II [7]	Set III [8]
1	-0.14195	−0.44530	−0.48499
2	-0.44688	−0.22726	−0.18538
3		0.11000	0.09418
4		−0.13458	−0.13432

4. TEST DATA

To test the phase approximations we will use classical RLC system functions, respectively functions characterized by a main lobe and also some secondary lobes [3].

4.1 Bode Circuit Function

The Bode circuit functions [1] have various practical applications. They are minimum phase functions, being the input impedances of some realizable circuits. Relation (9) corresponds to the circuit from Figure 1.

$$H_a(s) = Z_{in}(s) = \frac{1|}{|As}+\frac{1|}{|Bs}+\frac{1|}{|Cs}+\frac{1|}{|D} \tag{9}$$

For the Bode circuit function we have two test data sets [3]:

- *Bode 1* ($C_1 = 22\,\mu$F, $L_2 = 500\,$mH, $C_3 = 33\,\mu$F, $R_4 = 50\,\Omega$); the considered range is $\omega \in [0,\ 5\cdot10^3]$;
- *Bode 2* ($C_1 = 1.8\,\mu$F, $L_2 = 2.5\,$mH, $C_3 = 3\,$nF, $R_4 = 50\,\Omega$); the considered range is $\omega \in [0,\ 4\cdot10^8]$;

Outside this frequency ranges neither the gain, nor the phase presents important variations.

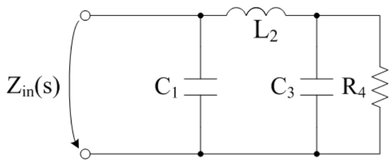

Figure 1: *π Bode circuit*

4.2 Gaussian Function

We consider also data modeled by a Gaussian function:

$$A(\omega) = G\exp\left[-a(\omega-\omega_0)^2\right], \tag{10}$$

Centering the Gaussian in the origin (i.e., $\omega_0 = 0$) and considering $G = 1$ [3], the HT of the Gaussian is [9]:

$$\mathscr{H}\{A(\omega)\} = -j\exp\left(-a\omega^2\right)\mathrm{erf}\left(j\sqrt{a}\omega\right) \tag{11}$$

For $\omega \in [0,20]$, using $a = 1$, the test data set obtained [3] is called *Gaussian*.

4.3 Lorentzian Function

In order to describe dielectric and atomic systems the transfer function of an harmonic-oscillator system is used, which has a Lorentzian form [4]:

$$A(\omega) = \frac{1}{\pi}\frac{a}{a^2+(\omega-\omega_0)^2}. \tag{12}$$

The HT of the Lorentzian can be obtained in closed form as:

$$\mathscr{H}\{A(\omega)\} = \frac{1}{\pi}\frac{(\omega-\omega_0)}{a^2+(\omega-\omega_0)^2}. \quad (13)$$

For $\omega \in [0,20]$, considering the constants $a = 1$ and $\omega_0 = 0$, the obtained data set [3] is called *Lorentzian*.

4.4 Attenuation Coefficient of a Transmitted Beam in a Thick Holographic Grating

The functions that appear in the case of a transmitted beam through a thick holographic grating near the Bragg mismatch angle, can be evaluated using KK relationships [10]. According to KK theory, the diffraction efficiency is:

$$\Psi(\omega) = \frac{\sin^2\omega}{\omega^2}, \quad (14)$$

respectively the corresponding real part:

$$\phi(\omega) = \frac{1}{\omega}\left[\frac{\sin(2\omega)}{2\omega} - 1\right]. \quad (15)$$

For $\omega \in [0,50]$ the test data are presented in [3] and they will be called further as *Attenuation coefficient*.

4.5 Test Data Altered by Perturbations

In order to study the robustness of the phase approximations from gain samples, at parameters changing, we have generated test data corrupted by perturbations [3]. For any ideal frequency response $H(\omega_k)$, noise corrupted data was taken as $\overline{H}(\omega_k) = H(\omega_k) + \Delta H(\omega_k)$, where $\mathrm{Re}\{\Delta H(\omega_k)\}$ and $\mathrm{Im}\{\Delta H(\omega_k)\}$, are normally distributed with $3\sigma = |H(j\omega_k)|\eta/100$ [11]; the percentage noise level η is 1%, respectively 5%.

The names for each data sets are the same as for the ones obtained for ideal frequency response data, followed by the level of perturbation [3]:

- *Bode 2* – $\eta = 1\%$, *Bode 2* – $\eta = 5\%$;
- *Lorentzian* – $\eta = 1\%$, *Lorentzian* – $\eta = 5\%$;
- *Gaussian* – $\eta = 1\%$, *Gaussian* – $\eta = 5\%$;
- *Attenuation coefficient* – $\eta = 1\%$, *Attenuation coefficient* – $\eta = 5\%$.

5. COMPARISON AND REMARKS

In order to compare the results given by the different phase approximation methods, in the logarithmic frequency domain, we have used in this paper the L_1 norm criterion [3].

The norm values are illustrated for each method presented in Section 3: in Table 2 considering as test data the ones obtained for frequency responses generated by ideal responses of the transfer function, respectively in Table 3 considering the ideal responses corrupted by noise. The methods are illustrated in tables in the descending order of the performances. The highest values of the norm highlight the poorest performances.

In Tables 2 and 3 next notations are used: LD – logarithmic derivative of gain; LD1 – first order logarithmic difference; LD2 – second order logarithmic difference; LD4-II – fourth order logarithmic difference (a_n from Set II); LD4-III – fourth order logarithmic difference (a_n from Set III); NC – Newton-Cotes; S – Simpson.

Table 2: Comparative results – ideal test data

Test data	Method	L_1 norm value
Bode 1	LD4-III	$1.4776673e-001$
	LD4-II	$1.4825894e-001$
	LD2	$1.6055015e-001$
	LD1	$1.6657468e-001$
	LD	$2.2919071e-001$
	S	$3.5930689e-001$
	NC	$3.6014563e-001$
Bode 2	LD4-III	$1.2400542e-001$
	LD4-II	$1.2425453e-001$
	LD1	$1.3096763e-001$
	LD2	$1.3624512e-001$
	LD	$1.6603472e-001$
	S	$2.4015429e-001$
	NC	$2.4128669e-001$
Lorentzian	NC	$1.2653202e-003$
	S	$1.2837740e-003$
	LD1	$2.6538180e-002$
	LD4-III	$2.6545693e-002$
	LD4-II	$2.6599078e-002$
	LD2	$2.8244093e-002$
	LD	$3.3933893e-002$
Gaussian	NC	$1.1984345e-003$
	S	$1.3207476e-003$
	LD1	$4.2549018e-002$
	LD4-III	$4.2557564e-002$
	LD4-II	$4.2644601e-002$
	LD2	$4.5327194e-002$
	LD	$5.4598002e-002$
Attenuation coefficient	NC	$1.5460832e-003$
	S	$1.8025268e-003$
	LD1	$3.4167457e-002$
	LD4-III	$3.4174345e-002$
	LD4-II	$3.4244494e-002$
	LD2	$3.6406583e-002$
	LD	$4.3878579e-002$

The results described in Tables 2 and 3 are graphically illustrated in Figure 2.

From the obtained results we can say that for those functions that present only a main lobe (i.e., Lorentzian and Gaussian functions), or a main lobe and some secondary lobes (i.e., attenuation coefficient of a transmitted beam through a thick holographic grating), in order to approximate the unknown component, is better to use Simpson or Newton-Cotes approaches. For classical RLC circuit functions (i.e., Bode functions) it is recommended to use the fourth order logarithmic difference in order to evaluate the phase from gain samples, even if we consider frequency response data generated by ideal responses of the transfer function, even if the ideal responses are altered by perturbations.

6. CONCLUSIONS

For all test data sets used, an approximation using 17 points offer the expected result. The phase approximation, HT approximation, respectively KK approximation is not justified to be evaluated using more than 17 points, at least in the studied cases [3], because the computational complexity increases by increasing the number of points used for approximation. Some problems can appear when the function is

Table 3: Comparative results – test data affected by perturbations

Test data	Method	L_1 norm value $\eta = 1\%$	L_1 norm value $\eta = 5\%$
Bode 2	LD4-II	$1.3481051e-001$	$1.4828613e-001$
	LD4-III	$1.3489117e-001$	$1.4857460e-001$
	LD2	$1.4421164e-001$	$1.5463774e-001$
	LD1	$1.5278487e-001$	$1.7725724e-001$
	LD	$1.9558145e-001$	$2.3055920e-001$
	S	$2.3839693e-001$	$2.3543214e-001$
	NC	$2.3975182e-001$	$2.3724263e-001$
Lorentzian	S	$7.1705714e-003$	$1.7260059e-002$
	NC	$7.8274635e-003$	$1.8713108e-002$
	LD4-II	$2.5185301e-002$	$2.5760931e-002$
	LD4-III	$2.5439969e-002$	$2.6393815e-002$
	LD2	$2.6957139e-002$	$2.9661244e-002$
	LD1	$3.1941981e-002$	$4.3167463e-002$
	LD	$4.2039582e-002$	$6.0006318e-002$
Gaussian	S	$1.5214601e-002$	$3.4079203e-002$
	NC	$1.5704758e-002$	$3.5154881e-002$
	LD4-III	$4.5748400e-002$	$5.1665896e-002$
	LD4-II	$4.5808138e-002$	$5.1836008e-002$
	LD2	$4.8598116e-002$	$5.4159992e-002$
	LD1	$5.2526140e-002$	$7.3331420e-002$
	LD	$7.3357580e-002$	$1.1382705e-001$
Attenuation coefficient	S	$2.1219558e-002$	$4.7648471e-002$
	NC	$2.1955541e-002$	$4.9500809e-002$
	LD2	$4.4118018e-002$	$5.5705973e-002$
	LD4-II	$4.4811632e-002$	$6.3102479e-002$
	LD4-III	$4.5014565e-002$	$6.4013063e-002$
	LD1	$6.2332884e-002$	$1.1277209e-001$
	LD	$8.4778758e-002$	$1.4516564e-001$

rapidly modifying in a relatively narrow frequency range; but an approximation using 17 points is adequate even if the function presents a large number of minima/maxima in a narrow frequency range.

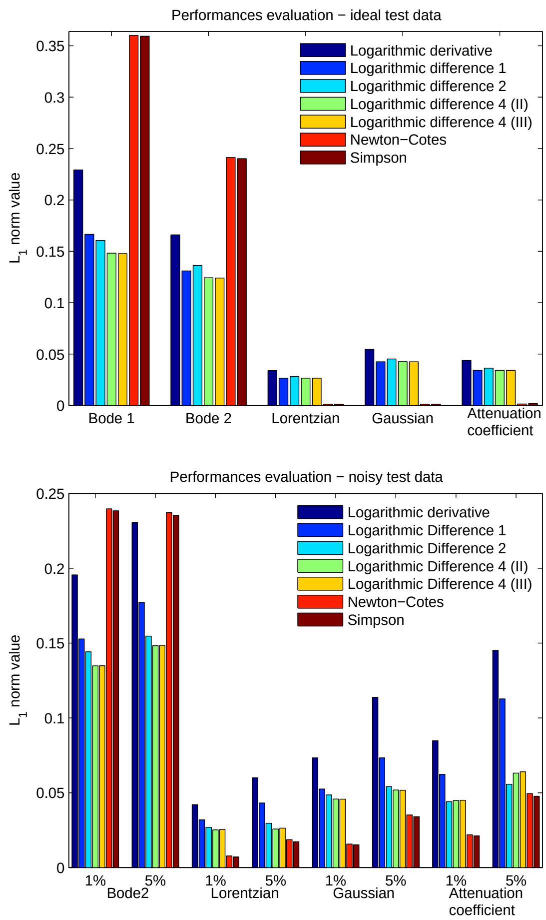

Figure 2*: Performances evaluation using L_1 norm*

REFERENCES

[1] H. W. Bode, *Network Analysis and Feedback Amplifier Design*. Princeton, New Jersy: D. Van Nostrand, 1945.

[2] C. Rusu and P. Kuosmanen, "Phase Approximation by Logarithmic Sampling of Gain," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 50, no. 2, pp. 93–101, February 2003.

[3] L. Grama, "Researches Regarding Phase Approximation from Frequency Response Gain Samples," Ph.D. dissertation, Technical University of Cluj-Napoca, 2009, (in Romanian).

[4] B. E. A. Saleh and M. C. Teich, *Fundamentals of Photonics*, 2nd ed. New York: Wiley, 2007.

[5] C. Rusu, I. Gavrea, and P. Kuosmanen, "On Phase Approximation by Gain Differences," in *Proceedings of ICECS'96*, Lisbon, Portugal, September 1998, pp. 497–500.

[6] M. Wübbenhorst and J. van Turnhout, "Analysis of Complex Dielectric Spectra. i. One-Dimensional Derivative Technique and Three-Dimensional Modelling," *Journal of Non-Crystal. Solids*, vol. 305, no. 1-3, pp. 40–49, July 2002.

[7] P. A. M. Steeman and J. van Turnhout, "A Numerical Kramers-Kronig Transform for the Calculation of Dielectric Relaxation Losses free from Ohmic Conduction Losses," *Journal of Colloid and Polymer Science*, vol. 275, pp. 106–115, 1997.

[8] V. Shtrauss, "FIR Kramers-Kronig Transformers for Relaxation Data Conversion," *Signal Processing*, vol. 86, no. 10, pp. 2887–2900, 2006.

[9] F. W. King, "Efficient Numerical Approach to the Evaluation of Kramers-Kronig Transforms," *Journal of Optical Society of America B*, vol. 19, no. 10, pp. 2427–2436, October 2002.

[10] M. Häckel, L. Kador, C. Frenz, and H. W. Schmidt, "Consequences of the Kramers-Kronig Relations for Light Diffraction on Thick Gratings," *Optics Letters*, vol. 29, no. 15, pp. 1721–1723, August 2004.

[11] A. H. Whitfield, "Transfer Function Synthesis using Frequency Response Data," *International Journal of Control*, vol. 43, no. 5, pp. 1413–1426, May 1986.

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